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Hyperbolic components in exponential parameter space

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Abstract

We discuss the space of complex exponential maps $E_\kappa : z \mapsto e^z + \kappa$. We prove that every hyperbolic component W has connected boundary, and there is a conformal isomorphism $\Phi_W : W \to \mathbb{H}^-$ which extends to a homeomorphism of pairs $\Phi_W : (\overline{W}, W) \to (\overline{\mathbb{H}^-}, \mathbb{H}^-)$. This solves a conjecture of Baker and Rippon, and of Eremenko and Lyubich, in the affirmative. We also prove a second conjecture of Eremenko and Lyubich. ***To cite this article: D. Schleicher, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Composantes hyperboliques dans l'espace des applications exponentielles. Nous étudions l'espace des applications exponentielles complexes $E_\kappa : z \mapsto e^z + \kappa$. Nous démontrons que pour chaque composante hyperbolique W, le bord ∂W est connexe, et qu'il y a un isomorphisme biholomorphe $\Phi_W : W \to \mathbb{H}^-$ qui s'étend en un homéomorphisme de paires $\Phi_W : (\overline{W}, W) \to (\overline{\mathbb{H}^-}, \mathbb{H}^-)$. Ceci établit une conjecture de Baker et Rippon, et de Eremenko et Lyubich. D'autre part, nous démontrons une autre conjecture de Eremenko et Lyubich. ***Pour citer cet article : D. Schleicher, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Version française abrégée

Dans l'espace des applications exponentielles $E_\kappa : z \mapsto e^z + \kappa$ (voir Fig. 1), chaque composante hyperbolique $W \subset \mathbb{C}$ est simplement connexe, et il y a un isomorphisme conforme $\Phi_W : W \to \mathbb{H}^-$ (le demi-plan gauche) tel que l'application des multiplicateurs $\mu : W \to \mathbb{D}^*$ se décompose comme $\mu = \exp \circ \Phi_W$. Il est assez facile de voir que Φ_W s'étend en une application continue $\Phi_W : (\overline{W}, W) \to (\overline{\mathbb{H}^-}, \mathbb{H}^-)$. Notre résultat principal est que ceci est un homéomorphisme. Pour chaque $h \in \mathbb{R}$, nous considérons le *rayon interne* $\Gamma_{W,h} : \mathbb{R}^- \to W$, $\Gamma_{W,h}(t) = \Phi_W^{-1}(t +$

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$2\pi \mathrm{i}h$). On démontre facilement que pour chaque h, $\Gamma_{W,h}(t)$ a une limite dans $\widehat{\mathbb{C}}$ (comme $t \nearrow 0$) ; la partie difficile est de démontrer que la limite est dans $\mathbb{C}$.

Pour les suites bornées $\underline{s} \in \mathcal{S} := \mathbb{Z}^{\mathbb{N}}$, nous introduisons les *rayons paramétriques à l'adresse externe* $\underline{s}$: ces sont des courbes $G_{\underline{s}} : (0, \infty) \to \mathbb{C}$ tel que pour $\kappa = G_{\underline{s}}(t)$, l'orbite de la valeur singulière κ sous E_κ converge vers ∞ (« l'orbite singulière s'échappe »). Plus précisement, $E_\kappa^{\circ n}(\kappa) = F^{\circ n}(t) + 2\pi \mathrm{i} s_{n+1} + \mathrm{o}(1)$, où $F(t) = \mathrm{e}^t - 1$ et $\underline{s} = s_1 s_2 s_3 \cdots$. En particulier, $G_{\underline{s}}(t) = t + 2\pi \mathrm{i} s_1 + \mathrm{O}(\mathrm{e}^{-t})$. Les rayons paramétriques ont un ordre vertical naturel dans leur approche vers $+\infty$; cet ordre est le même que l'ordre lexicographique de leurs adresses externes $\underline{s}$.

Si, pour W et h données, le rayon interne $\Gamma_{W,h}$ aboutit à ∞, alors $\mathrm{Re}(\Gamma_{W,h}(t)) \to +\infty$, et $\Gamma_{W,h}$ découpe l'espace des adresses externes en deux ensembles S^- et S^+ tels que $G_{\underline{s}}$ est dessus (ou dessous) $\Gamma_{W,h}$ si et seulement si $\underline{s} \in S^+$ (ou $\underline{s} \in S^-$). Pour la suite $\mathrm{addr}(\Gamma_{W,h}) := \inf(S^+) = \sup(S^-)$, il y a trois possibilités : (1) $\mathrm{addr}(\Gamma_{W,h}) \in \mathcal{S}$ est bornée ; (2) $\mathrm{addr}(\Gamma_{W,h}) \in \mathcal{S}$ est non bornée ; et (3) $\mathrm{addr}(\Gamma_{W,h}) = s_1 s_2 \cdots s_{n-2} s_{n-1}$ est une suite finie telle que $s_1, \ldots, s_{n-2} \in \mathbb{Z}$ et $s_{n-1} \in \mathbb{Z} + \frac{1}{2}$. Chacune de ces trois possibilités donnera une contradiction.

(1) Si $\underline{s} := \mathrm{addr}(\Gamma_{W,h})$ est bournée, alors des calculs asymptotiques impliquent qu'il n'y a pas de rayons paramétriques entre $\Gamma_{W,h}$ et $G_{\underline{s}}$; mais l'ordre vertical et des raisons combinatoires impliquent le contraire.

(2) Si $\mathrm{addr}(\Gamma_{W,h})$ n'est pas bournée, nous utilisons la dynamique symbolique (en forme des *kneading sequences* et des adresses internes) pour démontrer qu'il y a une autre composante hyperbolique $W' \neq W$ et deux rayons paramétriques $G_{\underline{s}^{(1)}}$ et $G_{\underline{s}^{(2)}}$ qui aboutissent sur $\partial W'$ et qui séparent W et $\Gamma_{W,h}$ des rayons paramétriques aux adresses près de $\mathrm{addr}(\Gamma_{W,h})$, ce qui est une autre contradiction.

(3) Si $\mathrm{addr}(\Gamma_{W,h}) = s_1 s_2 \cdots s_{n-2} s_{n-1}$ comme décrit ci-dessus, alors il y a une composante hyperbolique W' de période n qui s'étend vers ∞, telle que le rayon paramétrique $G_{\underline{s}'}$ s'approche vers ∞ dessus W' ssi $\underline{s}' \in S^+$. Encore une fois, il y a deux rayons paramétriques $G_{\underline{s}^{(1)}}$ et $G_{\underline{s}^{(2)}}$ qui aboutissent à W' et qui séparent $\Gamma_{W,h}$ des rayons paramétriques aux adresses externes près de $\mathrm{addr}(\Gamma_{W,h})$, encore une contradiction.

Il s'en suit que $\Gamma_{W,h}$ aboutit dans $\mathbb{C}$, et ceci suffit pour démontrer que Φ_W donne un homéomorphisme de ∂W sur $\partial \mathbb{H}^-$, et que ∂W est connexe.

Finalement, nous décrivons deux autres conjectures de Eremenko et Lyubich. Nous démontrons qu'il existe une collection dénombrable des composantes hyperboliques qui ne peuvent pas êtres jointes par des chaînes finies d'autres composantes hyperboliques telles que les componsantes voisines soient des bifurcations les unes des autres. Nous espérons que des méthodes semblables à celles de notre démonstration du Théorème 1.1 pourraient aider à démontrer que des composantes non-hyperboliques sont bournées.

1. Introduction

In this Note, we investigate the fundamental structure of the space of complex exponential maps $z \mapsto E_\kappa(z) = \mathrm{e}^z + \kappa$ with $\kappa \in \mathbb{C}$ (see Fig. 1). Translation by $-\kappa$ conjugates such a map to $\mathrm{e}^{z+\kappa} = \lambda \, \mathrm{e}^z$ with $\lambda = \mathrm{e}^\kappa$. The space of complex exponential maps has been investigated since the mid-1980s by Baker and Rippon [1], Eremenko and Lyubich [3–5], Devaney, Goldberg and Hubbard [2], and others.

A *hyperbolic component of period* n is a maximal open set $W \subset \mathbb{C}$ such that for $\kappa \in W$, the map E_κ has an attracting periodic orbit of period n; all other periodic orbits are then necessarily repelling. It is known from [5,1,2] that every hyperbolic component is simply connected, and it comes with a holomorphic multiplier map $\mu : W \to \mathbb{D}^*$ such that the attracting orbit of E_κ has multiplier $\mu(\kappa)$. The map μ is a universal covering map. Equivalently, there is a conformal isomorphism $\Phi_W : W \to \mathbb{H}^-$ (the left half plane) such that $\mu = \exp \circ \Phi_W$. The map Φ_W is unique up to translation by $2\pi \mathrm{i}\mathbb{Z}$. A preferred choice for Φ_W has been given in [11], but for our purposes any fixed choice will do.

The main result of this note is the following.

Theorem 1.1. *Every hyperbolic component has connected boundary, and* Φ_W *extends to a homeomorphism of pairs* $\mu : (\overline{W}, W) \to (\overline{\mathbb{H}^-}, \mathbb{H}^-)$.

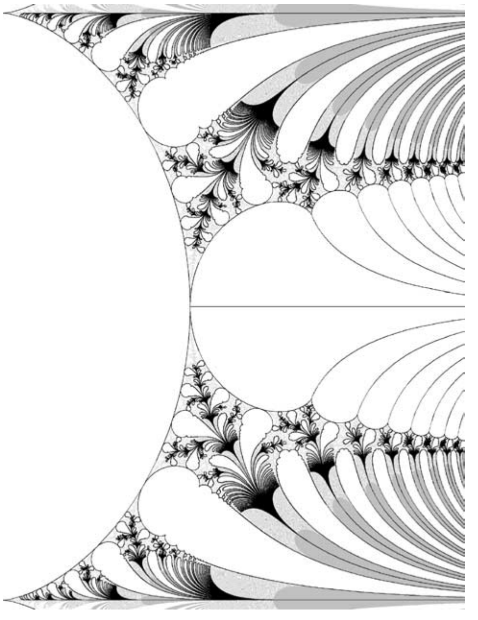

Fig. 1. The space of complex exponential maps $z \mapsto e^z + \kappa$. Many hyperbolic components of various periods are shown. It is clearly visible that each has connected boundary.
Fig. 1. L'espace des applications exponentials complexes $z \mapsto e^z + \kappa$. Beaucoup des composantes hyperboliques sont dessinées. Il est bien visible que leur bord est connexe.

(Note that $\overline{W}$ and $\overline{\mathbb{H}}^-$ etc. will denote closures in $\mathbb{C}$ throughout this paper; however, this theorem remains true if closures in the Riemann sphere $\widehat{\mathbb{C}}$ are taken.) This result had been conjectured by Baker and Rippon [1] and Eremenko and Lyubich [3] in the mid-1980s. The proof requires a substantial amount of knowledge on exponential parameter space; many of the required results are of interest in their own right. Among them is a description of parameters κ for which the singular orbit *escapes*, i.e. converges to ∞. We need the map $F:\mathbb{R}^+ \to \mathbb{R}^+$, $F(t) = e^t - 1$ and the notation $\mathcal{S} := \mathbb{Z}^{\mathbb{N}}$; sequences $\underline{s} \in \mathcal{S}$ will be called *external addresses*.

Theorem 1.2. *For every bounded external address $\underline{s} \in \mathcal{S}$, there exists a unique injective C^1-curve $G_{\underline{s}}:(0,\infty) \to \mathbb{C}^*$ (a* parameter ray*) so that for $\kappa = G_{\underline{s}}(t)$, the singular value κ escapes to ∞ such that*

$$E_\kappa^{\circ n}(\kappa) = F^{\circ n}(t) + 2\pi i s_{n+1} + o(1) \quad \text{as } n \to \infty \text{ or } t \to \infty.$$

The curve $G_{\underline{s}}$ satisfies $G_{\underline{s}}(t) = t + 2\pi i s_1 + O(e^{-t})$ as $t \to \infty$. All these curves are disjoint.

In fact, there are parameter rays $G_{\underline{s}}$ for all *exponentially bounded* external addresses $\underline{s} \in \mathcal{S}$ [6]. Exponential maps with escaping singular orbits are completely classified in terms of parameter rays [7].

Note that all parameter rays $G_{\underline{s}}$ come with a natural vertical order: since these rays are disjoint and $\operatorname{Re}(G_{\underline{s}}(t)) = +\infty$ as $t \to +\infty$, each ray cuts sufficiently far right half planes into two unbounded parts, so every other parameter ray must be *above* or *below* $G_{\underline{s}}$ (depending on in which unbounded part it converges to $+\infty$). The proof of Theorem 1.2 also shows that the vertical order coincides with the lexicographic order of the external address $\underline{s}$. We say that *the parameter ray $G_{\underline{s}}$ lands at $\kappa \in \mathbb{C}$* if $\lim_{t\searrow 0} G_{\underline{s}}(t) = \kappa$.

A periodic orbit is *indifferent* if it has a periodic orbit with multiplier $\mu \in \partial\mathbb{D}$. Every such parameter is on the boundary of a hyperbolic component. If μ is a root of unity, the parameter is called *parabolic*.

Theorem 1.3. *For every periodic $\underline{s} \in \mathcal{S}$, the parameter ray $G_{\underline{s}}$ lands at a parabolic parameter κ, and every parabolic κ is the landing point of one or two parameter rays at periodic external addresses.*

For the purposes of this note, only the second half of Theorem 1.3 will be needed; the proof of the first half needs Theorem 1.1. (Similarly, if $\underline{s}$ is preperiodic, then $G_{\underline{s}}$ lands at a *Misiurewicz parameter*: that is a κ such that the singular orbit of E_κ is strictly preperiodic; conversely, every Misiurewicz parameter is the landing point of a finite positive number of parameter rays at preperiodic external addresses.)

Most results in this note were obtained in the thesis [10]. A detailed proof of Theorem 1.1 will be given in [9], together with the necessary background about exponential parameter space and a number of further results.

2. Internal rays

Consider a hyperbolic component W with conformal isomorphism $\Phi_W : W \to \mathbb{H}^-$ as above. For every $k \in \mathbb{Z}$, the set $\Phi^{-1}(\{z \in \mathbb{H}^-\colon\ 2\pi k < \operatorname{Im}(z) < 2\pi(k+1)\}$ is called a *sector* of W. Moreover, for any $h \in \mathbb{R}$ we define the *internal ray at height h* to be the curve $\Gamma_{W,h} : \mathbb{R}^- \to W$, $t \mapsto \Phi_W^{-1}(t + 2\pi \mathrm{i} h)$. The component W is thus canonically foliated into internal rays.

Lemma 2.1. *The multiplier map $\mu : W \to \mathbb{D}^*$ and the conformal isomorphism $\Phi_W : W \to \mathbb{H}^-$ extend to continuous maps $\mu : \overline{W} \to \overline{\mathbb{D}^*}$ and $\Phi_W : \overline{W} \to \overline{\mathbb{H}^-}$. Every boundary component of W is a piecewise analytic curve. For every W and h, $\lim_{t\to-\infty} \Gamma_{W,h}(t) = +\infty$, while $\lim_{t\nearrow 0} \Gamma_{W,h}(t)$ exists in $\widehat{\mathbb{C}}$.*

The first two statements follow simply from the implicit function theorem. Since as $t \to -\infty$, $\mu(\Gamma_{W,h}(t)) \to 0$, but no exponential map has a periodic orbit with multiplier 0, it follows $\lim_{t\to-\infty} \Gamma_{W,h}(t) = \infty$. As $t \nearrow 0$, any limit parameter in $\mathbb{C}$ of $\Gamma_{W,h}(t)$ must have an indifferent orbit with multiplier $\exp(h)$; since such are discrete, the ray lands in $\widehat{\mathbb{C}}$. Theorem 1.1 is proved as soon as we know that all limits are in $\mathbb{C}$: this gives a continuous inverse $\Phi_W^{-1} : \overline{\mathbb{H}^-} \to \overline{W}$.

In order to prove that the internal ray $\Gamma_{W,h}$ lands in $\mathbb{C}$ (as $t \nearrow 0$), we use the (external) parameter rays $G_{\underline{s}}$ from Theorem 1.2. Suppose that $\Gamma_{W,h}$ lands at ∞ as $t \nearrow 0$. It is not hard to show that the real parts tend to $+\infty$ (the ray cannot cross other hyperbolic components or parameter rays $G_{\underline{s}}$), so the vertical order between the ray $\Gamma_{W,h}$ and each parameter ray $G_{\underline{s}}$ is well-defined. Therefore, $\Gamma_{W,h}$ cuts the space of bounded external addresses into two sets S^+ and S^- such that rays $G_{\underline{s}}$ with $s \in S^+$ are above $\Gamma_{W,h}$, while rays with $s \in S^-$ are below $\Gamma_{W,h}$. This defines a cutting sequence $\operatorname{addr}(\Gamma_{W,h}) := \inf\{S^+\} = \sup\{S^-\}$ for which there are three possibilities:

(1) $\operatorname{addr}(\Gamma_{W,h})$ is a bounded sequence in $\mathcal{S}$;
(2) $\operatorname{addr}(\Gamma_{W,h})$ is an unbounded sequence in $\mathcal{S}$;
(3) $\operatorname{addr}(\Gamma_{W,h}) = s_1 s_2 \cdots s_{n-2} s_{n-1}$, where $s_1, \ldots, s_{n-2} \in \mathbb{Z}$, while $s_{n-1} \in \mathbb{Z} + \frac{1}{2}$.

The third case needs some explanation: since $\operatorname{addr}(\Gamma_{W,h})$ is defined as a supremum over bounded sequences in S^-, there might be a position n such that the supremum of the first $n-1$ entries is a finite sequence $s_1 s_2 \cdots s_{n-2} s'_{n-1}$ of integers, while the n-th entries are unbounded above. Setting $s_{n-1} := s'_{n-1} + \frac{1}{2}$, the finite sequence $s_1 s_2 \cdots s_{n-2} s_{n-1}$ can then be considered the supremum over S^-, as well as the infimum over S^+. In order to prove that the internal ray $\Gamma_{W,h}$ cannot land at ∞, we have to exclude all three possibilities for $\operatorname{addr}(\Gamma_{W,h})$.

3. Squeezing of internal rays

Given an internal ray $\Gamma_{W,h}$ which lands at ∞, we exclude the three cases for $\operatorname{addr}(\Gamma_{W,h})$ in order.

(1) If $\underline{s} := \operatorname{addr}(\Gamma_{W,h})$ is bounded, then there is a parameter ray $G_{\underline{s}} : (0, \infty) \to \mathbb{C}$. To fix ideas, suppose that $\Gamma_{W,h}$ approaches $+\infty$ below $G_{\underline{s}}$. All parameter rays $G_{\underline{s}'}$ with $\underline{s}' \in S^-$ are below $\Gamma_{W,h}$. However, for every $\varepsilon > 0$

and $s \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that $|G_{\underline{s}}(t) - G_{\underline{s}'}(t)| < \varepsilon$ uniformly for all $t > 1$ provided the first n entries in $\underline{s}$ and $\underline{s}'$ coincide and all entries in $\underline{s}$ and $\underline{s}'$ are bounded by s. No matter how closely $\Gamma_{W,h}$ approaches $+\infty$ to $G_{\underline{s}}$, there is another ray $G_{\underline{s}'}$ closer to $G_{\underline{s}}$, and this is a contradiction.

(2) If $\mathrm{addr}(\Gamma_{W,h})$ is an unbounded sequence in $\mathcal{S}$, we need to know quite a bit more about exponential parameter space. The fundamental idea is easy, though: there are a hyperbolic component W' and two external addresses $\underline{s}^{(1)}$ and $\underline{s}^{(2)}$ such that the parameter rays $G_{\underline{s}^{(1)}}$ and $G_{\underline{s}^{(2)}}$ land at $\partial W'$, and $\Gamma_{W,h}$ is in a different connected component of $\mathbb{C} \setminus (\overline{W}' \cup G_{\underline{s}^{(1)}} \cup G_{\underline{s}^{(2)}})$ than all parameter rays $G_{\underline{s}'}$ at external addresses near $\mathrm{addr}(\Gamma_{W,h})$. This is a contradiction again.

The basic idea is to use symbolic dynamics in the form of *kneading sequences* (and the human-readable variant, *internal addresses*) [8]. For fixed $\underline{s} \in \mathcal{S}$ and all $\mathsf{k} \in \mathbb{Z}$, consider the sequences $\mathsf{k}\underline{s}$ (concatenation: the symbol k followed by the sequence $\underline{s}$). Then $\mathcal{S} \setminus \bigcup_{\mathsf{k}} \{\mathsf{k}\underline{s}\}$ is the union of the countably many intervals $\mathcal{S}_{\mathsf{k}} = (\mathsf{k}\underline{s}, (\mathsf{k}+1)\underline{s})$. We define the *kneading sequence* $\mathbb{K}(\underline{s})$ as the infinite sequence $\mathsf{k}_1\mathsf{k}_2\mathsf{k}_3\cdots$ such that $\mathsf{k}_i = \mathsf{k}$ iff $\sigma^{i-1}(\underline{s}) \in \mathcal{S}_{\mathsf{k}}$; in the boundary case when $\sigma^{i-1}(\underline{s}) = \mathsf{k}\underline{s}$, set $\mathsf{k}_i = {}^{\mathsf{k}}_{\mathsf{k}-1}$. Clearly, a boundary symbol ${}^{\mathsf{k}}_{\mathsf{k}-1}$ occurs if and only if $\underline{s}$ is periodic, and $\mathbb{K}(\underline{s})$ is bounded if and only if $\underline{s}$ is bounded.

Here is one aspect how kneading sequences help to describe the structure of parameter space.

Proposition 3.1. (a) *Every sector W'_k of a hyperbolic component W' of period n has an associated sequence $\underline{\mathsf{k}} = \mathsf{k}_1\mathsf{k}_2\mathsf{k}_3\cdots$ which is periodic of period n with the following property: if the parameter ray $G_{\underline{s}'}$ lands on $\partial W'_k$, and $\underline{s}'$ is periodic, then $\mathbb{K}(\underline{s}')$ is "almost equal" to $\underline{\mathsf{k}}$ in the following sense: the period of $\underline{s}'$ equals qn for some $q \in \mathbb{N}$, and the i-th entry of $\mathbb{K}(\underline{s}')$ equals k_i whenever i is not a multiple of qn; if it is, then the i-th entry of $\mathbb{K}(\underline{s})$ is either ${}^{\mathsf{k}_i}_{\mathsf{k}_i-1}$ or ${}^{\mathsf{k}_i+1}_{\mathsf{k}_i}$.*

(b) *Suppose that $\underline{s}'$ and $\underline{s}''$ are two bounded external addresses whose kneading sequences coincide in their first $n-1$ entries, while the n-th entries differ. Then there are a hyperbolic component W' of some period $n' \leqslant n$ and two external addresses $\underline{s}^{(1)}$ and $\underline{s}^{(2)}$ such that the parameter rays $G_{\underline{s}^{(1)}}$ and $G_{\underline{s}^{(2)}}$ land at $\partial W'$, and $G_{\underline{s}'}$ and $G_{\underline{s}''}$ are in different connected components of $\mathbb{C} \setminus (\overline{W}' \cup G_{\underline{s}^{(1)}} \cup G_{\underline{s}^{(2)}})$.*

We can now finish the proof that the internal ray $\Gamma_{W,h} \subset W$ cannot land at $+\infty$ so that $\mathrm{addr}(\Gamma_{W,h})$ is unbounded. The ray $\Gamma_{W,h}$ is contained in the closure of some sector W_k of W, and all parameter rays landing at ∂W_k have uniformly bounded kneading sequences. Since $\mathrm{addr}(\Gamma_{W,h})$ is unbounded, so is its kneading sequence, and there must be infinitely many pairs of hyperbolic components with associated parameter rays landing at them which separate $\Gamma_{W,h}$ from all parameter rays landing at ∂W_k.

(3) The third case is that $\mathrm{addr}(\Gamma_{W,h}) = s_1s_2\cdots s_{n-2}s_{n-1} =: \underline{s}$, where $s_1,\ldots,s_{n-2} \in \mathbb{Z}$, while $s_{n-1} \in \mathbb{Z} + \frac{1}{2}$. By the main result of [11], there exists a unique hyperbolic component W' of period n such that for every $h \in \mathbb{R}$, a parameter ray $G_{\underline{s}'}$ approaches $+\infty$ above (resp. below) the internal ray $\Gamma_{W',h'}(t)$ if and only if $\underline{s}' > \underline{s}$ (resp. $\underline{s}' < \underline{s}$) (recall that $\Gamma_{W',h'}$ satisfies $\lim_{t\to-\infty} \mathrm{Re}(\Gamma_{W',h'}(t)) = +\infty$). This means that the ray $\Gamma_{W,h}$ approaches $+\infty$ (as $t \to 0$) so close to the curve $\Gamma_{W',h'}$ (as $t \to -\infty$) that no parameter ray at bounded external address is between $\Gamma_{W,h}$ and $\Gamma_{W',h'}$. This is excluded by the following result.

Lemma 3.2. *For every hyperbolic component W' of period n with associated external address $s_1s_2\cdots s_{n-2}s_{n-1}$, and for every $\xi > 0$, there are two parameter rays $G_{\underline{s}^{(1)}}$ and $G_{\underline{s}^{(2)}}$ which both land at $\partial W'$, and the two landing points can be connected by a curve $\Gamma \subset W$ such that all points in $G_{\underline{s}^{(1)}} \cup G_{\underline{s}^{(2)}} \cup \overline{\Gamma}$ have real parts greater than ξ.*

The curve $G_{\underline{s}^{(1)}} \cup G_{\underline{s}^{(2)}} \cup \overline{\Gamma}$ disconnects $\mathbb{C}$ into two parts, say U and U', so that all real parts in U' are greater than ξ, and all parameter rays with external addresses between $\underline{s}^{(1)}$ and $\underline{s}^{(2)}$ are contained in U'.

If $W \neq W'$, then it is easy to see that $\Gamma_{W,h}$ cannot approach $+\infty$ so that $\mathrm{addr}(\Gamma_{W,h}) = s_1s_2\cdots s_{n-2}s_{n-1}$: the ray $\Gamma_{W,h}$ is disjoint from $G_{\underline{s}^{(1)}} \cup G_{\underline{s}^{(2)}} \cup \overline{\Gamma}$, so if ξ is sufficiently large, then $\Gamma_{W,h}$ cannot be contained in U'; but parameter rays at external addresses close to $\underline{s}$ must be contained in U', a contradiction.

Finally, if $W' = W$, then W has associated external address $s_1 s_2 \cdots s_{n-2} s_{n-1}$. The two ends of the curve $\Gamma_{W,h}$ (as $t \to 0$ and $t \to -\infty$) are not homotopic within W, so they enclose together some part of ∂W. But the surrounded part of ∂W contains infinitely many parabolic parameters, hence infinitely many parameter rays at periodic external addresses, and again $\operatorname{addr}(\Gamma_{W,h}) \neq s_1 s_2 \cdots s_{n-2} s_{n-1}$.

This concludes the proof that every internal ray lands in $\mathbb{C}$, and hence the proof of Theorem 1.1.

In addition to connectedness of the boundaries of hyperbolic components, [3] contains two more conjectures about exponential parameter space. The first states that there are countably many hyperbolic components of which no two can be connected by a finite chain of further hyperbolic components so that adjacent components in the chain are bifurcations from each other. This conjecture also follows from a systematic investigation of the bifurcation structure of hyperbolic components as given in [10,9]; in fact, there are countably many components of period 3 with imaginary parts in $(-\pi, +\pi)$, and among them is a single pair of components which can be connected by a chain of bifurcating components.

The third conjecture in [3] states that *hyperbolicity is dense in exponential parameter space*, in analogy to the main conjecture about quadratic polynomials. If this is false, then there is a *non-hyperbolic component* $W \subset \mathbb{C}$: this is a maximal open set containing no exponential map with an attracting periodic orbit. A similar argument as above shows that W cannot contain a curve to ∞ [9]; moreover, there are at most two external addresses $\underline{s}_W$ and $\underline{s}'_W$ such that every parameter ray $G_{\underline{s}}$ at external address $\underline{s} \notin \{\underline{s}_W, \underline{s}'_W\}$ is separated from W by a pair of periodic parameter rays landing at a common point. It might well be possible to close these routes to ∞ for W in a similar way, thus proving that every non-hyperbolic component (if any) was bounded. This nicely contrasts with the observation that many features of exponential parameter space are unbounded (such as all hyperbolic components). If one could prove that every non-hyperbolic component had to be unbounded, then this would prove the third conjecture.

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