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Probability Theory

A small step towards the hydrodynamic limit of a colored disordered lattice gas

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Abstract

We consider a system of colored particles in $\mathbb{Z}^d$ driven by a disordered Markov generator similar to that of Faggionato and Martinelli (Probab. Theory Related Fields 127 (2003) 535–608) and submitted to two external chemical potentials. We construct Gibbs measures such that the dynamics is time reversible. ***To cite this article: A. Dermoune, P. Heinrich, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Un petit pas vers la limite hydrodynamique d'un système désordonné de gaz coloré. On considère un système de particules colorées dans $\mathbb{Z}^d$ évoluant selon un générateur de Markov aléatoire analogue à celui de Faggionato et Martinelli (Probab. Theory Related Fields 127 (2003) 535–608) et soumis à deux potentiels chimiques externes. On construit des mesures de Gibbs rendant la dynamique réversible en temps. ***Pour citer cet article : A. Dermoune, P. Heinrich, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Version française abrégée

Cette Note constitue une première étape pour étendre les résultats de Faggionato et Martinelli [1] au cas des particules colorées. On considère un modéle de la dynamique de gaz coloré dans $\mathbb{Z}^d$ en présence de désordre noté α. L'interaction entre les particules dans un volume $\Lambda \subset \mathbb{Z}^d$ a lieu suivant le générateur $\mathcal{L}_\Lambda^\alpha$ (1). Dans le Théorème 1.1, on caractérise les mesures de Gibbs produits pour lesquelles $\mathcal{L}_\Lambda^\alpha$ est réversible. Cette note se termine par l'extension

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au cas des particules colorées des paramètres de potentiel chimique empirique, de potentiel chimique « annealed » et de compressibilité statique qui figurent dans Faggionato et Martinelli [1].

1. Introduction

Consider the d-dimensional lattice $\mathbb{Z}^d$ with sites $x=(x_1,\dots,x_d)$ and canonical basis $\mathcal{E}$. The disorder is described as in [1] by a collection of real i.i.d. random variables $\alpha=\{\alpha_x, x\in\mathbb{Z}^d\}$ such that $|\alpha_x|\leqslant B$ for some finite constant B. The corresponding product measure (resp. expectation) on $\Omega_D:=[-B,B]^{\mathbb{Z}^d}$ will be denoted by $\mathbb{P}$ (resp. $\mathbb{E}$). Consider two species of mechanically identical particles, say blue and white particles. A (particles) configuration η in $\mathbb{Z}^d$ or in a finite subset Λ of $\mathbb{Z}^d$ is given by

$$\eta_x=\begin{cases}+1 & \text{if there is a } \textit{blue} \text{ particle at } x,\\ 0 & \text{if there is } \textit{no} \text{ particle at } x,\\ -1 & \text{if there is a } \textit{white} \text{ particle at } x.\end{cases}$$

Considering a volume $\Lambda\subset\mathbb{Z}^d$, the dynamics of the particles can be described as follows. Each particle, say at x, waits a random time (exponentially distributed) and then jumps to an adjacent empty site, say at $x+e$ where $e\in\mathcal{E}$, with a probabilistic rate given by

$$c^{\alpha}_{x,x+e}(\eta)=f_e\big(\alpha_x,|\eta_x|,\alpha_{x+e},|\eta_{x+e}|\big), \tag{1}$$

where f_e is a bounded function on $(\mathbb{R}\times\{0,1\})^2$ satisfying the following conditions:

(i) $f_e(a,s,a',s')=f_e(a',s',a,s)$ (*symmetry* condition),
(ii) $ss'\neq 0\Rightarrow f_e(a,s,a',s')=0$ (*exclusion* condition),
(iii) $ss'=0\Rightarrow f_e(a,s,a',s')\geqslant\delta>0$ (*uniform bound* condition),
(iv) $f_e(a,s,a',s')=f_e(a,s',a',s)\exp(-(s'-s)(a'-a))$ (needed for the *detailed balance* condition).

These conditions allow us to define a disordered Markov generator whose properties will be described below. All throughout this note, disordered means depending on the r.v. α. The generator mentioned is given for bounded functions f on $\Omega_\Lambda:=\{-1,0,1\}^\Lambda$ by

$$\mathcal{L}^{\alpha}_{\Lambda}f(\eta)=\sum_{\substack{(x,e)\in\Lambda\times\mathcal{E}\\ \{x,x+e\}\subset\Lambda}}c^{\alpha}_{x,x+e}(\eta)\big[f(\eta^{x,x+e})-f(\eta)\big] \tag{2}$$

where $\eta^{x,x+e}$ is the configuration derived from η by turning only η_x into η_{x+e} and vice versa. Namely, $\eta^{x,x+e}_x=\eta_{x+e}$, $\eta^{x,x+e}_{x+e}=\eta_x$ and the rest is unchanged.

Remark 1. If we take the absolute value of the associated Markov process $(\eta(t))_{t\geqslant 0}$, the one with state space Ω_Λ and generator $\mathcal{L}^{\alpha}_{\Lambda}$, we are exactly in the setting of the work of Faggionato and Martinelli in [1].

Now, we fix a parameter $\lambda\in\mathbb{R}$ and a finite volume Λ. We are looking for disordered probability measures μ^α on Ω_Λ such that, $\mathbb{P}$-almost surely, under μ^α, the random variables $\eta\mapsto\eta_x$, $x\in\Lambda$, are independent and

$$\mu^{\alpha}\big(|\eta_x|\big)=\frac{e^{\alpha_x+\lambda}}{1+e^{\alpha_x+\lambda}}\quad(x\in\Lambda), \tag{3}$$

$$\mu^{\alpha}\big(\{\eta^{x,x+e}\}\big)\,c_{x,x+e}(\eta^{x,x+e})=\mu^{\alpha}\big(\{\eta\}\big)\,c_{x,x+e}(\eta)\quad(x\in\Lambda,\ e\in\mathcal{E},\ \eta\in\Omega_\Lambda). \tag{4}$$

It is well known that the so called detailed balance condition (4) implies the self-adjointness of $\mathcal{L}^{\alpha}_{\Lambda}$ in $L^2(\mu^\alpha)$ and the invariance of μ^α; conversely, the self-adjointness of $\mathcal{L}^{\alpha}_{\Lambda}$ in $L^2(\mu^\alpha)$ implies (4). We can now state a first result.

Theorem 1.1. *Set* $\eta_x^+ = \max(\eta_x, 0)$ *and* $\eta_x^- = \max(-\eta_x, 0)$. *Let* μ^α *be a disordered probability measure on* Ω_Λ *such that, for almost all* α*, under* μ^α*, the random variables* $\eta \mapsto \eta_x$, $x \in \Lambda$, *are independent. Then, for almost all* α*,* (3), (4) *hold if and only if there exists* $(\lambda^+, \lambda^-) \in \mathbb{R}^2$ *such that*

(i) $e^{\lambda^+} + e^{\lambda^-} = e^\lambda$,
(ii) $\mu^\alpha(\eta_x^+) = e^{\alpha_x+\lambda^+}/(1 + e^{\alpha_x+\lambda^+} + e^{\alpha_x+\lambda^-})$, $\mu^\alpha(\eta_x^-) = e^{\alpha_x+\lambda^-}/(1 + e^{\alpha_x+\lambda^+} + e^{\alpha_x+\lambda^-})$ $(x \in \Lambda)$.

Such a (random) measure μ will be denoted more precisely by $\mu_\Lambda^{\alpha,\lambda^+,\lambda^-}$ if the disorder configuration α is fixed. Consider then the so-called grand canonical Gibbs probability measure $\mu_\Lambda^{\alpha,\lambda}$ on $\{0,1\}^\Lambda$ as defined in [1] by

$$\mu_\Lambda^{\alpha,\lambda}\big(\{\eta\}\big) \propto \exp\left[\sum_{x\in\Lambda}(\alpha_x + \lambda)\eta_x\right] \quad \big(\eta \in \{0,1\}^\Lambda\big).$$

By the above proposition, it is easily seen that $\mu_\Lambda^{\alpha,\lambda}$ corresponds to the image of $\mu_\Lambda^{\alpha,\lambda^+,\lambda^-}$ by the transformation $\eta \mapsto |\eta|$ on Ω_Λ. In other words, $\mathbb{P}$-almost surely, for all bounded function f on $\{0,1\}^\Lambda$, we have

$$\mu_\Lambda^{\alpha,\lambda^+,\lambda^-}\big(f \circ |\cdot|\big) = \mu_\Lambda^{\alpha,\lambda}(f).$$

We shall define in the following the Hamiltonian of the system as well as the empirical and annealed chemical potentials and the static compressibility matrix in each volume Λ. Given a couple (λ^+, λ^-), let λ be such that $e^\lambda = e^{\lambda^+} + e^{\lambda^-}$. If we define the Hamiltonian of the system in the volume Λ by

$$H_\Lambda^{\alpha,\lambda^+,\lambda^-}(\eta) = -\sum_{x\in\Lambda}\big[\alpha_x|\eta_x| + \lambda^+\eta_x^+ + \lambda^-\eta_x^-\big],$$

then the corresponding *grand canonical Gibbs measure* on Ω_Λ coincides with $\mu_\Lambda^{\alpha,\lambda^+,\lambda^-}$, namely

$$\mu_\Lambda^{\alpha,\lambda^+,\lambda^-}\big(\{\eta\}\big) \propto \exp\big(-H_\Lambda^{\alpha,\lambda^+,\lambda^-}(\eta)\big).$$

Let α be a given disorder configuration. For each $(m^+, m^-) \in [0,1]^2$ such that $m^+ + m^- \leqslant 1$, we define also the *canonical measures* $\nu_{\Lambda,m^+,m^-}^\alpha$ as follows. Let $N_\Lambda^+(\eta)$ be the number of blue particles in the volume Λ and $m^+ \in \{0, \frac{1}{|\Lambda|}, \dots, 1\}$. Define similarly $N_\Lambda^-(\eta)$ and m^- for the white particles. Then

$$\nu_{\Lambda,m^+,m^-}^\alpha\big(\{\eta\}\big) = \mu_\Lambda^{\alpha,\lambda^+,\lambda^-}\big(\{\eta\} \mid N_\Lambda^+ = |\Lambda|m^+, N_\Lambda^- = |\Lambda|m^-\big).$$

Consider the random variables $m_\Lambda^+ = \frac{N_\Lambda^+}{|\Lambda|}$ and $m_\Lambda^- = \frac{N_\Lambda^-}{|\Lambda|}$ which are the densities of blue and white particles. Notice that $\nu_{\Lambda,m^+,m^-}^\alpha$ does not depend on the chemical potentials λ^+, λ^-.

Let $m \in \{0, \frac{1}{|\Lambda|}, \dots, 1\}$. From [1], we know that there exists a unique real λ called *empirical chemical potential* and written $\lambda_\Lambda(\alpha, m)$, such that

$$\mu_\Lambda^{\alpha,\lambda}(m_\Lambda^+ + m_\Lambda^-) = m.$$

Also, the *annealed chemical potential* $\lambda_0(m)$ is the unique real λ such that

$$\mathbb{E}\big[\mu_\Lambda^\lambda\big(|\eta_0|\big)\big] = m,$$

and the corresponding *static compressibility* $\chi(m)$ as $\chi(m) = \mathbb{E}[\mu_\Lambda^{\lambda_0(m)}(|\eta_0|; |\eta_0|)]$ (the notation $\mu(f; g)$ stands for the covariance of f and g w.r.t. μ). The proofs of the following results are similar to [1].

Proposition 1.2. *Fix $\alpha \in \Omega_D$. Let $(m^+, m^-) \in [0,1]^2$ such that $m^+ + m^- \leqslant 1$. Set $m = m^+ + m^-$.*

(i) *The system with the unknown variables (λ^+, λ^-)*

$$\mu_\Lambda^{\alpha,\lambda^+,\lambda^-}(m_\Lambda^+) = \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \frac{\exp(\alpha_x + \lambda^+)}{1 + \exp(\alpha_x + \lambda_\Lambda(\alpha, m^+, m^-))} = m^+,$$
$$\mu_\Lambda^{\alpha,\lambda^+,\lambda^-}(m_\Lambda^-) = \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \frac{\exp(\alpha_x + \lambda^-)}{1 + \exp(\alpha_x + \lambda_\Lambda(\alpha, m^+, m^-))} = m^-,$$
$$\exp(\lambda^+) + \exp(\lambda^-) = \exp\big(\lambda_\Lambda(\alpha, m)\big)$$

has a unique solution $(\lambda_\Lambda^+(\alpha, m^+, m^-), \lambda_\Lambda^-(\alpha, m^+, m^-))$, called the empirical chemical potential of color.

(ii) *The system with the unknown variables (λ^+, λ^-)*

$$\mathbb{E}\big[\mu^{\alpha,\lambda^+,\lambda^-}(\eta_0^+)\big] = \mathbb{E}\left[\frac{\exp(\alpha(0) + \lambda^+)}{1 + \exp(\alpha(0) + \lambda)}\right] = m^+,$$
$$\mathbb{E}\big[\mu^{\alpha,\lambda^+,\lambda^-}(\eta_0^-)\big] = \mathbb{E}\left[\frac{\exp(\alpha(0) + \lambda^-)}{1 + \exp(\alpha(0) + \lambda)}\right] = m^-,$$
$$\exp(\lambda^+) + \exp(\lambda^-) = \exp\big(\lambda_0(m)\big)$$

has a unique solution $(\lambda_0^+(m^+, m^-), \lambda_0^-(m^+, m^-))$ called the annealed chemical potential of color.

(iii) *Set for simplicity $\mu = \mu^{\alpha,\lambda_0^+(m^+,m^-),\lambda_0^-(m^+,m^-)}$. Let us define the colored matrix of static compressibility by*

$$\chi(m^+, m^-) = \mathbb{E}\big[\mu\big((\eta_0^+, \eta_0^-); (\eta_0^+, \eta_0^-)\big)\big]$$

where $\mu((\eta_0^+, \eta_0^-); (\eta_0^+, \eta_0^-))$ is the covariance matrix of the random vector (η_0^+, η_0^-) under the measure μ. Since for any local and bounded function f on $\Omega_D \times \Omega_\Lambda$ (local means that f depends only on finitely many η_x, α_x), we have

$$\frac{\partial}{\partial_{\lambda^+}} \mu_\Lambda^{\alpha,\lambda^+,\lambda^-}(f) = \mu_\Lambda^{\alpha,\lambda^+,\lambda^-}(f; N_\Lambda^+), \qquad \frac{\partial}{\partial_{\lambda^-}} \mu_\Lambda^{\alpha,\lambda^+,\lambda^-}(f) = \mu_\Lambda^{\alpha,\lambda^+,\lambda^-}(f; N_\Lambda^-),$$

we get the 2×2 matrix identities

$$\left[\frac{\partial}{\partial_{m^\varepsilon}} \lambda_\Lambda^{\varepsilon'}(\alpha, m^+, m^-)\colon\ \varepsilon, \varepsilon' \in \{+,-\}\right] = \left[\frac{\partial}{\partial_{\lambda^\varepsilon}} \mu_\Lambda^{\alpha,\lambda^+,\lambda^-}(m_\Lambda^{\varepsilon'})\colon\ \varepsilon, \varepsilon' \in \{+,-\}\right]^{-1},$$
$$\left[\frac{\partial}{\partial_{m^\varepsilon}} \lambda_0^{\varepsilon'}(m^+, m^-)\colon\ \varepsilon, \varepsilon' \in \{+,-\}\right] = \big[\chi(m^+, m^-)\big]^{-1}.$$

Concluding Remark. In the work under way, we define an inner product V_{m^+,m^-} on some space of local functions $\mathcal{G}$ and we derive a 'central limit theorem variance'. We hope to obtain the hydrodynamic limit of this colored disordered lattice gas, similar to [1].

2. Proof of Theorem 1.1

Set

$$m^+(a) = \frac{\mathrm{e}^{a+\lambda^+}}{1 + \mathrm{e}^{a+\lambda^+} + \mathrm{e}^{a+\lambda^-}}, \quad m^-(a) = \frac{\mathrm{e}^{a+\lambda^-}}{1 + \mathrm{e}^{a+\lambda^+} + \mathrm{e}^{a+\lambda^-}}, \quad m(a) = m^+(a) + m^-(a).$$

Assume that conditions (i) and (ii) of Proposition 1.2 hold. Let $x \in \Lambda$. Integrating the relation $\eta_x^+ + \eta_x^- = |\eta_x|$ w.r.t. μ^α gives immediately (3). Let moreover $e \in \mathcal{E}$ and $\eta \in \Omega_\Lambda$. To prove (4), we set $s = |\eta_x|$, $s' = |\eta_{x+e}|$, $a = \alpha_x$ and $a' = \alpha_{x+e}$. Then (4) can be expressed as

$$\mu^\alpha\big(\{\eta^{x,x+e}\}\big) f_e(a, s', a', s) = \mu^\alpha\big(\{\eta\}\big) f_e(a, s, a', s'). \tag{5}$$

If $ss' \neq 0$, (5) is obvious by the exclusion condition (ii) on f_e whereas if $ss' = 0$, the conditions (iii) and (iv) reduce (5) to

$$\mu^\alpha\big(\{\eta^{x,x+e}\}\big) = \mu^\alpha\big(\{\eta\}\big) \exp\big(-(s'-s)(a'-a)\big).$$

Assume for instance that $\eta_x = 0$ and $\eta_{x+e} = 1$. Then, by independence, we have

$$\mu^\alpha\big(\{\eta\}\big) = m^+(a')\big(1 - m(a)\big)\nu^\alpha\big(\big\{\eta_y, y \in \Lambda \setminus \{x, x+e\}\big\}\big),$$
$$\mu^\alpha\big(\{\eta^{x,x+e}\}\big) = m^+(a)\big(1 - m(a')\big)\nu^\alpha\big(\big\{\eta_y, y \in \Lambda \setminus \{x, x+e\}\big\}\big)$$

where ν^α is the the marginal measure of μ^α on $\Omega_{\Lambda \setminus \{x,x+e\}}$. In this case where $s = 0$ and $s' = 1$, it follows that (5) is equivalent to

$$\exp(a' - a) = \frac{m^+(a')(1 - m(a))}{m^+(a)(1 - m(a'))},$$

and this last equality is easily checked by using (ii) and by noting that $1 - m(a) = (1 + \exp(a + \lambda))^{-1}$ and similarly for a'. The other cases are similar by symmetry.

Conversely, assume that conditions (3) and (4) hold. Let $(x, e) \in \Lambda \times \mathcal{E}$. Consider η such that $\eta_x = 0$ and $\eta_{x+e} = 1$. Then, keeping the previous notations, in particular $a = \alpha_x$ and $a' = \alpha_{x+e}$, we have

$$\exp(a' - a) = \frac{m^+(a')(1 - m(a))}{m^+(a)(1 - m(a'))},$$

which can be written as

$$\exp(-a')(1 - m(a'))^{-1} m^+(a') = \exp(-a)(1 - m(a))^{-1} m^+(a).$$

Since the r.v. a' and a are independent w.r.t. $\mathbb{P}$, we deduce that there is a non random positive constant c^+ such that, $\mathbb{P}$-almost surely,

$$\frac{\exp(a)(1 - m(a))}{m^+(a)} = c^+.$$

Note that c^+ does not depend on x too since the law of $a = \alpha_x$ does not depend on x. It follows from the previous equality and $[1 - m(a)]^{-1} = 1 + \exp(a + \lambda)$ that $\mathbb{P}$-almost surely,

$$m^+(a) = c^+ \frac{\exp(-a)}{1 + \exp(a + \lambda)}.$$

Similarly, there exists a non random positive constant c^- such that $\mathbb{P}$-almost surely,

$$m^-(a) = c^- \frac{\exp(-a)}{1 + \exp(a + \lambda)}.$$

Take $\lambda^+ = \ln(c^+)$ and $\lambda^- = \ln(c^-)$. Then (i) and (ii) follow easily.

References

[1] A. Faggionato, F. Martinelli, hydrodynamic limit of a disordered lattice gas, Probab. Theory Related Fields 127 (2003) 535–608.