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On a Liouville comparison principle for entire weak solutions of quasilinear elliptic partial differential inequalities

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Abstract

We establish a new Liouville-type comparison principle for entire weak solutions of quasilinear elliptic partial differential inequalities of the form $A(u) \leqslant A(v)$ on $\mathbb{R}^n$, $n \geqslant 2$. Typical examples of the operator $A(w)$ are the p-Laplacian and its well-known modifications for $1 < p \leqslant 2$. ***To cite this article: V.V. Kurta, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Sur un principe de comparaison de type Liouville pour des solutions entières faibles d'inégalités aux dérivées partielles elliptiques quasi linéaires. On établit un nouveau principe de comparaison de type Liouville pour des solutions entières faibles d'inégalités aux dérivées partielles elliptiques quasi linéaires de la forme $A(u) \leqslant A(v)$ dans $\mathbb{R}^n$, $n \geqslant 2$. Le p-laplacien et ses modifications bien connues pour $1 < p \leqslant 2$ sont des exemples typiques de l'opérateur $A(w)$. ***Pour citer cet article : V.V. Kurta, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

1. Introduction

It is well known that, in order to obtain and even formulate Liouville's theorem, for example, for superharmonic functions on $\mathbb{R}^2$, one needs to compare an arbitrary superharmonic function with a constant which is, naturally, a trivial subharmonic function. Due to the linearity of the Laplacian one can reformulate this famous result in the form of a Liouville comparison principle: *Let (u, v) be an entire solution of the inequality*

$$\Delta u \leqslant \Delta v \tag{1}$$

on $\mathbb{R}^2$ such that $u(x) \geqslant v(x)$. Then $u(x) = v(x)$, up to a constant, on $\mathbb{R}^2$. On the other hand, it is also well known that for $n \geqslant 3$ there exist non-constant superharmonic functions on $\mathbb{R}^n$ bounded below by a constant. In [3], in

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particular, we established the 'sharp distance at infinity' between the non-constant superharmonic functions on $\mathbb{R}^n$, $n \geqslant 3$, bounded below by a constant and this constant itself in the framework of the Liouville theorem. Again, due to the linearity of the Laplacian one can reformulate this very special result from [3] in the form of a Liouville-type comparison principle: *Let* (u, v) *be an entire solution of inequality* (1) *on* $\mathbb{R}^n$, $n \geqslant 3$, *such that* $u(x) \geqslant v(x)$. *Then either* $u(x) = v(x)$ *on* $\mathbb{R}^n$ *or the relation*

$$\liminf_{r\to+\infty}\Big[\sup_{r\leqslant|x|\leqslant 2r}\big(u(x)-v(x)\big)\Big]r^{(n-2)/(1-\nu)} = +\infty \tag{2}$$

holds with any fixed $\nu \in (0, 1)$. It is important to note here that for $n \geqslant 3$ the pair (u, v) of functions

$$u(x) = \big(1 + |x|^2\big)^{(2-n)/2} \quad \text{and} \quad v(x) = 0 \tag{3}$$

is an entire classical solution of inequality (1) such that (2) holds with any fixed $\nu \in (0, 1)$ and, at the same time, the relation

$$\lim_{r\to+\infty}\Big[\sup_{r\leqslant|x|\leqslant 2r}\big(u(x)-v(x)\big)\Big]r^{n-2} = C, \tag{4}$$

with C a certain positive constant, also holds.

The main purpose of this Note is to characterize in terms of monotonicity basic properties of quasilinear elliptic partial differential operators which make it possible to obtain a Liouville comparison principle for entire weak solutions of quasilinear elliptic partial differential inequalities of the form

$$A(u) \leqslant A(v) \tag{5}$$

on $\mathbb{R}^n$, $n \geqslant 2$; as a by-product, we also include in our consideration the corresponding case of quasilinear ordinary differential inequalities. Note that such properties are inherent for a wide class of quasilinear differential operators, typical examples of which are the p-Laplacian

$$\Delta_p(w) := \operatorname{div}\big(|\nabla w|^{p-2}\nabla w\big) \tag{6}$$

and its well-known modification, see, e.g., [4], p. 155,

$$\tilde{\Delta}_p(w) := \sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial w}{\partial x_i}\right|^{p-2}\frac{\partial w}{\partial x_i}\right) \tag{7}$$

for $1 < p \leqslant 2$ and $n \geqslant 1$.

2. Definitions

Let $A(w)$ be a differential operator given formally by

$$A(w) = \sum_{i=1}^{n}\frac{\mathrm{d}}{\mathrm{d}x_i}A_i(x, \nabla w). \tag{8}$$

Here and in what follows, $n \geqslant 1$. Assume that the functions $A_i(x, \xi)$, $i = 1, \ldots, n$, satisfy the Carathéodory conditions on $\mathbb{R}^n \times \mathbb{R}^n$; namely, they are continuous in ξ at almost all $x \in \mathbb{R}^n$ and measurable in x at all $\xi \in \mathbb{R}^n$.

Definition 2.1. Let $\alpha > 1$ be a given number. The operator $A(w)$ given by (8) is said to be α-monotone if $A_i(x, 0) = 0$, $i = 1, \ldots, n$, at almost all $x \in \mathbb{R}^n$, and if for all $\xi^1, \xi^2 \in \mathbb{R}^n$ and almost all $x \in \mathbb{R}^n$ the following two inequalities hold:

$$0 \leqslant \sum_{i=1}^{n}\big(\xi_i^1 - \xi_i^2\big)\big(A_i(x, \xi^1) - A_i(x, \xi^2)\big), \tag{9}$$

with equality only if $\xi_1 = \xi_2$, and

$$\left(\sum_{i=1}^{n}\bigl(A_i(x,\xi^1) - A_i(x,\xi^2)\bigr)^2\right)^{\alpha/2} \leqslant \mathcal{K}\left(\sum_{i=1}^{n}(\xi_i^1 - \xi_i^2)\bigl(A_i(x,\xi^1) - A_i(x,\xi^2)\bigr)\right)^{\alpha-1} \tag{10}$$

with $\mathcal{K}$ a certain positive constant.

Remark 1. It is important to note, as was established in [1], that the p-Laplacian $\Delta_p(w)$ and its modification $\tilde{\Delta}_p(w)$ are α-monotone for $\alpha = p$ and $1 < p \leqslant 2$. The corresponding algebraic inequalities from which it follows immediately that the operators $\Delta_p(w)$ and $\tilde{\Delta}_p(w)$ satisfy α-monotonicity condition (10) with $\alpha = p$ and $1 < p \leqslant 2$ can be found e.g. in [1,2].

Remark 2. It is also important to note that there exist α-monotone operators with an arbitrary degeneracy of ellipticity. So, for example, the differential operator $\bar{\Delta}_p$ given formally by

$$\bar{\Delta}_p(w) := \operatorname{div}\bigl(a(x)|\nabla w|^{p-2}\nabla w\bigr), \tag{11}$$

with $p > 1$ and an arbitrary function $a(x)$ measurable, uniformly bounded and positive on $\mathbb{R}^n$, is α-monotone with $\alpha = p$ and $1 < p \leqslant 2$.

Definition 2.2. Let $\alpha > 1$ be a given number, and let the operator $A(w)$ given by (8) be α-monotone. By an entire weak solution of inequality (5) on $\mathbb{R}^n$ we understand a pair (u, v) of functions $u, v : \mathbb{R}^n \to \mathbb{R}^1$ such that $u, v \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $|\nabla u|, |\nabla v| \in L^{\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$ and the integral inequality

$$\int_{\mathbb{R}^n}\sum_{i=1}^{n}\varphi_{x_i}A_i(x,\nabla u)\,\mathrm{d}x \geqslant \int_{\mathbb{R}^n}\sum_{i=1}^{n}\varphi_{x_i}A_i(x,\nabla v)\,\mathrm{d}x \tag{12}$$

holds for every nonnegative function $\varphi \in W^1_{\alpha}(\mathbb{R}^n)$ with compact support.

Analogous definitions of entire weak solutions of the inequalities

$$A(u) \leqslant 0 \tag{13}$$

and

$$A(v) \geqslant 0, \tag{14}$$

which are special cases of inequality (5) for $v = 0$ and $u = 0$, respectively, can be immediately obtained from Definition 2.2.

Definition 2.3. Let $\alpha > 1$ be a given number, and let the operator $A(w)$ given by (8) be α-monotone. By an entire weak solution of inequality (13) or (14) on $\mathbb{R}^n$ we understand a function $w : \mathbb{R}^n \to \mathbb{R}^1$ such that $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $|\nabla w| \in L^{\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$ and the integral inequality

$$\int_{\mathbb{R}^n}\sum_{i=1}^{n}\varphi_{x_i}A_i(x,\nabla w)\,\mathrm{d}x \geqslant 0 \quad \text{or} \quad \int_{\mathbb{R}^n}\sum_{i=1}^{n}\varphi_{x_i}A_i(x,\nabla w)\,\mathrm{d}x \leqslant 0, \tag{15}$$

respectively, holds for every nonnegative function $\varphi \in W^1_{\alpha}(\mathbb{R}^n)$ with compact support.

3. Results

Theorem 3.1. *Let* $n \geqslant 2$, $2 \geqslant \alpha > 1$ *and* $n > \alpha$. *Let the operator* $A(w)$ *given by* (8) *be* α*-monotone, and let* (u, v) *be an entire weak solution of* (5) *on* $\mathbb{R}^n$ *such that* $u(x) \geqslant v(x)$ *and* $u - v \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$. *Then either* $u(x) = v(x)$ *on* $\mathbb{R}^n$ *or the relation*

$$\liminf_{r\to+\infty}\Big[\sup_{r\leqslant|x|\leqslant 2r}\big(u(x)-v(x)\big)\Big]r^{(n-\alpha)/(\alpha-1-\nu)} = +\infty \tag{16}$$

holds with any fixed $\nu \in (0, \alpha - 1)$.

Theorem 3.2. *Let* $n \geqslant 2$, $2 \geqslant \alpha > 1$ *and* $n > \alpha$. *Let the operator* $A(w)$ *given by* (8) *be* α*-monotone, and let* (u, v) *be an entire weak solution of* (5) *on* $\mathbb{R}^n$ *such that* $u(x) \geqslant v(x)$. *Then either* $u(x) = v(x)$ *on* $\mathbb{R}^n$ *or the relation*

$$\liminf_{r\to+\infty} r^{-\alpha}\int\limits_{r\leqslant|x|\leqslant 2r}\big(u(x)-v(x)\big)^{\alpha-1-\nu}\,\mathrm{d}x = +\infty \tag{17}$$

holds with any fixed $\nu \in (0, \alpha - 1)$.

Remark 3. Let $n \geqslant 2$, $2 \geqslant \alpha > 1$ and $n > \alpha$. Then the pair (u, v) of functions

$$u(x) = \big(1 + |x|^{\alpha/(\alpha-1)}\big)^{(\alpha-n)/\alpha} \quad \text{and} \quad v(x) = 0 \tag{18}$$

defined and locally bounded on $\mathbb{R}^n$ is an entire weak solution of inequality (5) on $\mathbb{R}^n$ for $A(w) = \Delta_p(w)$ and $A(w) = \tilde{\Delta}_p(w)$, with $p = \alpha$, such that (16) and (17) hold with any fixed $\nu \in (0, \alpha - 1)$ and, at the same time, the relations

$$\lim_{r\to+\infty}\Big[\sup_{r\leqslant|x|\leqslant 2r}\big(u(x)-v(x)\big)\Big]r^{(n-\alpha)/(\alpha-1)} = C_1 \tag{19}$$

and

$$\lim_{r\to+\infty} r^{-\alpha}\int\limits_{r\leqslant|x|\leqslant 2r}\big(u(x)-v(x)\big)^{\alpha-1}\,\mathrm{d}x = C_2, \tag{20}$$

with C_1, C_2 certain positive constants, also hold.

Theorem 3.3. *Let* $n = 1$ *and* $2 \geqslant \alpha > 1$ *or* $n = \alpha = 2$. *Let the operator* $A(w)$ *given by* (8) *be* α*-monotone, and let* (u, v) *be an entire weak solution of* (5) *on* $\mathbb{R}^n$ *such that* $u(x) \geqslant v(x)$. *Then* $u(x) = v(x)$*, up to a constant, on* $\mathbb{R}^n$.

Remark 4. To prove these results we further develop an approach that was proposed for solving similar problems in [1].

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