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Very hyperbolic and stably hyperbolic polynomials

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Abstract

A real polynomial in one variable is *hyperbolic* if it has only real roots. A hyperbolic polynomial is *very hyperbolic* if it has hyperbolic primitives of all orders. A polynomial P is *stably hyperbolic* if $x^k P + Q$ is hyperbolic for suitable $k \in \mathbf{N}$ and Q (polynomial of degree $\leqslant k-1$). We present some geometric properties of the domains of very hyperbolic and of stably hyperbolic polynomials in the family $x^n + a_1 x^{n-1} + \cdots + a_n$. ***To cite this article: V.P. Kostov, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Polynômes très hyperboliques et stablement hyperboliques. Un polynôme réel d'une variable est *hyperbolique* si toutes ses racines sont réelles. Un polynôme hyperbolique est *très hyperbolique* s'il a des primitives hyperboliques de tout ordre. Un polynôme P est *stablement hyperbolique* si $x^k P + Q$ est hyperbolique pour certains $k \in \mathbf{N}$ et Q (polynôme de degré $\leqslant k-1$). Nous présentons des propriétés géométriques des domaines de polynômes très hyperboliques et stablement hyperboliques dans la famille $x^n + a_1 x^{n-1} + \cdots + a_n$. ***Pour citer cet article : V.P. Kostov, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Version française abrégée

Considérons la famille de polynômes $P(x,a) = x^n + a_1 x^{n-1} + \cdots + a_n$, $a_i, x \in \mathbf{R}$. Un polynôme de cette famille est dit *hyperbolique* si toutes ses racines sont réelles. On désigne par Π_n le *domaine d'hyperbolicité*, c. à d. l'ensemble $\{a \in \mathbf{R}^n \mid P$ est hyperbolique$\}$. On appelle *dilatation* dans $\mathbf{R}^n \simeq Oa_1 \cdots a_n$ ou $\mathbf{R}^n \times \mathbf{R} \simeq Oa_1 \cdots a_n \times Ox$ une application linéaire de la forme $a_i \mapsto \beta_i a_i$, $i = 1, \ldots, n$, $x \mapsto \beta_{n+1} x$, $\beta_i > 0$. Dans ce qui suit on pose $a_1 = 0$, $a_2 = -1$.

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Définition 0.1. L'ensemble Π_n est une variété stratifiée les strates étant définies par les *vecteurs multiplicité* (*VM*), c. à d. vecteurs dont les composantes sont les multiplicités des racines distinctes du polynôme hyperbolique données dans l'ordre de décroissance. Exemple : si $n = 6$ et si pour les racines du polynôme on a $x_1 = x_2 > x_3 > x_4 = x_5 = x_6$, alors, son VM vaut $(2, 1, 3)$. Posons $\Pi_n(0) = \Pi_n$. Pour $k \in \mathbf{N}$ posons $\Pi_n(k) = \{P \in \Pi_n \mid \exists Q, \deg Q \leqslant k - 1, R(x) := x^k P + Q \in \Pi_{n+k}\}$. Alors, on a $\Pi_n(k + 1) \supset \Pi_n(k)$. On appelle l'ensemble $\Pi_n(\infty) = \overline{\bigcup_{k=0}^{\infty} \Pi_n(k)}$ le *domaine de polynômes stablement hyperboliques* de degré n.

Remarque 1. Un indice supplémentaire s indique projection dans $Oa_3 \cdots a_s$. On a $\Pi_n(k) = \Pi_{n+k;n}$ et $\Pi_n(\infty) = \Pi_{n+1}(\infty)_{;n}$. Les polynômes stablement hyperboliques apparaissent comme symboles d'opérateurs différentiels sur la droite qui préservent l'hyperbolicité. L'idée d'étudier ces polynômes a été donnée à l'auteur par B.Z. Shapiro et discutée avec lui et J. Borcea.

Définition 0.2. Une fonction réelle $f \in C^j$ est dite *primitive d'ordre* j de la fonction g si $f^{(j)} = g$. Un polynôme hyperbolique est dit *très hyperbolique* s'il a des primitives hyperboliques de tout ordre. On désigne par $\Pi_n^k \subset \Pi_n$ le sous-ensemble de polynômes ayant des primitives hyperboliques d'ordre $\geqslant k$. Posons $V\Pi_n := \{a \in \Pi_n \mid P$ est très hyperbolique$\}$.

Remarque 2. On a $\Pi_n^{k+1} \subset \Pi_n^k$ et $V\Pi_n = \bigcap_{k=0}^{\infty} \Pi_n^k$ $(*)$. L'inclusion $V\Pi_n \subseteq \Pi_n$ est une égalité pour $n \leqslant 3$; elle est stricte pour $n \geqslant 4$, voir [3].

Théorème 0.3. *Il existe des dilatations $\mathcal{S}$, $\mathcal{T}$ telles que $\mathcal{S}(\Pi_n^k) = \Pi_n(k)$ et $\mathcal{T}(\Pi_n(\infty)) = V\Pi_n$.*

Remarque 3. On peut regarder les domaines $\Pi_k(\infty)$ et $V\Pi_k$ comme des projections dans $Oa_3 \cdots a_k$ d'un domaine homéomorphe à un simplexe de dimension infinie. Ce domaine est l'image (par l'application « racines $\mapsto$ fonctions symétriques de Vieta ») de la chambre de Weyl de dimension infinie $\{(x_1, \ldots, x_n, \ldots) \in \mathbf{R}^\infty \mid x_1 \geqslant \cdots \geqslant x_n \geqslant \cdots\}$ intersectée par les hypersurfaces $x_1 + \cdots + x_n + \cdots = 0$ et $x_1^2 + \cdots + x_n^2 + \cdots = 2$. Les domaines $\Pi_k(\infty)$ et $V\Pi_k$ sont compacts (voir formule $(*)$ dans Remarque 2 et Théorème 0.3) et ont la propriété de Whitney (la distance curviligne d'être équivalente à la distance euclidienne, voir [2]).

Théorème 0.4 (voir [2]). *Pour $2 \leqslant k \leqslant n - 1$ l'ensemble $V\Pi_{n;k+1}$ est la réunion de tous les points sur et entre les graphes $\widetilde{H}_{k+1}^{\pm}$ de deux fonctions continues $\tilde{f}_{k+1}^{\pm} : V\Pi_{n;k} \to \mathbf{R}$, $\tilde{f}_{k+1}^{+} \geqslant \tilde{f}_{k+1}^{-}$, dont les valeurs coïncident sur et seulement sur $\partial V\Pi_{n;k}$, le bord de $V\Pi_{n;k}$. Une fibre non-vide de la projection $V\Pi_{n;k+1} \to V\Pi_{n;k}$ est un segment* (*un point*) *si elle est sur l'intérieur de $V\Pi_{n;k}$ (sur $\partial V\Pi_{n;k}$).*

Remarque 4. Le théorème implique que pour un polynôme de $\partial V\Pi_n$ il existe une suite unique de constantes d'intégration telles que les primitives respectives de tout ordre sont hyperboliques. D'après Théorèmes 0.3 et 0.4, pour un polynôme $P_n \in \partial\Pi_n(\infty)$ il existe une suite unique de constantes $c_j \in \mathbf{R}$ telles que $P_{n+k} := x^k P_n + c_1 x^{k-1} + c_2 x^{k-2} + \cdots + c_k \in \Pi_{n+k}(\infty)$, $k = 1, 2, \ldots$.

Théorème 0.5. *Le graphe $\widetilde{H}_{k+1}^{+}$ (resp. $\widetilde{H}_{k+1}^{-}$) est la limite pour $k \to \infty$ de la clôture de la réunion de strates de Π_n^k avec VM de la forme $(r', 1, r'', 1, \ldots)$ (resp. $(1, r', 1, r'', \ldots)$) et à k composantes dont une tend vers ∞ les autres étant fixées.*

Remarque 5. On présente l'ensemble $V\Pi_4$ sur Fig. 1 – c'est le polygone curviligne de sommets $(1, \infty)$, $(2, \infty)$, $\ldots, \Omega, \ldots, (\infty, 2)$, $(\infty, 1)$. Il n'est pas semi-algébrique – son bord consiste en une quantité dénombrable d'arcs – $(1, \infty, 1)$, $(s, 1, \infty)$ et $(\infty, 1, s)$ – dont les extrémités (s, ∞) et (∞, s) (qui sont des points singuliers pour $\partial V\Pi_4$) s'accumulent vers le point Ω de coordonnées $(0, 1/12)$, voir [3]. On montre sur Fig. 1 aussi la projection dans Oa_3a_4 du graphe $\widetilde{H}_5^{-}$. L'arc liant $(1, \infty)$ avec Ω est la limite des strates de Π_5 avec VM $(1, r', 1, r'')$ où

$r', r'' \to \infty$. On ne peut pas définir une stratification de Whitney autour d'un point de cet arc. Pour obtenir la projection dans Oa_3a_4 de $\widetilde{H}_5^+$ il faut faire le changement $a_3 \mapsto -a_3$ sur Fig. 1 et lire les VM à l'envers. La concavité des « strates » $(1, r, 1, \infty)$, $(1, \infty, 1, r)$, $(r, 1, \infty, 1)$, $(\infty, 1, r, 1)$ est vers l'intérieur de $V\Pi_5$.

1. Main results

Consider the family of polynomials $P(x,a) = x^n + a_1x^{n-1} + \cdots + a_n$, $a_i, x \in \mathbf{R}$.

Definition 1.1. A polynomial from the family is called (*strictly*) *hyperbolic* if it has only real (and distinct) roots. Denote by Π_n the *hyperbolicity domain* of the family, i.e. the set $\{a \in \mathbf{R}^n \mid P$ is hyperbolic$\}$. A *stretching* in $\mathbf{R}^n \simeq Oa_1 \cdots a_n$ or in $\mathbf{R}^n \times \mathbf{R} \simeq Oa_1 \cdots a_n \times Ox$ is a linear map of the form $a_i \mapsto \beta_i a_i$, $i = 1, \ldots, n$, $x \mapsto \beta_{n+1}x$, $\beta_i > 0$.

In what follows we set $a_1 = 0$ (one can make the shift $x \mapsto x - a_1/n$) and $a_2 = -1$ (one has $\Pi_n \cap \{a_1 = 0, a_2 > 0\} = \emptyset$ and $\Pi_n \cap \{a_1 = 0\}$ is a quasi-homogeneous cone over $\Pi_n \cap \{a_1 = 0, a_2 = -1\}$ defined by the stretchings $x \mapsto \mathrm{e}^t x$, $a_j \mapsto \mathrm{e}^{jt} a_j$, $t \in \mathbf{R}$).

Definition 1.2. The set Π_n is a stratified variety the strata being defined by *multiplicity vectors* (*MVs*), i.e. vectors whose components are the multiplicities of the distinct roots of a hyperbolic polynomial listed in decreasing order. Example: if $n = 6$ and if for the roots of the polynomial one has $x_1 = x_2 > x_3 > x_4 = x_5 = x_6$, then the MV is $(2, 1, 3)$. Set $\Pi_n(0) = \Pi_n$. For $k \in \mathbf{N}$ set $\Pi_n(k) = \{P \in \overline{\Pi_n} \mid \exists Q, \deg Q \leqslant k - 1, R(x) := x^k P + Q \in \Pi_{n+k}\}$. Hence, one has $\Pi_n(k+1) \supset \Pi_n(k)$. Call the set $\Pi_n(\infty) = \overline{\bigcup_{k=0}^{\infty} \Pi_n(k)}$ the *domain of stably hyperbolic polynomials* of degree n.

Remark 1. An additional lower index s separated by a semi-colon means projection in $Oa_3 \cdots a_s$. It is clear that $\Pi_n(k) = \Pi_{n+k;n}$. Hence, one has $\Pi_n(\infty) = \Pi_{n+1}(\infty)_{;n}$. Indeed,

$$\Pi_{n+1}(\infty)_{;n} = \left(\overline{\bigcup_{k=0}^{\infty} \Pi_{n+k+1;n+1}}\right)_{;n} = \overline{\bigcup_{k=0}^{\infty} \Pi_{n+k+1;n}} = \overline{\bigcup_{k=0}^{\infty} \Pi_{n+k;n}} = \Pi_n(\infty)$$

(omitting $\Pi_{n;n} = \Pi_n$ in the union does not change anything due to $\Pi_{n+k+1;n} \supset \Pi_{n+k;n}$).

Remark 2. The study of stably hyperbolic polynomials was suggested to the author by B.Z. Shapiro and discussed with him and J. Borcea. Stably hyperbolic polynomials appear as symbols of linear differential operators T on the line preserving hyperbolicity, i.e. $P \in \Pi_n \Rightarrow T(P) \in \Pi_n$.

Definition 1.3. A real-valued function $f \in C^j$ is called a *primitive of order* j of the real-valued function g if $f^{(j)} = g$. A hyperbolic polynomial is called *very hyperbolic* if it has hyperbolic primitives of all orders. Denote by $\Pi_n^k \subset \Pi_n$ the subset of polynomials having hyperbolic primitives of order $\geqslant k$. Set $V\Pi_n := \{a \in \Pi_n \mid P$ is very hyperbolic$\}$.

Remark 3. One has $\Pi_n^{k+1} \subset \Pi_n^k$ and $V\Pi_n = \bigcap_{k=0}^{\infty} \Pi_n^k$ $(*)$. The inclusion $V\Pi_n \subseteq \Pi_n$ is an equality for $n \leqslant 3$ and a strict inclusion for $n \geqslant 4$, see [3].

Lemma 1.4. *There exists a stretching* $\mathcal{S}$ *such that* $\mathcal{S}(\Pi_n^k) = \Pi_n(k)$ $(= \Pi_{n+k;n})$.

Indeed, the set Π_n^k is the projection in $Oa_3\cdots a_n$ of the hyperbolicity domain of the family

$$P^{(-k)} = \frac{n!x^{n+k}}{(n+k)!} - \frac{(n-2)!x^{n+k-2}}{(n+k-2)!} + \frac{(n-3)!a_3x^{n+k-3}}{(n+k-3)!} + \cdots + \frac{a_nx^k}{k!} + \cdots + a_{n+k} \tag{1}$$

obtained by k-fold integration of the family P. Up to a stretching this is the projection in $Oa_3\cdots a_n$ of the hyperbolicity domain of the family $x^{n+k} - x^{n+k-2} + a_3x^{n+k-3} + \cdots + a_{n+k}$. □

Theorem 1.5. *For $n \in \mathbf{N}$, $n \geqslant 2$, there exists a stretching $\mathcal{T}$ such that $\mathcal{T}(\Pi_n(\infty)) = V\Pi_n$.*

Proof.
1°. The domain $\Pi_n(\infty)$ of stably hyperbolic polynomials can be considered as a limit of the set $\Pi_n(k)$ for $k \to \infty$. The latter can be parametrized by $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_{n+k}$, the roots of the polynomial R from Definition 1.2. The conditions $a_1 = 0$, $a_2 = -1$ are equivalent to

$$x_1 + \cdots + x_{n+k} = 0, \qquad x_1^2 + \cdots + x_{n+k}^2 = 2 \tag{2}$$

and Vieta's formulas read $a_j = (-1)^j\sigma_j$ where $\sigma_j = \sum_{1\leqslant i_1<i_2<\cdots<i_j\leqslant n+k} x_{i_1}\cdots x_{i_j}$.

2°. The domain $V\Pi_n$ can be considered as a limit for $k \to \infty$ of the set $\Pi_n^k \subset \Pi_n$ (see Definition 1.3). One can parametrize the set Π_n^k by $y_1 \geqslant \cdots \geqslant y_{n+k}$, the roots of the polynomial $P^{(-k)}$, see (1). Vieta's formulas applied to $P^{(-k)}$ imply that one has

$$y_1 + \cdots + y_{n+k} = 0, \qquad y_1^2 + \cdots + y_{n+k}^2 = \frac{2(n-2)!(n+k)!}{(n+k-2)!n!} = \frac{2(n+k)(n+k-1)}{n(n-1)} \tag{3}$$

and

$$\frac{(n-j)!(n+k)!a_j}{(n+k-j)!n!} = \frac{(n+k)\cdots(n+k-j+1)a_j}{n(n-1)\cdots(n-j+1)} = (-1)^j \sum_{1\leqslant i_1<i_2<\cdots<i_j\leqslant n+k} y_{i_1}\cdots y_{i_j} \tag{4}$$

(we are interested only in the formulas with $j \leqslant n$). Set $y_j = b_{n,k}x_j$, $b_{n,k} = \sqrt{(n+k)(n+k-1)/n(n-1)}$. Then formulas (3) become formulas (2). Formulas (4) after dividing their three sides by $(b_{n,k})^j$ imply that one has $a_j\gamma_{j,k}/n(n-1)\cdots(n-j+1) = (-1)^j\sigma_j$ where $\lim_{k\to\infty}\gamma_{j,k} = (n(n-1))^{j/2}$. So one can set $\beta_j = (n(n-1))^{j/2}/n(n-1)\cdots(n-j+1)$. □

Remark 4. The domain Π_n is homeomorphic to a standard $(n-2)$-dimensional simplex. This domain is the image under the map "τ : roots $\mapsto$ Vieta symmetric functions of them" of Weyl's chamber $\{(x_1,\ldots,x_n) \in \mathbf{R}^n \mid x_1 \geqslant \cdots \geqslant x_n\}$ (related to the singularity A_{n-1}) intersected by the hypersurfaces $x_1 + \cdots + x_n = 0$ and $x_1^2 + \cdots + x_n^2 = 2$.

The domain Π_k^{n-k}, $3 \leqslant k \leqslant n$, is isomorphic to $\Pi_{n;k}$ and the isomorphism is a stretching: $\phi_j : a_j \mapsto \delta_ja_j$, $\delta_j = n(n-1)\cdots(n-j+1)/(n+k)(n+k-1)\cdots(n+k-j+1)$, see (1). Hence, the domain Π_k^{n-k} is the projection in $Oa_3\cdots a_k$ of a domain homeomorphic to an $(n-2)$-dimensional simplex. Notice that $\lim_{n\to\infty}\phi_j = \mathrm{id}$, therefore one has $\lim_{n\to\infty}\Pi_k^{n-k} = \Pi_{n;k}$.

Thus one can view the domain $\Pi_k(\infty)$ (and by Theorem 1.5 the domain $V\Pi_k$ as well) as a projection in $Oa_3\cdots a_k$ of a domain homeomorphic to an infinite dimensional simplex related to the "singularity A_∞" and which is the image under the map τ of the infinite dimensional Weyl's chamber $\{(x_1,\ldots,x_n,\ldots) \in \mathbf{R}^\infty \mid x_1 \geqslant \cdots \geqslant x_n \geqslant \cdots\}$ intersected by the hypersurfaces $x_1 + \cdots + x_n + \cdots = 0$ and $x_1^2 + \cdots + x_n^2 + \cdots = 2$. The domains $\Pi_k(\infty)$ and $V\Pi_k$ are compact, see formula $(*)$ from Remark 3 and Theorem 1.5; they have the *Whitney property* (the curvilinear distance to be equivalent to the Euclidian one, see [2]).

The following properties of $V\Pi_n$ (see their analogs for Π_n in [4] and [5]) are proved in [2]:

Theorem 1.6. *For $2 \leqslant k \leqslant n-1$ the set $V\Pi_{n;k+1}$ is the union of all points on and between the graphs $\widetilde{H}^{\pm}_{k+1}$ of two continuous functions $\tilde{f}^{\pm}_{k+1} : V\Pi_{n;k} \to \mathbf{R}$, $\tilde{f}^{+}_{k+1} \geqslant \tilde{f}^{-}_{k+1}$, whose values coincide on and only on $\partial V\Pi_{n;k}$, the boundary of $V\Pi_{n;k}$. Thus a non-empty fibre of the projection $V\Pi_{n;k+1} \to V\Pi_{n;k}$ is a segment (a point) if it is over the interior of $V\Pi_{n;k}$ (over $\partial V\Pi_{n;k}$).*

Remark 5. The theorem implies that for a polynomial from $\partial V\Pi_n$ there exists a unique sequence of constants of integration for which its primitives of respective orders are hyperbolic. By virtue of Theorem 1.5, Theorem 1.6 is true for the domain $\Pi_n(\infty)$ as well. Hence, for a polynomial $P_n \in \partial\Pi_n(\infty)$ there exists a unique sequence of constants $c_j \in \mathbf{R}$ such that $P_{n+k} := x^k P_n + c_1 x^{k-1} + c_2 x^{k-2} + \cdots + c_k \in \Pi_{n+k}(\infty)$, $k = 1, 2, \ldots$.

Theorem 1.7. *The graph $\widetilde{H}^{+}_{k+1}$ (resp. $\widetilde{H}^{-}_{k+1}$) is the limit for $k \to \infty$ of the closure of the union of strata of Π^k_n with MVs of the form $(r', 1, r'', 1, \ldots)$ (resp. $(1, r', 1, r'', \ldots)$) and with k components in which one of the components $r^{(i)}$ tends to ∞ while the others remain fixed.*

Remark 6. We present the set $V\Pi_4$ on Fig. 1 – this is the curvilinear polygon with vertices $(1,\infty), (2,\infty), \ldots, \Omega, \ldots, (\infty,2), (\infty,1)$. It is not semi-algebraic – its boundary consists of countably many arcs – $(1,\infty,1)$, $(s,1,\infty)$ and $(\infty,1,s)$ – whose endpoints (s,∞) and (∞,s) (which are singular points for $\partial V\Pi_4$) accumulate towards the point Ω with coordinates $(0, 1/12)$; see the details in [3]. On Fig. 1 we show also the projection in Oa_3a_4 of the graph $\widetilde{H}^{-}_5$. The arc joining $(1,\infty)$ with Ω is the limit of strata of Π_5 with MVs $(1, r', 1, r'')$ in which $r', r'' \to \infty$. One cannot define a Whitney stratification at a point of this arc. To obtain the projection in Oa_3a_4 of $\widetilde{H}^{+}_5$ one has to make the change $a_3 \mapsto -a_3$ on Fig. 1 and to read MVs from the back. The concavity of the "strata" $(1, r, 1, \infty)$, $(1, \infty, 1, r)$, $(r, 1, \infty, 1)$, $(\infty, 1, r, 1)$ is towards the interior of $V\Pi_5$.

In what follows we focus on computations of stably hyperbolic polynomials P_{n+k} for given $P_n \in \partial\Pi_n(\infty)$, see Remark 5. To extend them to the case of very hyperbolic ones one has use the stretching from Theorem 1.5. Compute the sequence of polynomials P_j defined in Remark 5 for $n = 4$ and $P_4 = \Omega$, see Fig. 1. The polynomials P_j can be defined as follows. Consider the polynomials $Q_r = (x^2 - 1/r)^r$. The MV of Q_r is (r, r). The polynomial Q_r defines the point of Π_{2r} which projects in Oa_3a_4 on the axis Oa_4 and has greatest a_4-coordinate, see [1]. Set $Q_r = \sum_{i=0}^{2r} d_i x^{2r-i}$, and set $R_{r,j} = \sum_{i=0}^{j} d_i x^{j-i}$. Hence, one has $P_j = \lim_{r\to\infty} R_{r,j}$ because $V\Pi_n = \lim_{k\to\infty} \Pi^k_n$ (the details are left for the reader). To make this computation more explicit write $Q_r = \sum_{k=0}^{r} (-1)^k C^k_r x^{2r-2k}/r^k$. When $r \to \infty$, one has $C^k_r/r^k \to 1/k!$. Thus one has $P_{2s} = \sum_{i=0}^{s} (-1)^i x^{2s-2i}/i!$, $P_{2s+1} = x P_{2s}$.

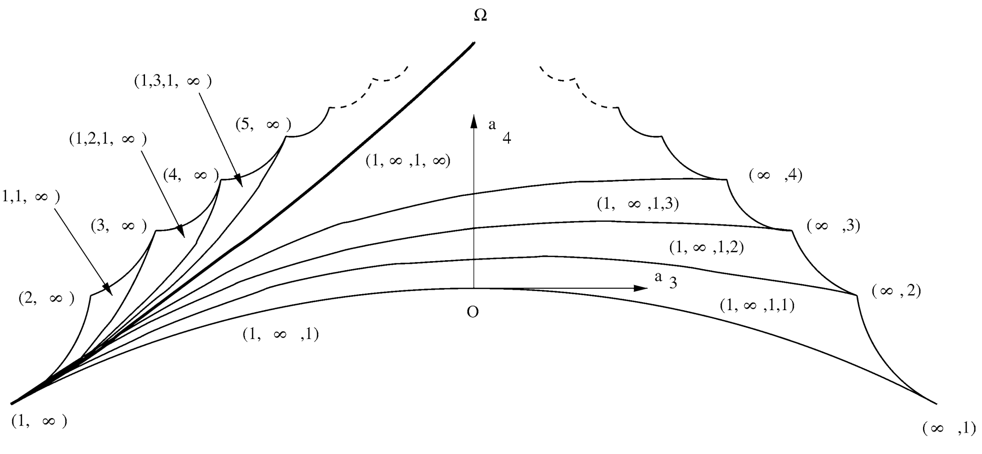

Fig. 1. The set $V\Pi_4$.

Lemma 1.8. *For s even the polynomial P_{2s} is elliptic, i.e. without real roots, for s odd P_{2s} has exactly two real roots which are simple.*

Proof. 1°. One checks the lemma directly for $s = 1$ and 2 (one has $P_2 = x^2 - 1$, $P_4 = x^4 - x^2 + 1/2$). Set $G_s(x) = \sum_{i=0}^{s}(-1)^i x^i/i!$. One has $P_{2s} = x^{2s}G_s(1/x^2)$.

2°. Show that the polynomial G_s is elliptic for s even and has a single real root for s odd. Indeed, for $s = 1$ and 2 this is to be checked directly. Observe that $G'_{s+1} = -G_s$. Hence, if for $s = s_0 \in 2\mathbf{N}$, G_s is elliptic, then G_{s_0+1} is a decreasing function and has a single real root β which is simple. This root is > 1 if $s_0 > 0$. Indeed, $G_{s_0+1}(1) = (1 - 1/1!) + (1/2! - 1/3!) + \cdots + (1/s_0! - 1/(s_0 + 1)!) > 0$ and the leading coefficient of G_{s_0+1} is negative. Hence, P_{2s_0+2} has exactly two simple real roots $\pm\alpha$ where $\alpha \in (0, 1)$.

3°. On the other hand, notice that $G_{s_0+2}(x) = G_{s_0+1}(x) + x^{s_0+2}/(s_0 + 2)!$. The polynomial G_{s_0+2} has a minimum at β and this minimum equals $G_{s_0+1}(\beta) + \beta^{s_0+2}/(s_0 + 2)! = \beta^{s_0+2}/(s_0 + 2)! > 0$. Hence, G_{s_0+2} is elliptic and so is P_{2s_0+4}. □

Remark 7. In a similar way one can compute the sequence of polynomials P_j for P_4 being one of the vertices of $\Pi_4(\infty)$, say, (s, ∞), see Fig. 1. (For (∞, s) the computation can be performed by analogy.) Consider the polynomials $U_r(x) = (x + u)^s(x - v)^{r-s}$, $u, v > 0$ (defining MVs $(r - s, s)$). Set $U_r = \sum_{i=0}^{r} g_i x^{r-i}$, $V_{r,j} = \sum_{i=0}^{j} g_i x^{j-i}$. Impose the conditions

$$su - (r - s)v = 0, \qquad \big(s(s - 1)/2\big)u^2 - s(r - s)uv + \big((r - s)(r - s - 1)/2\big)v^2 = -1.$$

(These conditions result from Vieta's formulas.) Hence, one has $v = \sqrt{2s/(r - s)r}$, $u = \sqrt{2(r - s)/sr}$. Then set (for $j \geqslant s$ and with these values of u, v) $P_j = \lim_{r\to\infty} V_{r,j}$.

Remark 8. One can perform a similar computation for a polynomial P_4 from one of the strata $(\infty, 1, s)$ (or $(s, 1, \infty)$), see Fig. 1. Denote the roots of a polynomial defining a MV $(n - s - 1, 1, s)$ by $\xi < \eta < \zeta$. Vieta's formulas imply the equations

$$(n - s - 1)\xi + \eta + s\zeta = 0, \qquad (n - s - 1)\xi^2 + \eta^2 + s\zeta^2 = 2. \tag{5}$$

Again by Vieta's formulas, the fourth coefficient a_4 of the polynomial equals

$$-\big(C^3_{n-s-1}\xi^3 + C^2_{n-s-1}\xi^2\eta + C^2_{n-s-1}s\xi^2\zeta + (n - s - 1)s\xi\eta\zeta + (n - s - 1)C^2_s\xi\zeta^2 + C^2_s\eta\zeta^2 + C^3_s\zeta^3\big).$$

The second of Eqs. (5) implies that the quantities η and ζ are bounded. Set $\xi = \varphi/n$. In the limit when $n \to \infty$ one has $\varphi + \eta + s\zeta = 0$, $\eta^2 + s\zeta^2 = 2$. The first of these equalities implies that φ is also bounded. The limit when $n \to \infty$ of a_4 equals $a_4 = -(\varphi^3/6 + \varphi^2\eta/2 + s\varphi^2\zeta/2 + s\varphi\eta\zeta + C^2_s\varphi\zeta^2 + C^2_s\eta\zeta^2 + C^3_s\zeta^3)$. Knowing the first four coefficients (the first three of them equal 1, 0, −1), one finds φ, η, ζ from the last three equations. Then from Vieta's formulas, using limits for $n \to \infty$, one finds the next coefficients of a polynomial P_j, i.e. one finds the constants $c_1, c_2, \ldots$.

Using the same ideas one can compute polynomials P_{n+k} for $n \geqslant 5$ as well.

References

[1] V.P. Kostov, On the hyperbolicity domain of the polynomial $x^n + a_1x^{n-1} + \cdots + a_n$, Serdica Math. J. 25 (1) (1999) 47–70.

[2] V.P. Kostov, Very hyperbolic polynomials in one variable, Manuscript, 10 p.

[3] V.P. Kostov, Very hyperbolic polynomials, Funct. Anal. Appl., in press.

[4] V.P. Kostov, On the geometric properties of Vandermonde's mapping and on the problem of moments, Proc. Roy. Soc. Edinburgh 112 (3–4) (1989) 203–211.

[5] I. Meguerditchian, Géométrie du discriminant réel et des polynômes hyperboliques, Thèse de doctorat, Univ. de Rennes I, soutenue le 24.01.1991.