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Numerical Analysis

A simultaneous directions parallel algorithm for the Navier–Stokes equations

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Abstract

We use a parallel algorithm (in time and space) to solve the incompressible Navier–Stokes problem. Adapting some previous ideas by Lu, Neittaanmäki and Tai, the task is reduced to solving a (large) family of independent second-order one-dimensional linear systems. We also present some numerical experiments. ***To cite this article: J.R. Galo et al., C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Un algorithme parallèlle de directions simultanées pour les équations de Navier–Stokes. On utilise un algorithme parallèlle en espace et en temps de type directions simultanées pour résoudre les équations de Navier–Stokes. On adapte quelques idées de Lu, Neittaanmäki et Tai, ce qui conduit à une (grande) famille de systèmes différentiels ordinaires linéaires du second ordre qui son indépendants. On présente les résultats de quelques expériences numériques. ***Pour citer cet article : J.R. Galo et al., C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

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Cette Note est consacrée à la résolution numérique du problème incompressible de Navier–Stokes

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} & \text{dans } Q = \Omega \times (0,T), \\ \nabla\cdot\mathbf{u} = 0 & \text{dans } \Omega \times (0,T), \\ \mathbf{u} = \mathbf{g} & \text{sur } \Sigma = \partial\Omega \times (0,T), \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(x) & \text{dans } \Omega \end{cases} \tag{1}$$

avec un algorithme parallèle en temps et en espace.

Dans (1), $\Omega \subset \mathbf{R}^d$ est un domaine borné de frontière $\partial\Omega$ régulière ($d = 2$ ou 3), $\nu > 0$ est la viscosité cinématique, $\mathbf{f} = \mathbf{f}(\mathbf{x}, t)$ est un champ de forces exterieures et $\mathbf{g} = \mathbf{g}(\mathbf{x})$ est un champ de vitesses donné à divergence nulle dans Ω. Par simplicité, on a supposé que la densité du fluide est $\rho = 1$ et que $\mathbf{g}$ ne dépend pas de t.

On propose dans cette Note une méthode numérique qui repose sur la parallélisation au plus haut niveau possible et a été inspirée par [10].

Pour l'approximation en temps de (1), on divise l'intervalle $[0, T]$ en M intervalles de longueur k ($k = T/M$) et on fixe les paramètres σ, θ, μ, a et b, avec $0 < \sigma \leqslant 1$, $0 \leqslant \theta, \mu \leqslant 1$ et $a, b > 0$.

Tout d'abord, on prend

$$\mathbf{u}^0 = \mathbf{u}_0. \tag{2}$$

Ensuite, pour $m \geqslant 0$ et $\mathbf{u}^m \in H_0^1(\Omega)^d$ donnés, des approximations de $\mathbf{u}$ et p au temps $t_{m+1} = (m+1)k$ sont calculées : On résout en parallèle les systèmes

$$\begin{cases} \dfrac{\mathbf{u}^{m+a} - \mathbf{u}^m}{ak} - \nu\Delta\big(\sigma\mathbf{u}^{m+a} + (1-\sigma)\mathbf{u}^m\big) + \dfrac{2\theta}{a}(\mathbf{u}^*\cdot\nabla)\mathbf{u}^{**} = \dfrac{2\mu}{a}\mathbf{f}^{m+a} - \nabla p^m, \\ u^{m+a} \in H_0^1(\Omega)^d + \mathbf{g}, \end{cases} \tag{3}$$

$$\begin{cases} \dfrac{\mathbf{u}^{m+b} - \mathbf{u}^m}{bk} - \nu\Delta\big(\sigma\mathbf{u}^m + (1-\sigma)\mathbf{u}^{m+b}\big) + \nabla p^{m+b} = \dfrac{2(1-\mu)}{b}\mathbf{f}^{m+b} - \dfrac{2(1-\theta)}{b}(\mathbf{u}^m\cdot\nabla)\mathbf{u}^m, \\ \mathbf{u}^{m+b} \in H_0^1(\Omega)^d + \mathbf{g}, \quad \nabla\cdot\mathbf{u}^{m+b} = 0, \quad p^{m+b} \in L^2(\Omega), \end{cases} \tag{4}$$

ce qui donne respectivement $\mathbf{u}^{m+a}$ et $(\mathbf{u}^{m+b}, p^{m+b})$; finalement, on calcule $\mathbf{u}^{m+1}$ par la relation

$$\mathbf{u}^{m+1} = \frac{1}{2}\big(\mathbf{u}^{m+a} + \mathbf{u}^{m+b}\big). \tag{5}$$

On note que (3) est un système elliptique de type Burgers. Il peut être linéaire ou non linéaire, selon le choix de $\mathbf{u}^*$ et $\mathbf{u}^{**}$. D'autre part, (4) est un système linéaire de type Stokes.

Pour la résolution numérique de (3) et (4), on peut appliquer techniques bien connues, basées sur la reformulation moindres carrés et des algorithmes de type gradient conjugué, voir [7] et [8]. À la fin, on doit résoudre un grand nombre de problèmes de Poisson

$$\begin{cases} -\mu\Delta u + \alpha u = f, \\ u \in H_0^1(\Omega) + g. \end{cases} \tag{6}$$

Ici, on applique une méthode de directions simultanées : On part d'une fonction U^0 arbitraire qui satisfait $U^0 = g$ sur $\partial\Omega$; ensuite, pour chaque $m \geqslant 0$, on résout en parallèle les problèmes différentiels ordinaires

$$(I + \tau L_n)U^{m+1,n} = \left(I - \tau\sum_{j\neq n} L_j\right)U^m + \tau f, \tag{7}$$

complétés par des conditions appropriées sur le bord (L_n et L_j sont donnés par (15)) et on utilise la formule

$$U^{m+1} = \frac{\omega}{d}\sum_{n=1}^{N} U^{m+1,n} + (1-\omega)U^m. \tag{8}$$

Dans (7) et (8), les paramètres τ et ω doivent être choisis convenablement, voir [3] et [4].

Le comportement numérique de cette méthode est illustré à la fin de cette Note avec quelques expériences 2D et 3D. Pour une analyse détaillée et d'autres résultats numériques, voir [1–4].

1. Time approximation

This Note deals with a parallel algorithm (in time and space) which is used to solve the incompressible Navier–Stokes problem:

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} & \text{in } Q = \Omega\times(0,T), \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega\times(0,T), \\ \mathbf{u}(\mathbf{x},t) = \mathbf{g}(\mathbf{x}) & \text{on } \Sigma = \partial\Omega\times(0,T), \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(x) & \text{in } \Omega. \end{cases} \tag{9}$$

Here, $\Omega\subset\mathbf{R}^d$ is a bounded regular domain ($d=2$ or 3), $\nu>0$ is the kinematic viscosity (a positive constant), $\mathbf{f}=\mathbf{f}(\mathbf{x},t)$ is the density function of a field of external forces and $\mathbf{g}=\mathbf{g}(\mathbf{x})$ is a prescribed solenoidal velocity field. For simplicity, we have assumed in (9) that the fluid has unit mass density and the velocity field is time-independent on the lateral boundary Σ.

It is known that the numerical solution of (9) can be a very difficult task, in particular if the goal is to describe accurately the behavior of a 3D fluid (i.e. $d=3$) with small viscosity. Here, we propose a method relying on *parallelization* at the highest possible level. A similar method was introduced in [10], where the seminal ideas can be found for this approach.

We will first present the time-approximation of (9). Let us divide the interval $[0,T]$ in M subintervals of length k ($k=T/M$) and let us assume that the parameters σ, θ, μ, a and b are given, with $0<\sigma\leqslant 1$, $0\leqslant\theta,\mu\leqslant 1$ and $a,b>0$. We first put

$$\mathbf{u}^0 = \mathbf{u}_0. \tag{10}$$

Then, for given $m\geqslant 0$ and $\mathbf{u}^m\in H_0^1(\Omega)^d$ (an approximation of $\mathbf{u}$ at time $t_m=mk$), we compute $\mathbf{u}^{m+a}$, $\mathbf{u}^{m+b}$ and then $\mathbf{u}^{m+1}$ as follows: First, we solve in parallel

$$\begin{cases} \dfrac{\mathbf{u}^{m+a}-\mathbf{u}^m}{ak} - \nu\Delta\big(\sigma\mathbf{u}^{m+a} + (1-\sigma)\mathbf{u}^m\big) + \dfrac{2\theta}{a}(\mathbf{u}^*\cdot\nabla)\mathbf{u}^{**} = \dfrac{2\mu}{a}\mathbf{f}^{m+a} - \nabla p^m, \\ u^{m+a}\in H_0^1(\Omega)^d + \mathbf{g}, \end{cases} \tag{11}$$

$$\begin{cases} \dfrac{\mathbf{u}^{m+b}-\mathbf{u}^m}{bk} - \nu\Delta\big(\sigma\mathbf{u}^m + (1-\sigma)\mathbf{u}^{m+b}\big) + \nabla p^{m+b} = \dfrac{2(1-\mu)}{b}\mathbf{f}^{m+b} - \dfrac{2(1-\theta)}{b}(\mathbf{u}^m\cdot\nabla)\mathbf{u}^m, \\ \mathbf{u}^{m+b}\in H_0^1(\Omega)^d + \mathbf{g}, \quad \nabla\cdot\mathbf{u}^{m+b} = 0, \quad p^{m+b}\in L^2(\Omega) \end{cases} \tag{12}$$

(an elliptic system of the Burgers kind and a linear system of the Stokes kind); then, we set

$$\mathbf{u}^{m+1} = \frac{1}{2}\big(\mathbf{u}^{m+a}+\mathbf{u}^{m+b}\big). \tag{13}$$

In (11) and (12), $\mathbf{f}^{m+a}$ and $\mathbf{f}^{m+b}$ are appropriate approximations of $\mathbf{f}$. Furthermore, several different definitions of $\mathbf{u}^*$ and $\mathbf{u}^{**}$ are possible (it seems natural to put $\mathbf{u}^{**}=\alpha\mathbf{u}^{m+a}+(1-\alpha)\mathbf{u}^m$ for some α). On the other hand, the

particular $\mathbf{u}^*$ we use determines the degree of linearity we keep in (11). Using (more or less) standard arguments, we can deduce existence and uniqueness results for (11) and (12), at least when ν is not too small. For more details, see [1–4], where some convergence results and error estimates are deduced.

2. Space approximation and the numerical solution in practice

After time approximation, we must solve the independent stationary problems (11) (that can be linear or not depending on the definition of $\mathbf{u}^*$) and (12). Now, *least squares* reformulation and *conjugate gradient* techniques can be efficiently applied in this context (see [7] and [8]). These reduce the task to the solution of (a large number of) linear Poisson problems of the kind

$$\begin{cases} -\mu\Delta u + \alpha u = f, \\ u \in H_0^1(\Omega) + g, \end{cases} \tag{14}$$

for different data f and g. Here, we will apply *simultaneous directions* techniques. As a result, we will only find (many) independent 1D differential problems.

The numerical method is the following. Let us set $L = -\mu\Delta + \alpha I$ and

$$L_n = -\mu\frac{\partial^2}{\partial x_n^2} + \frac{\alpha}{d}I \quad \text{for } 1 \leqslant n \leqslant d. \tag{15}$$

We start with an arbitrary U^0 satisfying $U^0 = 0$ on $\partial\Omega$. Then, for any $m \geqslant 0$, U^{m+1} is found from U^m as follows: First, we compute $U^{m+1,1}, \ldots, U^{m+1,d}$ by solving in parallel the 'one-dimensional' problems

$$(I + \tau L_n)U^{m+1,n} = \left(I - \tau\sum_{j\neq n} L_j\right)U^m + \tau f \tag{16}$$

(completed with appropriate boundary conditions); secondly, we set

$$U^{m+1} = \frac{\omega}{d}\sum_{n=1}^{N} U^{m+1,n} + (1-\omega)U^m. \tag{17}$$

Here, τ and ω are parameters that must be determined in order to improve convergence properties, see [3].

For instance, when $d = 2$, we see that for each fixed x_2 the unknown function

$$U^{m+1,1}(\cdot\,, x_2)$$

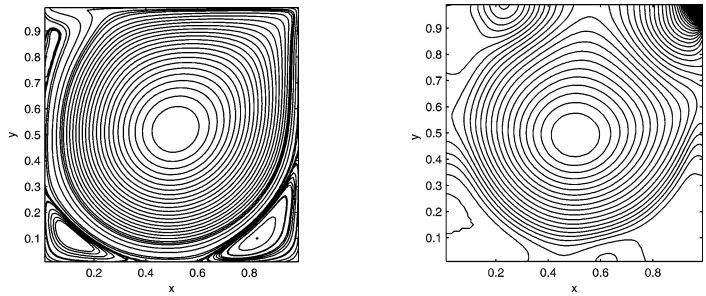
must solve an ordinary differential equation. In practice, it seems reasonable to fix a finite set of x_2 values and, then, solve numerically the corresponding problems (16). In this way, we will obtain approximations to the values of the unknown $U^{m+1,1}$ in a finite set of grid points (x_1^ℓ, x_2^j).

Of course, similar things can be said for $U^{m+1,1}(x_1, \cdot)$. Since both unknowns $U^{m+1,1}$ and $U^{m+1,2}$ have to be used for the computation of U^{m+1} in (17), it is desirable to use the same coordinates x_1^ℓ and x_2^j. Accordingly, we are led to use rectangular grids.

It is important to emphasize that the level of difficulty is not increased in the case $d = 3$ since, at the end, the task is reduced to the numerical solution of (many) one-dimensional problems like (16).

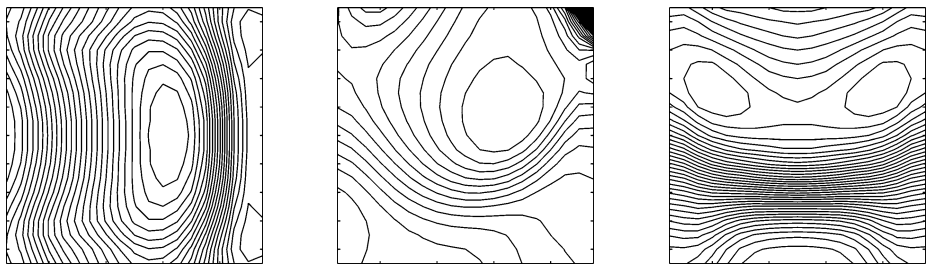
3. Some numerical results

We first consider a 2D square cavity of unit side filled with a fluid. We assume that the upper wall slides with constant velocity and we try to determine which is the effect of this sliding on the fluid (see [6]). This test has



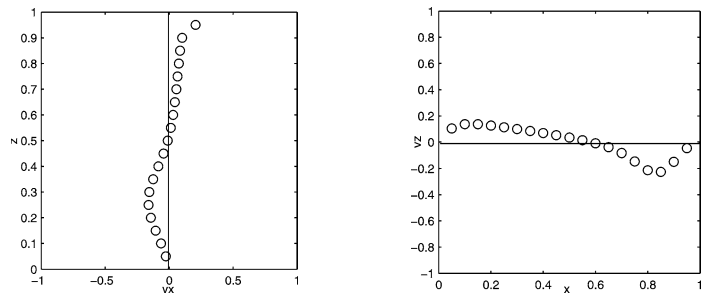
(a) (b)

Fig. 1. The 2D cavity test with $Re = 4000$, $T = 50$, $h = 0.01$ and $k = 0.005$. (a) The streamlines; (b) The isobars.



(a) (b) (c)

Fig. 2. The isobars for the 3D cavity test with $Re = 1000$ and $T = 10$. (a) $z = 0.5$; (b) $y = 0.5$; (c) $x = 0.5$.



(a) (b)

Fig. 3. The velocity profiles for the 3D cavity test with $Re = 1000$ and $T = 10$ at $y = 0.5$. (a) V_x at $x = 0.5$; (b) V_z at $z = 0.5$.

been performed for a Reynolds number $Re = 4000$, with spatial mesh size $h = 0.01$ and time approximation step $k = 0.005$. We have taken $a = b = 0.5$ and $\sigma = 0.51$. In order to avoid the occurrence of spurious pressures, a regularizing scheme and a discrete filtering operator has been introduced, see [3] and [5]. The numerical results corresponding to the particular case $\theta = 0$ at time $T = 50$ are given in Fig. 1(a) and 1(b).

For the similar 3D cavity test, we have compared our results to those provided in [9]. The results obtained for $Re = 1000$ at $T = 10$ have been presented in Figs. 2 and 3. In all cases, good agreement is found.

Other numerical results will be given in [4].

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