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Differential Geometry

Geometry of generalized Einstein manifolds

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Abstract

A formula linking the horizontal Laplacian $\bar{\Delta}\varphi$ of a function φ on the fibre bundle W of unitary tangent vectors to a Finslerian compact manifold without boundary (M, g), to the square of a symmetric 2-tensor and Finslerian curvature. From it an estimate, under a certain condition, is obtained for the function $\lambda : \bar{\Delta}\varphi = \lambda\varphi$. If $\lambda = nk$ where k is a positive constant and M simply connected, then M is homeomorphic to an n-sphere. Let $F^\circ(g_t)$ be a deformation of (M, g) preserving the volume of W. One proves that the critical points $g_0 \in F^\circ(g_t)$ of the integral $I(g_t)$ of a certain Finslerian scalar curvature on W define a generalized Einstein manifold. One calculates the second variationals at the critical points first in the general case, then, for an infinitesimal conformal deformation and one shows that in certain cases one has $I''(g_0) \geqslant 0$. We also study the case when the scalar curvature is non-positive constant. ***To cite this article: H. Akbar-Zadeh, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Géométrie des espaces d'Einstein généralisés. On établit une formule reliant le laplacien horizontal $\bar{\Delta}\varphi$ d'une fonction φ sur le fibré W des vecteurs unitaires tangents à une variété finslérienne (M, g) compacte sans bord, au carré d'un 2-tenseur symétrique et la courbure finslérienne. On en déduit, selon une certaine condition, une estimée pour la fonction $\lambda : \bar{\Delta}\varphi = \lambda\varphi$. Si $\lambda = n \cdot k$ où k est constante positive et M simplement connexe, alors M est homéomorphe à une n-sphère. Soit $F^\circ(g_t)$ une déformation de (M, g) préservant le volume de W. On prouve que les points critiques $g_0 \in F^\circ(g_t)$ de l'intégrale $I(g_t)$ d'une certaine courbure scalaire finslérienne sur W, définissent un espace d'Einstein généralisé. On calcule les variations secondes au point critique g_0 d'abord dans le cas général, puis pour une déformation infinitésimale conforme et on montre que dans certains cas on a $I''(g_0) \geqslant 0$. Nous étudions aussi le cas où la courbure scalaire est constante non-positive. ***Pour citer cet article : H. Akbar-Zadeh, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

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Préliminaires

Soit M une variété différentiable de dimension n et de classe C^∞. Nous désignons par $T(M)$ le fibré tangent à M, par $p: V(M) \to M$ le fibré des vecteurs non-nuls tangents à M, par $p^{-1}T(M) \to V(M)$ le fibré induit sur $V(M)$ par p. Un point de $V(M)$ est désigné par $z = (x, v)$ où $x = pz$ et $v \in T_{pz}$. Notons par V_z, $z \in V(M)$ l'ensemble des vecteurs verticaux en z, c'est à dire des vecteurs qui sont tangents à la fibre passant par z. On désigne par ∇ une dérivation covariante dans le fibré vectoriel $p^{-1}T(M) \to V(M)$. Soit $\mu_z : T_z V(M) \to T(M)$ une application linéaire définie par $\mu_z(\widehat{X}) = \nabla_{\widehat{X}} v$, où $\widehat{X} \in TV(M)$. Nous dirons que ∇ est régulière si pour chaque z $\in V(M)$, μ_z définit un isomorphisme de V_z sur T_{pz}. S'il en est ainsi, l'espace tangent en chaque point de $V(M)$ est somme directe d'un espace horizontal H_z et d'un espace vertical V_z. On désigne par τ et Ω respectivement la torsion et la courbure de ∇. Par la décomposition précédente, on a deux tenseurs de torsion notés S et T et trois tenseurs de courbure notés R, P et Q pour une connexion régulière. Ils satisfont à cinq identités dites de Bianchi (voir [1,3]).

Variétés finsleriennes [1]

Soit $(x^i)(i = 1, 2, \dots, n)$ une carte locale du domaine $U \subset M$ et (x^i, v^i) la carte locale induite sur $p^{-1}(U)$ où $v = v^i \frac{\delta}{\delta x^i} \in T_{pz}$. Une variété finslerienne est définie par la donnée d'une fonction positive F sur $T(M)$ positivement homogène de degré 1 en v et de classe C^∞ sur $V(M)$ telle que la forme quadratique g_{ij} :

$$g_{ij}(x, v) = \frac{1}{2} \frac{\partial^2 F^2}{\partial v^i \partial v^j}$$

soit définie positive. Nous dirons que (M, g) est pseudo-finslérien si g_{ij} définit une forme quadratique non dégénérée (dét $(g_{ij}) \neq 0$) Si g_{ij} est indépendant de v, nous avons alors une métrique riemannienne sur M. Soit S_x l'ensemble des vecteurs unitaires tangents en $x \in M(u \mid F(x, u) = 1)$ et $W = \bigcup_{x\in M} S_x$. W sera appelé le fibré des vecteurs unitaires.

1. Connection of directions attached to a Finslerian manifold

Let ω be the 1-form of a regular connection [1]. We call ω a connection of directions if it is invariant by positive homothetic transformations $v \to \lambda v$ $(\lambda > 0)$.

We now state the fundamental theorem of Finslerian geometry: *There exists a unique regular Euclidean connection of directions such that the torsion tensor S is zero and the torsion tensor T satisfies a condition of symmetry.* This characterization give us quite naturally the Euclidean connection defined by Elie Cartan (see [1,6,7]). On writing the 1-form of connection of directions of Elie Cartan in a local coordinates on $p^{-1}(U)$, we have [8]

$$\omega^i_j = \overset{*}{\Gamma}{}^i{}_{jk}(x, v)\,\mathrm{d}x^k + T^i{}_{jk}(x, v), \nabla v^k.$$

The coefficients of the horizontal and vertical parts of this connection are determined by the above theorem:

$$2\,\overset{*}{\Gamma}_{kij} = (\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}), \quad \overset{*}{\Gamma}_{kij} = g_{kr}\,\overset{*}{\Gamma}{}^r{}_{ij},$$
$$\partial_j = \delta_j - \overset{*}{\Gamma}{}^k{}_{oj}\delta^\bullet{}_k, \quad T_{ijk} = \frac{1}{2}\delta^\bullet{}_k g_{ij} \quad \left(\delta_j = \frac{\delta}{\delta x^j}, \delta^\bullet{}_k = \frac{\delta}{\delta v^k}\right)$$

where o denotes the multiplication contracted by v and the ∂_j represent the n horizontal vector fields over δ_j. The Berwald connection will be denoted by D [3,5]. It has no torsion and has two curvature tensors denoted by G and H.

Let π be a connection of directions without torsion that defines the same splitting on the tangent fibre bundle of $V(M)$ as the Berwald or Cartan connection of directions. Then the geodesics and flag curvature of Cartan or Berwald connections coincide with those of connection π.

2. The Laplacian on *W*

Let $u \to W$ be a unitary vector field and $\omega = u_i\,\mathrm{d}x^i$ be the corresponding 1-form. We denote by $(\mathrm{d}\omega)^{n-1}$ the $(n-1)$th exterior power of $\mathrm{d}\omega$, by η the volume element of W, proportional to $\Phi = \omega \wedge (\mathrm{d}\omega)^{n-1}$. Let δ be the co-differential, the formal adjoint of d with respect to the global scalar defined on W [1,2]. If φ is a differentiable function on W, we define the Laplacian:

$$\Delta\varphi = \bar{\Delta}\varphi + \dot{\Delta}\varphi, \qquad \bar{\Delta}\varphi = -g^{ij} D_i D_j \varphi, \qquad \dot{\Delta}\varphi = -F^2 g^{ij} \partial_i^{\bullet} \partial_j^{\bullet} \varphi$$

where $\bar{\Delta}$ is called the horizontal Laplacian and $\dot{\Delta}$ the vertical Laplacian and D_i is the horizontal covariant derivation in the Berwald connection.

3. Comparison theorem

Let us consider the symmetric tensor A_{ij} defined by:

$$A_{ij}(\varphi) = D_i\varphi_j + D_j\varphi_i + \frac{2}{n}\rho g_{ij} \quad (\varphi_i = D_i\varphi).$$

We choose ρ in such a way that the trace of A vanishes. Let us put

$$(A, A) = \frac{1}{2} A^{ij} A_{ij}.$$

Lemma 3.1. *Let (M, g) be a Finslerian manifold of dimension n. We have*

$$\frac{1}{2}(A, A) = \left(1 - \frac{1}{n}\right)(\bar{\Delta}\varphi, \bar{\Delta}\varphi) - \Phi(\varphi_*, \varphi_*) + \textit{Divergence on } M$$

where $(\cdot,\cdot)$ denotes the scalar product and ϕ is a quadratic form in φ_i and $\dot{\varphi}_i = \partial_i^{\bullet}\varphi$ defined by

$$\Phi(\varphi, \varphi) = H_*(\varphi^*, \varphi^*) - g^{jl} D_j H^i{}_{olk} \varphi^k \dot{\varphi}_i - \frac{3}{4} H_o^{ikl} H^j{}_{okl} \dot{\varphi}_i \dot{\varphi}_j$$

where $H(\varphi^*, \varphi^*) = g^{jl} H_{ijkl} \varphi^i \varphi^k$.

From this it follows:

Theorem 3.2. *Let (M, g) be a compact, simply connected Finslerian manifold without boundary. We suppose that the curvature tensor of the Berwald connection satisfies*

$$\big(H(\varphi^*, u)u, \varphi^*\big) \geqslant K\big[\|\varphi^*\|^2 - (\varphi^*, u)^2\big] \quad \textit{or} \quad H_*(\varphi^*, \varphi^*) \geqslant (n-1)K\|\varphi^*\|^2$$

where φ^* is the horizontal derivative of φ and φ_* its dual, and k is a positive constant. Moreover we suppose that the vertical derivative of φ vanishes. Then the function λ $(\Delta\varphi = \lambda\varphi)$ cannot always be between zero and $n \cdot k$. If $\lambda = nk$, (M, g) is homeomorphic to an n-sphere.

We study the case of a manifold with constant sectional curvature. We find: on a compact Finslerian manifold without boundary with non-zero constant sectional curvature in the Berwald connection every horizontally harmonic function on W is constant. This result holds true in the case when the flag curvature is non-zero everywhere. If (M, g) is Riemannian the above formula gives the theorem of Lichnerowicz–Obata [11,13].

4. Generalized Einstein manifolds

To the Ricci tensor of the Finslerian connection is associated a scalar-valued function H on W called the directional Ricci curvature (it is the trace of the flag curvature). This function is the same for Finsler and Berwald connections and also for the connection π defined at the end of Section 1. From it we deduce by vertical derivation a symmetric covariant 2-tensor

$$\widetilde{H}_{jk} = \cdot\frac{1}{2}\frac{\partial^2 H(v,v)}{\partial v^j \partial v^k}, \qquad H(v,v) = H_{ij}(x,v)v^i v^j.$$

Let $t \in [-\varepsilon, \varepsilon]$ where ε is sufficiently small > 0. By the deformation of a Finslerian metric we mean a 1-parameter family of this metric. For such metric $\omega = u_i\,\mathrm{d}x^i$, the volume element as well as the connections and curvatures attached to g depend on t. We calculate the variational of $\widetilde{H}_{jk}$ as well as its trace $\widetilde{H} = g^{jk}\widetilde{H}_{jk}$. We denote by $t_{ij} = (g_{ij})'$ the derivative of g with respect to t.

Then we have the following lemma:

Lemma 4.1. *The first variational of the volume element η of W of the fibre bundle of unitary tangent vectors to a Finslerian manifold (M, g) is given by:*

$$\eta' = \left(g^{ij} - \frac{n}{2}u^i u^j\right)g'_{ij}\eta \quad \left(u = \frac{v}{F}\right)$$

where the notation $'$ denotes the derivation with respect to t [4].

Lemma 4.2 (Fundamental). *Let (M, g) be a Finslerian manifold of dimension $n \neq 2$. Let Ψ be a differentiable function, homogeneous of degree zero in v, defined on W and $t_{ij} = (g_{ij})'$ then we have*

$$\Psi \operatorname{trace} t - n\Psi \cdot t(u,u) + \frac{F^2}{(n-2)}\left(t^{jl} - g^{jl}\operatorname{trace} t\right)\partial_{\dot{j}\dot{l}}\Psi = \textit{Divergence on } W$$

where $t(u,u) = t_{ij}u^i u^j$ and $\partial_{\dot{j}} = \partial/\partial v^j$.

Lemma 4.3. *Let (M, g_t) be a deformation of a Finslerian manifold of dimension n: we have*

$$g^{jk}\widetilde{H}'_{jk} = n\tau \cdot t(u,u) + \textit{Divergence on } W$$

where τ is defined by

$$\tau = g^{ij}(D_i D_0 T_j + \partial_{\dot{i}} D_0 D_0 T_j), \quad t(u,u) = t_{ij}u^i u^j.$$

We suppose M compact and without boundary and we denote by $F^\circ(g_t)$ a deformation of the Finslerian metric which leaves invariant the volume element of W and we consider

$$I(g_t) = \int_W \widetilde{H}_t \eta_t, \qquad \operatorname{vol} W = \int_W \eta_t = 1.$$

Among the elements of $F^\circ(g_t)$ we look for a metric which makes $I(g_t)$ extremal. We then prove that they are precisely the manifolds for which the tensor $\widetilde{H}_{jk}$ is proportional to the metric tensor $\widetilde{H}_{jk} = C(z)g_{jk},\ z \in W(t = 0,$

$g_0 \in F^\circ(g_t))$. We show that C is independent of the direction. Such a manifold is called a generalized Einstein manifold. We have [4,9]:

Theorem 4.4. *For a Finslerian compact manifold without boundary of dimension $n \neq 2$, the Finslerian metric which makes the integral $I(g_t)$ critical at the point $(t = 0,\ g = g_0 \in F^\circ(g_t))$ defines at this point a generalized Einstein manifold.*

We give the conditions for C to be constant. In the same way we look for, among the metrics of $F^\circ(g_t)$, those that render critical the integral bearing the directional Ricci curvature $H(u, u)$. We find that these are again generalized Einstein manifolds. We calculate the second variational of the integral $I(g_t)$. In view of reducing calculations we suppose that the trace of the torsion tensor is invariant by deformation and we give the formula for the second variational. We study the case of an infinitesimal conformal deformation and we obtain in the case where $\widetilde{H}$ is a constant and

$$\int_W F^2 Q_{ij}\varphi^i\varphi^j\eta \geqslant 0 \qquad (*)$$

where Q_{ij} is the Ricci vertical curvature the theorem for the second variationals [10,12]

Theorem 4.5. *Let (M, g) be a compact Finslerian manifold without boundary $(n \neq 2)$. We suppose that the scalar curvature $\widetilde{H}$ is a constant and the Ricci vertical Q_{ij} satisfies the inequality* (*)*. Then at the critical point $g_0 \in F^\circ(g_t)(t = 0)$ of the integral $I(g_t)$ and for a conformal infinitesimal deformation the second variational is non-negative.*

5. Scalar curvature and Conformal Infinitesimal Transformations

Let X be a vector field on M; and $\exp(uX)$ the 1-parameter group of local transformations generated by X. We denote by $\exp(u\widehat{X})$ its extension to $V(M)$ and by $L(\widehat{X})$ the Lie derivative (see Chapter III [1]). We say that X is an infinitesimal conformal Finslerian transformation if there exists a function φ on M such that

$$L(\widehat{X})g_{jk} = 2\varphi g_{jk}.$$

We assume M to be compact and without boundary. Let us suppose now that the scalar curvature is a non positive constant

$$\widetilde{H} = g^{jk}\widetilde{H}_{jk} = \text{constant} \leqslant 0.$$

The calculations being the same as in the case of variation of scalar curvatures, and using the fact that $\widetilde{H}$ is constant and $\tau = 0$, we obtain the equality

$$\int_{W(M)} q_{ij}\varphi^i\varphi^j\eta = \frac{\widetilde{H}}{n-1}\int_{W(M)} \varphi^2\eta$$

where we have put

$$q_{ij} = g_{ij} + \frac{n}{(n-1)(n-2)}F^2 Q_{ij}.$$

Theorem 5.1. *Let (M, g) be a compact Finslerian manifold without boundary of dimension $n > 2$. Let us suppose that the scalar curvature $\widetilde{H}$ be a non-positive constant and τ vanishes everywhere. If the quadratic form q_{ij} is positive definite everywhere on W then the largest connected group of infinitesimal conformal transformations $C_0(M)$ coincides with the largest connected group of isometries $I_0(M)$.*

This theorem generalizes the Riemannian case (see Lichnérowicz [11]).

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