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Topology/Group Theory

Integral cohomology of the Milnor fibre of the discriminant bundle associated with a finite Coxeter group

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Abstract

Let W be a finite Coxeter group generated by real reflections in a complex vector space. We compute the integral cohomology of the Milnor fibre of the discriminant bundle $\Delta : \mathbb{C}^n / W \to \mathbb{C}$, together with the action of the monodromy, for the whole list of exceptional groups. Here Δ is the map induced by the square of the polynomial defining the *arrangement* of reflection hyperplanes of W. The computation is equivalent to that of the cohomology, with suitable local coefficients, of the corresponding *Artin* group. These computations complete, for the exceptional cases, those performed by De Concini et al. for rational coefficients. ***To cite this article: F. Callegaro, M. Salvetti, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Cohomologie entière de la fibre de Milnor du fibré discriminant d'un groupe de Coxeter fini. Soit W un groupe de Coxeter fini engendré par des réflexions réelles dans un espace vectoriel complexe. On calcule la cohomologie entière de la fibre de Milnor du fibré discriminant $\Delta : \mathbb{C}^n / W \to \mathbb{C}$ et l'action de la monodromie, pour tous les groupes exceptionnels. Ici Δ est l'application induite par le carré du polynôme qui définit *l'arrangement* des hyperplans de réflexion de W. Le calcul équivaut à celui de la cohomologie, à coefficients locaux bien choisis, du *groupe d'Artin* correspondant. Ces calculs complètent, pour les cas exceptionnels, ceux de De Concini et al. à coefficients rationnels. ***Pour citer cet article : F. Callegaro, M. Salvetti, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

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Soit (W, S) un système de Coxeter qui agit sur un espace vectoriel complexe V comme groupe de réflexions réelles, et soit $\mathcal{A}(W)$ *l'arrangement* des hyperplans de reflexion de W. L'application

$$\delta := \prod_{H\in\mathbb{A}(W)} \alpha_H^2,$$

où α_H est une forme linéaire qui définit H, est W-invariante et donc induit

$$\Delta : V/W \to \mathbb{C}.$$

Le revêtement $\pi : V \to V/W$ ramifie sur l'*hypersurface discriminante* $\Sigma := \Delta^{-1}(0)$ donc Δ définit un *fibré discriminant* en dehors de Σ, avec fibre de Milnor $F_1 := \Delta^{-1}(1)$. L'action de monodromie donne à la cohomologie de F_1 la structure de $R := \mathbb{Z}[q, q^{-1}]$-module (qui est un invariant plus fin que la fonction zêta de la monodromie ; voir [12], où l'on considère aussi le cas des groupes de réflexions complexes).

Dans ce travail nous calculons la cohomologie entière, comme R-module, de la fibre de Milnor F_1, pour tous les groupes exceptionnels (Section 4).

Notre méthode est basée sur le calcul suivant.

Soit $G_W = \pi_1((V/W) \setminus \Sigma)$ le *groupe d'Artin* associé à W. Alors on a un isomorphisme (voir [5–7,13])

$$H^*(F_1; A) \cong H^{*+1}(G_W; R_q)$$

(avec un décalage dans les dimensions) où R_q est le G_W-module R avec l'action qui associe à un générateur standard, la multiplication par q.

On utilise ensuite le complexe algébrique (C^*, d^*) suivant dans [17] (voir aussi [8,16]) qui calcule la cohomologie de G_W. Si on identifie l'ensemble des générateurs de Coxeter S avec $\{1, \dots, n\}$ on a :

$$C^k := \oplus R.e_\Gamma, \quad \text{où la somme est prise sur tous } \Gamma \subset \{1, \dots, n\} \text{ avec } |\Gamma| = k,$$

$$d^k(e_\Gamma) := \sum_{j\notin\Gamma} (-1)^{\sigma(j,\Gamma)} \frac{W_{\Gamma\cup\{j\}}(q)}{W_\Gamma(q)} e_{\Gamma\cup\{j\}},$$

où $\sigma(j, \Gamma) := \#\{i \in \Gamma \cup \{j\} \mid i \leqslant j\}$ et $W_\Gamma(q) := \sum_{w\in W_\Gamma} q^{l(w)}$ est le polynôme de Poincaré du sous-groupe parabolique W_Γ [2,14].

Alors :

$$H^*(C^*, \delta) = H^*(G_W; R_{(-q)}).$$

Les coefficients dans la différentielle sont des polynômes à coefficients entiers calculables explicitement en utilisant les *exposants* du groupe.

Nous utilisons ensuite une filtration naturelle du complexe décrit ci-dessus, en définissant F^pC^* comme le sous-complexe engendré par les sous-ensembles de $\{n\}$ qui contiennent les derniers p éléments. On continue par récurrence, puisque le complexe quotient $F^pC^*/F^{p+1}C^*$ est isomorphe, modulo un décalage d'indices, à un sous-complexe $F^rC'^*$, où C'^* est le complexe associé à un graphe de Coxeter avec moins de sommets.

La suite spectrale se termine après un nombre fini d'étapes, et finalement nous résolvons les extensions pour déterminer $H^*(C^*, \delta)$.

Notons que cette méthode permet aussi de construire explicitement des générateurs des groupes de cohomologie considérés (ce dont nous ne parlons pas, par souci de brièveté).

1. Introduction

Let V be a complex vector space where a finite Coxeter system (W,S) acts as a real reflection group. Let $\mathcal{A}(W)$ be the *arrangement* of reflection hyperplanes of W in V, and choose a linear form α_H for each hyperplane $H \in \mathcal{A}(W)$. The map

$$\delta := \prod_{H \in \mathcal{A}(W)} \alpha_H^2$$

is W-equivariant so it induces a map on the *orbit space* V/W (an affine manifold, by a classical result of Shephard–Todd, see e.g. [2], Chapter V, §5)

$$\Delta : V/W \to \mathbb{C}.$$

The covering $\pi : V \to V/W$ is ramified over $\Sigma := \Delta^{-1}(0)$, so Δ is the *discriminant* of the group W and Σ the *discriminant hypersurface*. One has

$$\pi^{-1}(\Sigma) = \bigcup_{H \in \mathcal{A}(W)} H$$

and

$$\pi : V \setminus \left(\pi^{-1}(\Sigma)\right) \to (V/W) \setminus \Sigma$$

is a regular covering of degree $|W|$.

The discriminant hypersurface Σ has complicated (in general, non-isolated) singularities; classically, one tries to study how complicated it looks by considering the *discriminant bundle*

$$\Delta : (V/W) \setminus \Sigma \to \mathbb{C}^*$$

and looking at the topology of the associated (global) Milnor fibre $F_1 := \Delta^{-1}(1)$. The action of the monodromy operator gives to the cohomology of F_1 a structure of R*-module*, where $R = \mathbb{Z}[q, q^{-1}]$ is the group algebra of $\mathbb{Z}$.

While for isolated singularities the theory is well understood [15,1], not much is known in general, and the above discriminant bundle turns out to be one of the most important cases to be understood.

In this Note we compute the integral cohomology, together with its R-module structure, of the Milnor fibre of the discriminant bundle for *the whole list* of exceptional groups. Our computation uses the isomorphism (see next section for details) between the previous cohomology and the cohomology of the fundamental group of the complement to the discriminant (which is called an *Artin group*) with coefficients in a suitable representation over R.

Our results complete in these cases papers [9,10], where the $\mathbb{Q}[q, q^{-1}]$-module structure of the cohomology of all Artin groups was determined.

The fact that in our case R is not a PID produces several difficulties: we solve them by using the spectral sequence associated to a suitable filtration and a case by case computation. On the other hand, this method allows us to write the full list of generators of each cohomology group which we found.

It is also interesting to remark that (because of the isomorphism in Section 2) the $\mathbb{Q}[q, q^{-1}]$-module structure of the (co-)homology is a finer invariant than the zeta function of the monodromy (see [12], where the case of complex reflection groups is considered too). Our computations here concern an even finer invariant, namely the *integral* (co)-homology of the Milnor fibre of the discriminant bundle, and its R-module structure.

Notice that $\mathbb{Z}$-torsion appears in the integral cohomology of the Milnor fibre of Δ, while it is conjectured that the Milnor fibre of δ (restricted to the complement to the arrangement $\mathcal{A}(W)$) is torsion free.

2. Milnor fibre cohomology and algebraic complexes

We keep the notations used in the introduction. Recall that the fundamental group

$$G_W := \pi_1\Bigg(\Bigg(V \setminus \bigcup_{H\in\mathcal{A}(W)} H\Bigg)/W\Bigg) = \pi_1\big((V/W)\setminus \Sigma\big)$$

is called the *Artin group* of type W. The standard presentation of G_W is obtained from the Coxeter presentation of W:

$$G_W = \big\langle g_s,\ s\in S \mid g_s g_{s'} g_s \cdots = g_{s'} g_s g_{s'} \cdots \big(m(s,s') \text{ factors}\big)\big\rangle$$

(see [3,4]).

Let A be any ring, let $R := A[q,q^{-1}]$ be the ring of Laurent polynomials in q with coefficients in A and $R' := A[[q,q^{-1}]]$ that of Laurent series. Let $R_q,\ R'_q$ be the G_W-modules $R,\ R'$ respectively, endowed with a module structure defined by taking standard generators g_s of G_W into q-multiplication. Since $(V/W)\setminus\Sigma$ is a $K(G_W,1)$-space (see [11]), there are isomorphisms:

$$H^*(F_1;A) \cong H^*(G_W;R'_q) \cong H^{*+1}(G_W;R_q);$$

the first one follows from standard results in group cohomology (Shapiro's lemma [5]; see also [7,13]) while the second isomorphism is shown in [6] (notice the shift in dimensions). Actually, these are isomorphisms of R-modules, where the action over the first member is induced by monodromy of the fibering induced by Δ: multiplication by q in the other members corresponds to the action of the standard generator of $\pi_1(\mathbb{C}^*)$ on the first one.

One has similar isomorphism $H_*(F_1;A)\cong H_*(G_W;R_q)$ for the homology (without any shift in dimensions).

We now consider a finite Coxeter group W of rank n and its associated Artin group G_W, as defined above. Identify the set $\{1,\dots,n\}$ with the (ordered) vertices of the Coxeter diagram of W.

We recall from [17] (see also [8,16]) the following result:

Theorem 2.1. *Consider the following algebraic complex (C^*,d^*) of free R-modules*:

$$C^k := \oplus R.e_\Gamma, \quad \text{where the sum is taken over all } \Gamma\subset\{1,\dots,n\} \text{ with } |\Gamma| = k,$$

$$d^k(e_\Gamma) := \sum_{j\notin\Gamma} (-1)^{\sigma(j,\Gamma)} \frac{W_{\Gamma\cup\{j\}}(q)}{W_\Gamma(q)} e_{\Gamma\cup\{j\}},$$

where $\sigma(j,\Gamma) := \#\{i\in\Gamma\cup\{j\} \mid i\leqslant j\}$ and $W_\Gamma(q) := \sum_{w\in W_\Gamma} q^{l(w)}$ is the Poincaré polynomial of the parabolic subgroup W_Γ (see [2,14]).

Then

$$H^*(C^*) = H^*(G_W;R_{(-q)}).$$

It is well-known [2,14] that if $\Gamma'\subset\Gamma$ then $W_{\Gamma'}(q)$ divides $W_\Gamma(q)$ so the coefficients in the above formula are integer polynomials. Recall also that the Poincaré series of a Coxeter group W having *exponents* $e_1,\dots,e_k$ is

$$W(q) = \prod_{i=1}^{k} [e_i].$$

Here we introduce the q-analogue of the number n as

$$[n] := \frac{q^n - 1}{q - 1}.$$

Therefore if $|\Gamma| = k$ and the exponents of $W_{\Gamma\cup\{j\}}$, W_Γ are respectively $l_1, \ldots, l_{k+1}$ and $h_1, \ldots, h_k$ then

$$\frac{W_{\Gamma\cup\{j\}}(q)}{W_\Gamma(q)} = \frac{[l_1]\cdots[l_{k+1}]}{[h_1]\cdots[h_k]}.$$

For example, it is easy to see that in case A_n all these coefficients have the shape of Gauss q-binomials

$$\begin{bmatrix} m \\ k \end{bmatrix} := \frac{[m]\cdots[m-k+1]}{[k]\cdots[1]}.$$

3. Calculation by spectral sequences

There is a natural filtration of C^*: namely, let F_n^p be the subcomplex generated by all subsets of $\{n\}$ containing the last p elements (this is clearly a subcomplex from the boundary formula). So $F_n^0 = C^* \supset \cdots \supset F_n^n = R.e_{\{n\}}$.

Our computations use the spectral sequence of this filtration. One can explicitly describe by induction the first term $E_0^{p,*} = F^pC^*/F^{p+1}C^*$ of the spectral sequence, which is isomorphic, modulo an index shift, to a subcomplex $F^rC'^*$, where C'^* is the complex associated to a Coxeter graph with less vertices.

For example, for the complex associated to A_n we have

$$F^iA_n/F^{i+1}A_n \simeq A_{n-i-1}$$

with $0 \leqslant i \leqslant n-2$ and

$$F^{n-1}A_n/F^nA_n \simeq F^nA_n \simeq R.$$

In a finite number of steps the spectral sequence collapses and at last we have to solve extensions (which in some cases are non-trivial).

4. Results

In the next tables we report the results of our computations, which are complete for all exceptional cases (of course, we write only the non-zero cohomology). All computations are made by hand, using the above mentioned spectral sequence.

The following ideals appear in the tables (here φ_k denote the k-th cyclotomic polynomial):

$$I_4 = (\varphi_2, 2)[60]/\varphi_{60}; \quad J_4 = \big([24]/\varphi_{24}\big);$$
$$I_6 = (\varphi_3\varphi_6\varphi_9\varphi_{12}); \quad I_7 = (\varphi_2\varphi_6\varphi_{14}\varphi_{18});$$
$$I_8 = (\varphi_2, \varphi_4)\varphi_{20}[24][30]/[6].$$

We remark that we also have generators for all the cohomology groups which we computed; we do not write them here for brevity.

Notation. We abbreviate notations by writing a module of the shape $R/(\varphi_k)$, simply as the number k.

Table 1
Exceptional cases

	$I_2(m)$	H_3	H_4	F_4
H^0	0	0	0	0
H^1	2	2	2	2
H^2	$R/[m]$	0	0	2
H^3		$R/(\varphi_2\varphi_6\varphi_{10})$	0	$R/(\varphi_2\varphi_3\varphi_6)$
H^4			R/I_4	R/J_4

	E_6	E_7	E_8
H^0	0	0	0
H^1	2	2	2
H^2	0	0	0
H^3	0	0	0
H^4	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
H^5	$6 \oplus 8$	$6 \oplus \mathbb{Z}/2$	4
H^6	R/I_6	$\mathbb{Z}/3 \oplus (\varphi_2, 3)R/\varphi_6(\varphi_2, 2)$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$
H^7		R/I_7	$8 \oplus 12 \oplus \mathbb{Z}/3$
H^8			R/I_8

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