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Partial Differential Equations

Uniqueness to elliptic and parabolic Hamilton–Jacobi–Bellman equations with non-smooth boundary

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Abstract

In the framework of viscosity solutions, we give an extension of the strong comparison result for Hamilton–Jacobi–Bellman (HJB) equations with Dirichlet boundary conditions to the case of some non-smooth domains. In particular, it may be applied to parabolic problems on cylindrical domains. ***To cite this article: S. Chaumont, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Unicité aux équations d'Hamilton–Jacobi–Bellman elliptiques et paraboliques avec frontière irrégulière. Dans le cadre de la théorie des solutions de viscosité, on donne une extension du principe de comparaison fort pour l'équation d'Hamilton–Jacobi–Bellman (HJB) avec condition au bord de type Dirichlet au cas de certains domaines irréguliers. En particulier, ce résultat est applicable aux problèmes paraboliques posés dans des domaines cylindriques. ***Pour citer cet article : S. Chaumont, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Version française abrégée

On s'intéresse au problème de Dirichlet pour l'équation de Hamilton–Jacobi–Bellman (HJB) du second ordre dégénérée, qui apparaît notamment dans les problèmes de contrôle optimal stochastique avec temps de sortie.

$$\begin{cases} \sup_{\alpha\in A}\left\{-L^{\alpha}u(x)+c(x,\alpha)u(x)-f(x,\alpha)\right\}=0 & \text{dans } \Omega, \\ u(x)=\varphi(x) & \text{sur } \partial\Omega. \end{cases} \tag{1}$$

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Ω est un ouvert de $\mathbb{R}^d$ (avec $d \geqslant 1$). La solution u est une fonction à valeurs réelles définie sur $\overline{\Omega}$. $(L^\alpha)_\alpha$ est une famille d'opérateurs linéaires elliptiques, indexée par un paramètre α variant dans un compact d'un espace métrique A, définie par

$$L^\alpha \psi(x) = b(x,\alpha) \cdot D\psi(x) + \frac{1}{2}\operatorname{trace}\big\{\sigma(x,\alpha)\sigma^*(x,\alpha)D^2\psi(x)\big\}, \quad \forall \alpha \in A, \forall \psi \in C^2(\overline{\Omega}),$$

où $D\psi$ désigne le gradient et $D^2\psi$ la matrice hessienne de ψ. Les coefficients b, σ, f et c sont définis sur $\overline{\Omega} \times A$ et sont respectivement à valeurs dans $\mathbb{R}^d$, $\mathbb{R}^d \otimes \mathbb{R}^p$ (avec $p \geqslant 1$), $\mathbb{R}$ et $]0, +\infty[$, et $\sigma^*(x,\alpha)$ est la matrice transposée de $\sigma(x,\alpha)$. La condition au bord φ est une fonction à valeurs réelles définie sur $\partial\Omega$.

On présente dans cette Note une extension du « principe de comparaison fort » (i.e. un principe de comparaison pour les solutions de viscosité semi-continues : sous des conditions assez faibles, toute sous-solution est inférieure à toute sur-solution) au cas de domaines avec une frontière irrégulière[1], plus précisemment au cas de domaines présentant des irrégularités pointant vers l'extérieur du domaine, l'exemple typique étant une intersection d'ouverts réguliers.

Ce type de résultat est un argument clé pour montrer que la fonction valeur d'un problème de contrôle optimal stochastique est continue et est l'unique solution de viscosité du système de Bellman–Dirichlet associé. Ceci permet en particulier de conclure à la convergence de schémas d'approximation numérique (voir Barles et Souganidis [6]).

Une application intéressante de cette extension est un résultat de comparaison fort pour les problèmes paraboliques, posés sur un cylindre (de type $]0, T[\times\mathcal{D}$). On peut alors conclure à l'unicité et à la continuité de la solution, sans avoir besoin de montrer a priori que la fonction valeur associée est continue.

1. Introduction

We study the Dirichlet problem for the following second-order degenerate Hamilton–Jacobi–Bellman (HJB) equation, arising in stochastic optimal control with exit time:[2]

$$\begin{cases} \sup_{\alpha \in A}\big\{-L^\alpha u(x) + c(x,\alpha)u(x) - f(x,\alpha)\big\} = 0 & \text{in } \Omega, \\ u(x) = \varphi(x) & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Ω is an open subset of $\mathbb{R}^d$ (with $d \geqslant 1$). The solution u is a real-valued function defined on $\overline{\Omega}$. $(L^\alpha)_\alpha$ is a family of linear elliptic operators, indexed by a parameter α taking its values in a compact separable metric space A, defined by

$$L^\alpha \psi(x) = b(x,\alpha) \cdot D\psi(x) + \frac{1}{2}\operatorname{trace}\big\{\sigma(x,\alpha)\sigma^*(x,\alpha)D^2\psi(x)\big\}, \quad \forall \alpha \in A,\ \forall \psi \in C^2(\overline{\Omega}),$$

where $D\psi$ denotes the gradient and $D^2\psi$ the Hessian matrix of ψ. The coefficients b, σ, f and c are defined on $\overline{\Omega} \times A$ and take their values in $\mathbb{R}^d$, $\mathbb{R}^d \otimes \mathbb{R}^p$ (with $p \geqslant 1$), $\mathbb{R}$ and $]0, +\infty[$ respectively. We also denote by $\sigma^*(x,\alpha)$ the transposed matrix of $\sigma(x,\alpha)$. Finally, the boundary condition φ is a real-valued function defined on $\partial\Omega$.

We provide an extension of the 'strong comparison result' (i.e. a comparison type result for semicontinuous viscosity solutions[3]), proven in Barles and Rouy [5], Theorem 2.1, p. 2001 (see also Barles and Burdeau [4]) to the case of domains with a non-smooth boundary, more precisely to the case of domains with irregularities pointing outwards the domain (the typical example being an intersection of regular open sets).

[1] Le cas de domaines avec une frontière régulière a été largement étudié par de nombreux auteurs pour différents types de conditions au bord (Dirichlet [4,5], Neumann [2], contraintes d'état [8]).

[2] See for instance Lions [10], Krylov [9] for a general presentation.

[3] See Crandall, Ishii, Lions [7] and Barles [1] for a presentation of the notion of viscosity solution.

This kind of result is a key argument to establish that the value function of a stochastic exit time control problem is continuous and that it is the unique viscosity solution of the associated Bellman–Dirichlet problem. It is also used to prove the convergence of approximation schemes (cf. Barles and Souganidis [6]). The case when the boundary $\partial\Omega$ is regular has been deeply studied, for different kinds of boundary conditions (Dirichlet [4,5], Neumann [2], state constraints [8]).

An interesting application of this extension is a strong comparison result for parabolic problems, on cylindrical domains (such as $]0,T[\times\mathcal{D}$). Our result allows to conclude to uniqueness and continuity of the solution, and does not require to prove, as usual, that the value function is continuous (see for example [3] where probabilistic arguments are used).

Remark 1. Since we shall always use the notion of viscosity solutions here, we will drop the term 'viscosity' in the whole sequel and simply refer to subsolutions, supersolutions and solutions. We do not recall the definition of these objects which can be found in [4], Definition 1.1, p. 136 for instance.

2. The strong comparison result

We make the following assumptions:

(H1) Ω is an open bounded subset of $\mathbb{R}^d$ and A is a compact separable metric space. The functions σ, b, c and f are continuous on $\overline{\Omega}\times A$. For any $\alpha\in A$, $\sigma(\cdot,\alpha)$ and $b(\cdot,\alpha)$ are Lipschitz continuous functions on $\overline{\Omega}$, moreover

$$\sup_{\alpha\in A}\bigl\|\phi(\cdot,\alpha)\bigr\|_{C^{0,1}(\overline{\Omega})}<\infty,$$

for $\phi=\sigma_{i,j},b_i$ $(1\leqslant i\leqslant d,\ 1\leqslant j\leqslant p)$.

(H2) $c>0$ on $\overline{\Omega}\times A$.

(H3) $\varphi\in C(\partial\Omega)$.

(H4) For any $x\in\partial\Omega$, the set

$$Z(x)=\bigl\{\zeta\in C^2(\mathbb{R}^d)\mid\zeta(x)=0,\ \zeta>0\text{ in }\Omega\text{ and }D\zeta(x)\neq 0\bigr\}$$

is nonempty.

Note that (H4) holds under the exterior ball condition:

$$\forall x\in\partial\Omega,\exists y\in\mathbb{R}^d\setminus\{0\}\text{ such that }\overline{B\bigl(x+y,|y|\bigr)}\cap\overline{\Omega}=\{x\},$$

where $B(x,r)$ denotes the ball with center x and radius r.

Let us note d the distance to the boundary $\partial\Omega$, i.e. $d(x)=\inf_{y\in\partial\Omega}|x-y|$, for all $x\in\overline{\Omega}$. Following the notations of [5], we now introduce:

$$\Gamma_{\text{in}}=\left\{x\in\partial\Omega\ \middle|\ \begin{cases}d\text{ is }C^2\text{ in a neighborhood of }x\text{ and}\\ \forall\alpha\in A,\sigma^*(x,\alpha)Dd(x)=0\text{ and }L^\alpha d(x)\geqslant 0\end{cases}\right\},$$

and

$$\Gamma_{\text{out}}=\left\{x\in\partial\Omega\ \middle|\ \begin{cases}\exists\zeta\in Z(x)\text{ such that}\\ \forall\alpha\in A,\sigma^*(x,\alpha)D\zeta(x)\neq 0\text{ or }L^\alpha\zeta(x)<0\end{cases}\right\}.$$

(H5) Γ_{in} is an open subset of $\partial\Omega$ and $\Gamma=\partial\Omega\setminus(\Gamma_{\text{in}}\cup\Gamma_{\text{out}})$ is an open subset of $\partial\Omega$ satisfying (H6), p. 2000 in [5].

Γ_{in} is defined as in [5], it is a subset of the smooth part of $\partial\Omega$. The definition of Γ_{out} is slightly more general and applies to non-smooth boundaries. This allows us to weaken the assumptions of Theorem 2.1, p. 2001 in [5], now Γ_{in} and Γ_{out} are not necessarily unions of connected components of $\partial\Omega$.

Remark 2. (H5) implies that Γ_{out} is a closed subset of $\partial\Omega$.

Remark 3. If $\partial\Omega$ is $W^{3,\infty}$, as in [5], then Γ_{in} is a closed subset of the boundary by its very definition. So, in this case, (H5) implies that Γ_{in} is a union of connected components of $\partial\Omega$. This means that, in our case, $\partial(\Gamma_{\text{in}})$ is a subset of the non-smooth part of $\partial\Omega$.

Theorem 2.1 (Strong Comparison Result). *Assume that* (H1)–(H5) *hold. If* u (*resp.* v) *is a subsolution* (*resp. supersolution*) *of* (2), *then*

$$u \leqslant v \quad \text{in } \Omega.$$

Example 1 (*Intersection of regular open sets*). Assume that the bounded open set Ω can be written as the intersection of two open sets Ω_1 and Ω_2 such that the distance d_i to the boundary $\partial\Omega_i$ is a C^2 function in a neighborhood of this boundary, for $i = 1$ and 2. If we define the sets

$$\Gamma_{\text{out}}^i = \left\{x \in \partial\Omega_i \mid \forall\alpha \in A, \sigma^*(x,\alpha)Dd_i(x) \neq 0 \text{ or } L^\alpha d_i(x) < 0\right\} \quad \text{for } i = 1 \text{ and } 2,$$

we have

$$x \in (\Gamma_{\text{out}}^1 \cup \Gamma_{\text{out}}^2) \cap \partial\Omega \implies x \in \Gamma_{\text{out}}.$$

Indeed, if $x \in \partial\Omega_i$ for some i then d_i obviously belongs to $Z(x)$. In particular, if $x \in \partial\Omega_1 \cap \partial\Omega_2$ (the non-smooth part of $\partial\Omega$), then d_1 and d_2 belong to $Z(x)$.

Note also that this theorem applies to the case of non-countable intersections, like a cone in $\mathbb{R}^3$.

Proof of Theorem 2.1. The proof is similar to Theorem 2.1, p. 2001 in [5]. Roughly speaking, it consists in considering a maximum point x_0 of $u - v$ on $\overline{\Omega}$ and obtain a contradiction by assuming $u(x_0) - v(x_0) > 0$. Under the assumptions of the theorem, this maximum point is either in the open set $\Omega \cup \Gamma_{\text{in}}$ or on Γ (and then the contradiction is proven in [5], Proposition 4.2, p. 2006 and Theorem 4.1, p. 2009), or in Γ_{out}. In the last case, we conclude using the following result:

Proposition 2.1. *Assume that* (H1)–(H5) *hold. If* u (*resp.* v) *is a subsolution* (*resp. supersolution*) *of* (2), *then*

$$u \leqslant \varphi \leqslant v \quad \text{on } \Gamma_{\text{out}},$$

i.e. the Dirichlet boundary condition holds in the classical sense on Γ_{out}.

Proof of Proposition 2.1. The proof is similar to the regular case (Proposition 4.1, p. 2006 in [5], see also Proposition 1.1, p. 140 in [4]), the only idea is to replace systematically the distance d to the boundary by a function $\zeta \in Z(x_0)$, for each $x_0 \in \Gamma_{\text{out}}$.

3. Application to parabolic problems

Let $T > 0$ and let $\mathcal{D}$ be an open bounded subset of $\mathbb{R}^n$ (with $n \in \mathbb{N}$). We now consider the following parabolic equation with an initial condition for $t = 0$ and a transversal boundary condition for $t \in]0, T[$:

$$\begin{cases} \frac{\partial u}{\partial t}(t, y) + \sup_{\alpha \in A}\{-\Lambda^\alpha u(t, y) + c(t, y, \alpha)u(t, y) - f(t, y, \alpha)\} = 0 & \text{in }]0, T[\times \mathcal{D}, \\ u(0, y) = \varphi(0, y) & \text{on } \overline{\mathcal{D}}, \\ u(t, y) = \varphi(t, y) & \text{on }]0, T[\times \partial\mathcal{D}. \end{cases} \tag{3}$$

For each $t_0 \in [0, T]$, $\Lambda^\alpha(t_0, \cdot)$ is defined by

$$\Lambda^\alpha \psi(t_0, y) = B(t_0, y, \alpha) \cdot D\psi(y) + \frac{1}{2}\operatorname{trace}\{S(t_0, y, \alpha)S^*(t_0, y, \alpha)D^2\psi(y)\}, \quad \forall \alpha \in A,\ \forall \psi \in C^2(\overline{\mathcal{D}}).$$

Note that this equation may be written in the elliptic form (2), by setting

$$d = n + 1, \quad \Omega =]0, T[\times \mathcal{D}, \quad x = (t, y), \quad b = \begin{pmatrix} -1 \\ B \end{pmatrix} \quad \text{and} \quad \sigma = \begin{pmatrix} 0 & \ldots & 0 \\ & S & \end{pmatrix},$$

and considering that φ is defined on $\partial\Omega$ (i.e. even on the terminal boundary $\{T\} \times \mathcal{D}$).

Corollary 3.1 (Strong Comparison Result for parabolic problems). *Assume that* (H1)–(H3) *hold. Let δ be the distance to the boundary $\partial\mathcal{D}$, assume δ is C^2 in a neighborhood of this boundary in $\mathcal{D}$ and*

$$\forall t \in]0, T],\ \forall y \in \partial\mathcal{D},\ \forall \alpha \in A, \quad S^*(t, y, \alpha)D\delta(y) \neq 0 \quad \text{or} \quad \Lambda^\alpha \delta(y) < 0. \tag{4}$$

If u (resp. v) is a subsolution (resp. supersolution) of (3)*, then*

$$u \leqslant v \quad \text{in }]0, T[\times \mathcal{D}.$$

Remark 4. (H3) implies that φ is continuous and satisfies the usual compatibility condition on the initial boundary.

Remark 5. The assumptions of Corollary 3.1 imply that $\Gamma = \emptyset$. The case when Γ is non-empty requires a more complicated proof and we will not treat this case here.

Remark 6. By classical arguments, (H2) may be replaced in this corollary by

(H2′) $c > \lambda$ on $\overline{\Omega} \times A$, for some $\lambda \in \mathbb{R}$.

Proof of Corollary 3.1. (H4) is straightforward, since Ω may be written as the intersection of the smooth domains $]0, T[\times\mathbb{R}^n$ and $\mathbb{R} \times \mathcal{D}$, so we just have to check (H5).

Let us note d_0 the distance to the closed initial boundary $(\{0\} \times \overline{\mathcal{D}})$, we have $d_0(t, y) = t$ and therefore

$$L^\alpha d_0(t, y) = -\frac{\partial d_0(t, y)}{\partial t} + \Lambda^\alpha d_0(t, y) = -1 \quad \text{on } (\{0\} \times \overline{\mathcal{D}}), \text{ for any } \alpha \in A.$$

Using Example 1, this implies $(\{0\} \times \overline{\mathcal{D}}) \subset \Gamma_{\text{out}}$. For the transversal boundary, using again the notations of Section 2 (see also Example 1), assumption (4) reads $(]0, T] \times \partial\mathcal{D}) \subset \Gamma_{\text{out}}$. In a neighborhood of the open terminal boundary $(\{T\} \times \mathcal{D})$, the distance d to $\partial\Omega$ satisfies $d(t, y) = T - t$ (it is thus C^2), so we have

$$S^*(t, y, \alpha)Dd(t, y) = 0 \quad \text{and} \quad -\frac{\partial d(t, y)}{\partial t} + \Lambda^\alpha d(t, y) = 1 > 0 \quad \text{on } (\{T\} \times \mathcal{D}), \text{ for any } \alpha \in A,$$

i.e. $(\{T\} \times \mathcal{D}) \subset \Gamma_{\text{in}}$.

It is now clear that (H5) holds, since $\Gamma_{\text{in}} = (\{T\} \times \mathcal{D})$ is an open subset of $\partial\Omega$ and $\Gamma_{\text{in}} \cup \Gamma_{\text{out}} = (\{0\} \times \overline{\mathcal{D}}) \cup (]0, T] \times \partial\mathcal{D}) \cup (\{T\} \times \mathcal{D}) = \partial\Omega$.

Remark 7. This result may be extended to more general domains. Assume that Ω is the intersection

$$\Omega = \big(]0,T[\times\mathbb{R}^n\big)\cap\left(\bigcap_{i=1}^{q}\Omega_i\right),$$

where for all $i\in\{1,\dots,q\}$, Ω_i is an open subset of $\mathbb{R}^{n+1}$, such that the distance d_i to the boundary $\partial\Omega_i$ is C^2 in a neighborhood of this boundary in Ω_i, and $\frac{\partial d_i}{\partial t}\neq 0$ on $\partial\Omega_i$ for $t\in[0,T]$. Corollary 3.1 still holds with assumption (4) replaced by: for all $(t,y)\in\partial\Omega$, for some $i\in\{1,\dots,q\}$ such that $(t,y)\in\partial\Omega_i$,

$$\forall\alpha\in A,\quad S^*(t,y,\alpha)Dd_i(t,y)\neq 0\quad\text{or}\quad -\frac{\partial d_i(t,y)}{\partial t}+\Lambda^\alpha d_i(t,y)<0. \tag{5}$$

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