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Mathematical Analysis

Orthogonal polynomials and a generalized Szegő condition

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Abstract

Asymptotical properties of orthogonal polynomials from the so-called Szegő class are very well-studied. We obtain asymptotics of orthogonal polynomials from a considerably larger class and we apply this information to the study of their spectral behavior. ***To cite this article: S. Denisov, S. Kupin, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Polynômes orthogonaux et la condition de Szegő généralisée. Les propriétés asymptotiques des polynômes orthogonaux de la classe de Szegő sont très bien étudiées. Nous obtenons les asymptotiques des polynômes orthogonaux appartenant à une classe considérablement plus large. Ensuite, nous appliquons cette information à l'étude du comportement spectral de ces derniers. ***Pour citer cet article : S. Denisov, S. Kupin, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

1. Introduction

In this Note, we prove asymptotics for orthogonal polynomials from the Szegő class with a polynomial weight and we apply the information to the study of their spectral behavior.

Let σ be a non-trivial Borel probability measure on the unit circle $\mathbb{T} = \{z\colon\ |z| = 1\}$. Consider orthonormal polynomials $\{\varphi_n\}$ with respect to the measure, $\int_{\mathbb{T}} \varphi_n \overline{\varphi_m}\, d\sigma = \delta_{nm}$ where δ_{nm} is the Kronecker's symbol. It is very well known [3,4,6,7] that polynomials $\{\varphi_n\}$ generate a sequence $\{\alpha_k\}$, $|\alpha_k| < 1$, of the so-called Verblunsky coefficients through special recurrence relations. Conversely, the measure σ (and polynomials $\{\varphi_n\}$) are completely determined by the sequence $\{\alpha_k\}$. Hence, it is natural to express properties of the sequence $\{\alpha_k\}$ and polynomials $\{\varphi_n\}$ in terms of σ and vice versa.

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We say that σ is a Szegő measure ($\sigma \in$ (S), for the sake of brevity), if $\mathrm{d}\sigma = \sigma'_{\mathrm{ac}}\,\mathrm{d}m + \mathrm{d}\sigma_{\mathrm{s}}$ and the density σ'_{ac} of the absolutely continuous part of σ is such that

$$\int_{\mathbb{T}} \log \sigma'_{\mathrm{ac}}\,\mathrm{d}m > -\infty.$$

Here, the singular part of σ is denoted by σ_{s}, and m is the probability Lebesgue measure on $\mathbb{T}$, $\mathrm{d}m(t) = \mathrm{d}t/(2\pi\mathrm{i}t) = 1/(2\pi)\,\mathrm{d}\theta$, $t = \mathrm{e}^{\mathrm{i}\theta} \in \mathbb{T}$.

For instance [3,7], a measure σ belongs to the Szegő class if and only if the corresponding sequence $\{\alpha_k\}$ is in l^2. Moreover, this happens if and only if analytic polynomials are not dense in $L^2(\sigma)$. Asymptotic properties of orthogonal polynomials connected to $\sigma \in$ (S) can be easily described in terms of the function

$$D(z) = \exp\left(\frac{1}{2}\int_{\mathbb{T}} \frac{t+z}{t-z} \log \sigma'_{\mathrm{ac}}(t)\,\mathrm{d}m(t)\right)$$

lying in the Hardy class $H^2(\mathbb{D})$ on the unit disk $\mathbb{D} = \{z\colon |z| < 1\}$. Namely, we have

$$\lim_{n\to\infty} \int_{\mathbb{T}} |D\varphi_n^* - 1|^2\,\mathrm{d}m = 0$$

and, in particular, $\lim_{n\to\infty} D(z)\varphi_n^*(z) = 1$ for every $z \in \mathbb{D}$. Above, $\varphi_n^*(z) = z^n\overline{\varphi_n(1/\bar{z})}$. A modern presentation and recent advances in this direction can be found in [4,6].

It is extremely interesting and important to obtain similar results for different classes of measures. Consider a trigonometric polynomial p, $p(t) \geqslant 0$, $t \in \mathbb{T}$, given by

$$p(t) = \prod_{j=1}^{N} |t - \zeta_j|^{2\kappa_j}. \tag{1}$$

Here $\{\zeta_j\}$ are points lying on $\mathbb{T}$ and κ_j are their "multiplicities". We say that σ is in the polynomial Szegő class (i.e., σ is a (pS)-measure or $\sigma \in$ (pS), to be brief), if $\mathrm{d}\sigma = \sigma'_{\mathrm{ac}}\,\mathrm{d}m + \mathrm{d}\sigma_{\mathrm{s}}$, σ_{s} being the singular part of the measure, and

$$\int_{\mathbb{T}} p(t) \log \sigma'_{\mathrm{ac}}(t)\,\mathrm{d}m(t) > -\infty. \tag{2}$$

The asymptotic behavior of orthogonal polynomials for $\sigma \in$ (pS) is completely described in Theorems 2.2 and 2.3. This information is used to construct wave operators for the so-called CMV-representations in Theorem 2.4. The approximation by analytic polynomials in $L^2(\sigma)$, $\sigma \in$ (pS), is addressed in Theorem 2.5.

2. Results

We fix the polynomial p (1) for the rest of this paper. For the sake of transparency we assume $\kappa_j = 1$; the discussion of the general case follows the same lines. Let $\mathcal{C}$ and $\mathcal{C}_0$ be the CMV-representations connected to σ and m (see [1,6, Chapter 4]), and $\operatorname{rank}(\mathcal{C} - \mathcal{C}_0) < \infty$.

We set $\Phi(\mathcal{C}) = \int_{\mathbb{T}} p(t) \log \sigma'_{\mathrm{ac}}(t)\,\mathrm{d}m(t)$.

Lemma 2.1. *Let* $\operatorname{rank}(\mathcal{C} - \mathcal{C}_0) < \infty$. *Then there is a polynomial* P *such that*

$$\int_{\mathbb{T}} p \log \sigma'_{\mathrm{ac}}\,\mathrm{d}m = a_0 t_0 + \operatorname{Re}\operatorname{tr}\big(P(\mathcal{C}) - P(\mathcal{C}_0)\big) \tag{3}$$

where $a_0 = 2\int_{\mathbb{T}} p\,\mathrm{d}m$, $t_0 = \sum_k \log \rho_k$, *and* $\rho_k = (1 - |\alpha_k|^2)^{1/2}$.

We denote the right-hand side of equality (3) by $\Psi(\mathcal{C})$ and we rewrite it in a different form. To this end, we consider the shift $S: l^2(\mathbb{Z}_+) \to l^2(\mathbb{Z}_+)$, given by $Se_k = e_{k+1}$ and, for a bounded operator A on $l^2(\mathbb{Z}_+)$, we look at $\tau(A) = S^*AS$. Consequently, we see that

$$\Psi(\mathcal{C}) = \sum_{k=0}^{2N+1} \left\{a_0 \log \rho_k + \operatorname{Re}\big(\big(P(\mathcal{C}) - P(\mathcal{C}_0)\big)e_k, e_k\big)\right\} + \sum_{k=0}^{\infty} \psi \circ \tau^k(\mathcal{C})$$

where $\psi(\mathcal{C}) = a_0 \log \rho_{2N+2} + \operatorname{Re}((P(\mathcal{C}) - P(\mathcal{C}_0))e_{2N+2}, e_{2N+2})$. It turns out that there exist functions η and γ, depending on a finite number of arguments, such that for any $\mathcal{C}$ with rank$(\mathcal{C} - \mathcal{C}_0) < \infty$

$$\Psi(\mathcal{C}) = \tilde{\Psi}(\mathcal{C}) = \sum_{k=0}^{2N+1} \left\{a_0 \log \rho_k + \operatorname{Re}\big(\big(P(\mathcal{C}) - P(\mathcal{C}_0)\big)e_k, e_k\big)\right\} + \sum_{k=0}^{\infty} \eta \circ \tau^k(\mathcal{C}) + \gamma(\mathcal{C})$$

and, moreover, η is nonpositive (see [5], Lemma 3.1).

Theorem 2.2 [5, Theorem 1.4]. *A measure σ is polynomially Szegő* (*see* (2)) *if and only if* $\tilde{\Psi}(\mathcal{C}) > -\infty$. *Moreover, in this case* $\Phi(\mathcal{C}) = \tilde{\Psi}(\mathcal{C}) = \Psi(\mathcal{C})$.

We turn now to the description of asymptotic properties of orthogonal polynomials for (pS)-measures. Consider a modified Schwarz kernel $K(t,z) = \frac{t+z}{t-z}\frac{q(t)}{q(z)}$ where $q(t) = C(\prod_j (t-\zeta_j)^2)/t^N$, and the constant C, $|C| = 1$, is chosen in a way that $q(t) \in \mathbb{R}$ for $t \in \mathbb{T}$ (i.e., $C = (\prod_j(-\zeta_j))^{-1}$). Furthermore, define

$$\widetilde{D}(z) = \exp\left(\frac{1}{2}\int_{\mathbb{T}} K(t,z) \log \sigma'_{\mathrm{ac}}(t)\, \mathrm{d}m(t)\right), \qquad \tilde{\varphi}_n^*(z) = \exp\left(\int_{\mathbb{T}} K(t,z) \log\left|\varphi_n^*(t)\right| \mathrm{d}m(t)\right).$$

The functions $\{\tilde{\varphi}_n^*\}$ are called (reversed) modified orthogonal polynomials with respect to σ. It can be readily seen that $|\widetilde{D}|^2 = \sigma'_{\mathrm{ac}}$ and $|\tilde{\varphi}_n^*| = |\varphi_n^*| = |\varphi_n|$ a.e. on $\mathbb{T}$. Furthermore, we see that $\tilde{\varphi}_n^* = \psi_n \varphi_n^*$, where

$$\psi_n(z) = \exp\left(A_{0n} + \sum_{j=1}^{N}\left(A_{jn}\frac{z+\zeta_j}{z-\zeta_j} + B_{jn}\left\{\frac{z+\zeta_j}{z-\zeta_j}\right\}^2\right)\right) \tag{4}$$

and $A_{0n}, B_{jn} \in \mathrm{i}\mathbb{R}$, $A_{jn} \in \mathbb{R}$. The coefficients $\{A_{0n}, A_{jn}, B_{jn}\}_{j,n}$ can be expressed in a closed form through Verblunsky coefficients $\{\alpha_k\}$.

Theorem 2.3. *Let* $\sigma \in (\mathrm{pS})$. *Then*

$$\lim_{n\to\infty} \int_{\mathbb{T}} |\widetilde{D}\tilde{\varphi}_n^* - 1|^2 \, \mathrm{d}m = 0$$

and, in particular, $\lim_{n\to\infty} \widetilde{D}(z)\tilde{\varphi}_n^*(z) = \lim_{n\to\infty} \widetilde{D}(z)\psi_n(z)\varphi_n^*(z) = 1$ *for any* $z \in \mathbb{D}$.

Special versions of this result for Jacobi matrices are obtained in [2,5]. The proof of the theorem is partially based on the sum rules proved in Theorem 2.2. The second important observation is that, for an $\varepsilon > 0$ small enough, $|\widetilde{D}\tilde{\varphi}_n^*(z)| \leqslant \frac{C_\varepsilon}{\sqrt{1-|z|}}$ where $z \in \mathbb{D}\backslash(\bigcup_k B_\varepsilon(\zeta_k))$, $B_\varepsilon(\zeta) = \{z\colon |z-\zeta| < \varepsilon\}$.

We use asymptotics described above, to construct modified wave operators. Let $\mathcal{F}_0: L^2(m) \to l^2(\mathbb{Z}_+)$, $\mathcal{F}: L^2(\sigma) \to l^2(\mathbb{Z}_+)$ be the Fourier transforms associated to the CMV-representations $\mathcal{C}$ and $\mathcal{C}_0$, see [6, Chapter 4]. Recall that $\mathcal{C} = \mathcal{F}z\mathcal{F}^{-1}$, $\mathcal{C}_0 = \mathcal{F}_0 z \mathcal{F}_0^{-1}$.

Theorem 2.4. *Let* $\sigma \in (\mathrm{pS})$. *The limits*

$$\widetilde{\Omega}_{\pm} = \text{s-}\lim_{n\to\pm\infty} \mathrm{e}^{W(2n,\mathcal{C})}\mathcal{C}^n\mathcal{C}_0^{-n}$$

exist. Here

$$W(\mathcal{C}, n) = A_{0n} + \sum_{j=1}^{N}\left(A_{jn}\frac{\mathcal{C}+\zeta_j}{\mathcal{C}-\zeta_j} + B_{jn}\left\{\frac{\mathcal{C}+\zeta_j}{\mathcal{C}-\zeta_j}\right\}^2\right)$$

and coefficients $\{A_{0n}, A_{jn}, B_{jn}\}$ *are defined in* (4). *We also have*

$$\mathcal{F}^{-1}\widetilde{\Omega}_{+}\mathcal{F}_0 = \chi_{E_{\mathrm{ac}}}\frac{1}{\widetilde{D}}, \qquad \mathcal{F}^{-1}\widetilde{\Omega}_{-}\mathcal{F}_0 = \chi_{E_{\mathrm{ac}}}\frac{1}{\overline{\widetilde{D}}}$$

where $E_{\mathrm{ac}} = \mathbb{T}\setminus \operatorname{supp}\sigma_{\mathrm{s}}$.

The proof of the above theorem mainly follows [6, Chapter 10].

We now briefly discuss approximation by analytic polynomials in $L^2(\sigma)$ with $\sigma \in (\mathrm{pS})$. We put $\mathcal{P}_0'$ to be the set of analytic on $\mathbb{D}$ polynomials g with the property $g \neq 0$ on $\mathbb{D}$; normalize them by the condition $g(0) > 0$. Furthermore, for a $g \in \mathcal{P}_0'$, we set $\lambda(g) = \exp(\int_{\mathbb{T}} p\log|g|\,\mathrm{d}m)$ and define $\mathcal{P}_1' = \{g\colon\ g \in \mathcal{P}_0',\ \lambda(g) = 1\}$.

Theorem 2.5. *Let* $\mathrm{d}\sigma = w\,\mathrm{d}m + \mathrm{d}\sigma_{\mathrm{s}}$. *Then*

$$\exp\left(\int_{\mathbb{T}} p\log\frac{w}{p}\,\mathrm{d}m\right) \leqslant \inf_{g\in\mathcal{P}_1'}\|g\|_\sigma^2 = \inf_{g\in\mathcal{P}_0',\ \|g\|_\sigma\leqslant 1}\frac{1}{|\lambda(g)|^2} \leqslant \exp\left(\int_{\mathbb{T}} p\log w\,\mathrm{d}m\right).$$

Remind that σ is a Szegő measure if and only if the system $\{\mathrm{e}^{\mathrm{i}ks}\}_{k\in\mathbb{Z}}$ is uniformly minimal in $L^2(\sigma)$. Saying that σ is a (pS)-measure translates into the uniform minimality of another system, $\{\mathrm{e}^{\mathrm{i}k\nu(s)}\}_{k\in\mathbb{Z}}$, in the same space $L^2(\sigma)$. Above, $\nu(s) = C_0\int_0^s p(\mathrm{e}^{\mathrm{i}s'})\,\mathrm{d}s'$ where $s, s' \in [0, 2\pi]$ and the constant C_0 comes from the condition $C_0\int_{\mathbb{T}} p\,\mathrm{d}m = 1$, see [5], Lemma 2.2.

We conclude the note with a few examples. For instance, classical Pollaczek polynomials [7] belong to the (pS)-class with $p(\mathrm{e}^{\mathrm{i}\theta}) = \sin^2\theta$. It was proved recently in [6, Chapter 2] that $\sigma \in (\mathrm{p_0S})$ with $p_0(\mathrm{e}^{\mathrm{i}\theta}) = 1 - \cos\theta$ if and only if $\{\alpha_k\} \in l^4$ and $\{\alpha_{k+1} - \alpha_k\} \in l^2$. Theorems 2.2–2.5 also apply to this case and yield explicit formulas for $\{\tilde{\varphi}_n^*\}$, $\{\psi_n\}$, $\widetilde{D}$ and coefficients $\{A_{0n}, A_{jn}, B_{jn}\}$.

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