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Mathematical Analysis

Invertible extensions and growth conditions

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Abstract

We study invertible extensions of Banach and Hilbert space bounded linear operators with prescribed growth conditions for the norm of inverses. In particular, the solutions of some open problems are obtained. ***To cite this article: C. Badea, V. Müller, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Extensions inversibles et conditions de croissance. Nous étudions les extensions inversibles des opérateurs linéaires et bornés sur un espace de Banach ou de Hilbert avec des conditions de croissance données pour les normes des inverses. Nous obtenons en particulier la réponse à plusieurs problèmes ouverts formulés dans la literature. ***Pour citer cet article : C. Badea, V. Müller, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Version française abrégée

Un thème classique en théorie des opérateurs (linéaires et bornés) est l'existence des dilatations et extensions ayant de bonnes propriétés spectrales. Dans le cas des opérateurs agissant sur l'espace de Hilbert, un exemple célèbre est le théorème de dilatation de Sz.-Nagy [21], qui affirme que chaque contraction a une dilatation unitaire. Notons aussi [21], que chaque contraction a une extension co-isométrique. Un autre exemple hilbertien est la notion d'opérateur sous-normal. Un exemple banachique [6], est le théorème suivant dû à Douglas : chaque isométrie sur un espace de Banach admet comme extension une isométrie surjective. La construction de Douglas est hilbertienne : si l'isométrie agit sur un espace de Hilbert, alors son extension, a posteriori un opérateur unitaire, agit aussi sur un espace de Hilbert. Dans le cadre d'une algèbre de Banach commutative $\mathcal{A}$, d'après un théorème d'Arens, [1], un élément $u \in \mathcal{A}$ est inversible dans une algèbre de Banach commutative contenant $\mathcal{A}$ si et seulement si u n'est pas un diviseur topologique de zéro.

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Nous nous intéressons ici aux extensions inversibles des opérateurs avec des conditions de croissance données pour les normes des inverses. On considère notament la condition de croissance de type polynomial ($(P(s))$), la condition (B) de Beurling, et la condition de croissance de type exponentiel ($(E(s))$) (les conditions sont définies dans la version anglaise). Si T vérifie une de ces trois conditions, alors T a une extension inversible, S, agissant sur un espace plus grand, et qui vérifie le même type de condition de croissance. Toutes les constructions sont hilbertiennes. On obtient ainsi la réponse à trois problèmes ouverts formulés dans [13, Problem 6.1.15] et [5,14,15].

On présente aussi deux conséquences concernant les opérateurs sur l'espace de Hilbert à spectre dénombrable (ou un ensemble de Carleson) et une version hilbertienne du théorème d'Arens mentioné précédemment.

Cette Note d'annonce ne contient aucune preuve, celles-ci seront publiées ultérieurement [3].

1. Introduction

We intoduce some notations and specify the terminology.

Operators. In this Note X (and Y) will denote complex Banach spaces and H (and K) will denote Hilbert spaces. Denote by $B(X)$ the algebra of all bounded linear operators on the Banach space X. By an operator we always mean a bounded linear operator.

For an operator $T \in B(X)$ acting on a Banach space X, we denote $m(T) = \inf\{\|Tx\|\colon\ x \in X,\ \|x\| = 1\}$. This quantity is called the *minimum modulus* of T [9] or the *lower bound* of T [13]. We write for short $v_n(T) = \max\{\|T^n\|, m(T^n)^{-1}\}$ $(n \geqslant 0)$.

We denote by $\sigma(T)$ and $\sigma_{ap}(T)$ the spectrum and the approximate point spectrum of a bounded linear operator $T \in B(X)$, respectively. The latter is given by

$$\sigma_{ap}(T) = \bigl\{\lambda \in \mathbb{C}\colon\ \inf\bigl\{\bigl\|(T-\lambda)x\bigr\|\colon\ \|x\| = 1\bigr\} = 0\bigr\}.$$

Note that $m(T) > 0$ if and only if $T \in B(X)$ is one-to-one and of closed range. If T is a Hilbert space operator, then $\sigma_{ap}(T)$ coincides with the left spectrum and $m(T) > 0$ if and only if T is left invertible.

We say that $S \in B(Y)$ is an extension of $T \in B(X)$ if there is an isometry $\pi : X \to Y$ such that $S\pi = \pi T$. We can also consider X as a subspace of Y and write $T = S_{|X}$.

Banach spaces of class SQ_p. Let $p \geqslant 1$ be a real number. A Banach space E is said to be a SQ_p-space if it is a quotient of a subspace of an L_p-space. Let X be a Banach space. A Banach space E is said to be a $SQ_p(X)$-space if it is (isometric to) a quotient of a subspace of an ultraproduct of spaces of the form $L_p(\Omega, \mu, X)$, for some measure space (Ω, μ). Since ultraproducts of L_p-spaces are L_p-spaces, the latter definition is consistent with the former one. Note that any Banach space is isometric to a subspace (resp. a quotient) of an L_∞-space (resp. an L_1-space). Also, if H is a Hilbert space, then each $SQ_2(H)$-space is a Hilbert space too. Note also that SQ_p-spaces are precisely the p-spaces in the sense of Herz, [10]. We refer to [12] for more information.

Growth conditions. We consider the following growth conditions for the operator T:

$(P(s))$ (Polynomial growth condition) there are $C > 0$ and $s \geqslant 0$ such that $v_n(T) \leqslant Cn^s$ $(n \geqslant 1)$;

(B) (Beurling-type condition) $\sum_{n=1}^{\infty} \frac{\log v_n(T)}{n^2} < \infty$;

$(E(s))$ (Exponential growth) there are $C > 0$ and $0 < s < 1$ such that $v_n(T) \leqslant C\,\mathrm{e}^{n^s}$ $(n \geqslant 1)$.

Note that condition $(P(s))$ implies $(E(s'))$ (for any $s' > 0$), which implies (B). Also, if T satisfies (B) and T is invertible, then $\sigma(T) = \sigma_{ap}(T) \subset \mathbb{T}$. Here $\mathbb{T} = \{z\colon\ |z| = 1\}$. If T satisfies (B) and $0 \in \sigma(T)$, then $\sigma_{ap}(T) = \mathbb{T}$ and $\sigma(T) = \{z\colon\ |z| \leqslant 1\}$.

We consider in this Note the following problems. Let T be an operator satisfying one of the above three conditions. Then T has an invertible extension S satisfying the same type of growth condition. Note that $m(S^n)^{-1} = \|S^{-n}\|$ for invertible operators S and so the growth conditions for invertible S are in fact growth conditions for the norm of iterates of S and S^{-1}. The existence of such invertible extensions solves several open problems mentioned in the literature, as we explain below. Proofs will be given elsewhere [3].

2. $\mathcal{E}(\mathbb{T})$-subscalar operators

We denote by $\mathcal{E}(\mathbb{C}) = C^\infty(\mathbb{C})$ the usual Fréchet algebra of all C^∞-functions on $\mathbb{C}$ with the topology of uniform convergence of derivatives of all orders on compact subsets of $\mathbb{C}$. An operator $S \in B(X)$ is said to be *generalized scalar* (or $\mathcal{E}(\mathbb{C})$-scalar) if there is a continuous algebra homomorphism $\Phi : \mathcal{E}(\mathbb{C}) \to B(X)$ for which $\Phi(1) = I$ and $\Phi(z) = S$ (see [4,13]). Such a homomorphism may be interpreted as an operator-valued distribution with compact support [13]. A bounded linear operator is $\mathcal{E}(\mathbb{C})$-*subscalar* if it is similar to the restriction of a $\mathcal{E}(\mathbb{C})$-scalar operator to one of its closed invariant subspaces. According to a result by J. Eschmeier and M. Putinar (see [7, Section 6.4]), a Banach space operator T is $\mathcal{E}(\mathbb{C})$-subscalar if and only if T has property $(\beta)_{\mathcal{E}}$, i.e., for every open set $U \subset \mathbb{C}$, the operator T_U on $\mathcal{E}(U, X)$ (the space of C^∞-functions from U into X), defined by $T_U(f)(z) = (T - z)f(z)$, is injective and has closed range.

The following statements are equivalent (see [4]):

(1) T is $\mathcal{E}(\mathbb{T})$-scalar, i.e., it has a continuous functional calculus on the Fréchet algebra $\mathcal{E}(\mathbb{T})$ of smooth functions on the unit circle $\mathbb{T}$;

(2) T is generalized scalar with $\sigma(T) \subset \mathbb{T}$;

(3) T is invertible, and there exist constants $C > 0$ and $s \geqslant 0$ such that

$$\left\|T^n\right\| \leqslant C\left(1 + |n|\right)^s \quad (n \in \mathbb{Z}).$$

Laursen and Neumann [13, Problem 6.1.15] and M. Didas [5] asked if $\mathcal{E}(\mathbb{T})$-subscalar operators are characterized by the polynomial growth condition $(P(s))$ above. One implication is easy. We refer to [5,13–17] for several partial results. By [6] the hard implication holds for $s = 0$ and $C = 1$.

Since condition $(P(s))$ implies that $\sigma_{ap}(T) \subset \mathbb{T}$, it follows [18,19] that T has an invertible extension S such that $\sigma(S) = \sigma_{ap}(T) \subset \mathbb{T}$. By [20], if T acts on a Hilbert space, then S acts also on a Hilbert space. However, no control on the norms of inverses is guaranteed by this method.

The following result gives a complete positive answer.

Theorem 2.1. (1) *An operator $T \in B(X)$ is $\mathcal{E}(\mathbb{T})$-subscalar if and only if there exist constants $C > 0$ and $s \geqslant 0$ such that*

$$\big(P(s)\big) \quad \frac{1}{Cn^s}\|x\| \leqslant \left\|T^n x\right\| \leqslant Cn^s\|x\| \quad (x \in X;\ n \in \mathbb{N}).$$

Moreover, given $p \geqslant 1$, there exist a $SQ_p(X)$-space Y, an invertible $\mathcal{E}(\mathbb{T})$-scalar operator S on Y and a closed subspace $M \subset Y$ invariant with respect to S such that T is similar to the restriction $S_{|M}$. We also have $\sigma(S) = \sigma_{ap}(T)$.

For $p = 1$ the operator S is an extension of T.

(2) *If the Hilbert space operator $T \in B(H)$ verifies*

$$\big(P(s)\big) \quad \frac{1}{Cn^s}\|h\| \leqslant \left\|T^n h\right\| \leqslant Cn^s\|h\| \quad (h \in H;\ n \in \mathbb{N}),$$

then there exists a Hilbert space K and a $\mathcal{E}(\mathbb{T})$-scalar extension $S \in B(K)$ with $\sigma(S) = \sigma_{ap}(T)$.

3. Operators with Bishop's property ($\boldsymbol{\beta}$)

Recall that an equivalent definition of decomposable operators is the following: $T \in B(X)$ is *decomposable* if for every open cover $\mathbb{C} = U \cup V$, there are closed invariant (for T) subspaces Y and Z of X such that $X = Y + Z$

and $\sigma(T \mid Y) \subset U$, $\sigma(T \mid Z) \subset V$. We refer for instance to [4] and [13]. An operator $T \in B(X)$ has *Bishop's property* (β) if, for every open set $U \subset \mathbb{C}$, the operator T_U defined by $T_U(f)(z) = (T - z)f(z)$ on the set $\mathcal{O}(U, X)$ of holomorphic functions from U into X is injective and has closed range. According to a result by Albrecht and Eschmeier (see [13,7]), $T \in B(X)$ is *subdecomposable* (i.e., T is similar to the restriction of a decomposable operator) if and only if T has Bishop's property (β).

It was proved in [4, 5.3.2] that an invertible operator $S \in B(X)$ is decomposable provided that

$$\sum_{n=-\infty}^{\infty} \frac{\log \|S^n\|}{1+n^2} < \infty.$$

The following result answers in the affirmative a question from [14] and [15].

Theorem 3.1. *Let $T \in B(X)$ be a Banach space operator such that*

$$\sum_{n=1}^{\infty} \frac{\log v_n(T)}{n^2} < \infty.$$

Then there exists a Banach space $Y \supset X$ and an invertible operator $S \in B(Y)$ such that $T = S_{|X}$ and S satisfies

$$\sum_{n=-\infty}^{\infty} \frac{\log \|S^n\|}{1+n^2} < \infty.$$

In particular, T has Bishop's property (β). Moreover, $\sigma(S) = \sigma_{ap}(T) = \sigma(T) \cap \mathbb{T}$.

If $X = H$ is a Hilbert space, then $Y = K$ can be chosen to be a Hilbert space too.

4. Condition $(E(s))$

The following result answers an open question from [14].

Theorem 4.1. *Let $T \in B(X)$ satisfy $(E(s))$. Then there exist a Banach space $Y \supset X$ and an invertible operator S on a larger space such that T is a restriction of S and S satisfies $(E(s'))$ for suitable $s' < 1$. The construction is Hilbertian, i.e. if $X = H$ is Hilbert, then $Y = K$ can be chosen a Hilbert space too.*

5. Applications

We also obtain the following consequences.

5.1. Operators with countable spectrum

Using the above results and [22] we obtain the following characterization of operators which are similar to unitaries with a countable spectrum. Recall that a Hilbert space operator $T \in B(H)$ is said to be similar to a unitary if there is an invertible operator $L \in B(H)$ such that $L^{-1}TL$ is a unitary operator.

Corollary 5.1. *Let $T \in B(H)$ be a Hilbert space operator such that* $\sup_{n\geqslant 1} \|T^n\| < \infty$. *Suppose that there are positive constants C and $s < \frac{1}{2}$ such that*

$$m\big(T^n\big)^{-1} \leqslant C\,\mathrm{e}^{n^s} \quad (n \geqslant 1)$$

and that $\sigma(T)$ is countable. Then T is invertible and it is similar to a unitary operator.

We note that a classical similarity criterion of Sz.-Nagy states that an invertible Hilbert space operator T is similar to a unitary if and only if $\sup_{m\in\mathbb{Z}} \|T^m\| < \infty$.

5.2. Contractions with spectrum a Carleson set

Recall that a closed set E of $\mathbb{T}$ is said to be a *Carleson set* if

$$\int_0^{2\pi} \log\left(\frac{2}{\operatorname{dist}(\mathrm{e}^{\mathrm{i}t}, E)}\right) \mathrm{d}t < +\infty.$$

Using the above results and [8] (see also [11]) we obtain the following consequence.

Corollary 5.2. *Let $T \in B(H)$ be a Hilbert space contraction such that $\sigma_{ap}(T) \subset \mathbb{T}$ is a Carleson set. Suppose that there exist $C > 0$ and $s \geqslant 0$ such that $m(T^n)^{-1} \leqslant Cn^s$ for all n. Then T is an isometry.*

5.3. A Hilbertian counterpart of Arens' result

We also obtain the following Hilbertian counterpart of Arens' result. We refer to [18,19] for versions of Arens' result in the case of the (non-commutative) Banach algebra $B(X)$.

Corollary 5.3. *Let $T \in B(H)$ be an operator on Hilbert space with $m(T) > 0$. Then there exist a Hilbert space K, an isometric embedding $\pi : H \mapsto K$ and an invertible operator $S \in B(K)$ such that $S\pi = \pi T$, $\|S^j\| \leqslant \|T^j\|$ $(j \geqslant 1)$, $\|S^{-1}\| \leqslant 2/m(T)$ and*

$$\left\| \sum_{j=0}^{N} S^{-j}\pi(x_j) \right\|^2 \leqslant 2\sum_{j=0}^{N} \left(\frac{\sqrt{2}}{m(T)}\right)^{2j} \|x_j\|^2$$

for every $N \in \mathbb{N}$ and all $x_j \in H$.

The last condition says, in the terminology of [2], that S^{-1} is quadratically near the null operator modulo $\pi(H)$.

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