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Group Theory

Some finiteness results for groups with bounded algebraic entropy

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Abstract

We prove the following fact: the number of elements of any generating set S of a discrete group G is bounded from above if we assume that the algebraic entropy of G with respect to S is smaller than some universal constant and the existence of a finite index subgroup of G with some hyperbolicity properties. We deduce some finiteness results for the pairs (G, S) when there exists a system of relations of (universally) bounded length, as it is the case for word hyperbolic groups or fundamental groups of manifolds. In this last case, the results are of geometric interest. ***To cite this article: F. Zuddas, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Quelques résultats de finitude pour les groupes d'entropie algébrique bornée. On démontre le fait suivant : le nombre d'éléments de toute partie génératrice S de n'importe quel groupe discret G est majoré si on suppose l'entropie algébrique de G par rapport à S plus petite qu'une certaine constante universelle et l'existence d'un sous-groupe de G d'indice fini ayant certaines propriétés d'hyperbolicité. On en déduit des résultats de finitude pour les couples (G, S) lorsqu'il existe un système de relations de longueurs (universellement) bornées, comme c'est le cas pour les groupes hyperboliques ou pour les groupes fondamentaux de variétés. Dans ce dernier cas, les résultats ont un intérêt géométrique. ***Pour citer cet article : F. Zuddas, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Version française abrégée

Soit G un groupe finiment engendré, et S une partie génératrice finie. Dans la suite, on suppose S toujours *symétrique*, i.e. l'élément neutre $e \notin S$ et si $s \in S$, alors $s^{-1} \in S$. Le couple (G, S) est appelé *groupe marqué* (Definition 2.4). L'*entropie* de (G, S) est la limite $\mathrm{Ent}_S(G) = \lim_{n\to\infty}(1/n)\,\mathrm{Log}\, f(n)$ où $f(n)$ est le nombre des éléments de G contenus dans la boule de rayon n (centrée en l'élément neutre) pour la distance des mots d_S. L'*entropie algébrique* $\mathrm{Ent}_{\mathrm{alg}}(G)$ est définie comme l'infimum de $\mathrm{Ent}_S(G)$ pour toutes les parties génératrices S.

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Dans cette Note on démontre des résultats de finitude pour les couples (G,S) lorsque leur entropie est majorée par une constante universelle. Nos résultats s'appliquent aux extensions finies de groupes N-non abéliens (Definition 1.1). Les exemples principaux de groupes N-non abéliens sont donnés par les groupes δ-épais (un tel groupe est alors N-non abélien pour tout $N \geqslant 4/\delta$, voir la Proposition 1.14 de [1]), i.e. les groupes fondamentaux de variétés riemanniennes compactes X dont la courbure sectionnelle et le rayon d'injectivité vérifient $K \leqslant -1$ et $\mathrm{InjRad}(X) \geqslant \delta$ (les groupes δ-épais ont de plus la *propriété de transitivité de la commutation*, i.e. pour tous les γ_1, γ_2, γ_3 différents de l'élément neutre, si γ_1 commute avec γ_2 et si γ_2 commute avec γ_3, alors γ_1 commute avec γ_3) ; d'autres exemples de groupes N-non abéliens qui ont la propriété de transitivité de la commutation sont donnés par les produits libres de tels groupes entre eux ou avec des groupes abéliens sans torsion. Notre théorème principal est le suivant :

Théorème 0.1. *Soit N un entier positif, soit G un groupe finiment engendré et soit Γ un sous-groupe N-non abélien qui a la propriété de transitivité de la commutation, alors*

(i) *pour toute partie génératrice S de G, on a* $\mathrm{Ent}_S(G) > 0$;
(ii) *pour toute partie génératrice S de G, il existe un $g \in G$ tel que (si $\Gamma^* = \Gamma \setminus \{e\}$)*

$$d_S\big(e, g\Gamma^* g^{-1}\big) \geqslant \frac{1}{N} \cdot \frac{\mathrm{Log}\, 2}{\mathrm{Ent}_S(G)} - 1;$$

(iii) *si Γ est d'indice d fini dans G, toute partie génératrice S de G telle que* $\mathrm{Ent}_S(G) < \frac{1}{N} \cdot \frac{\mathrm{Log}\, 2}{3}$ *contient au plus $d-1$ éléments* ;
(iv) *pour toute partie génératrice S, on a* $\mathrm{Ent}_S(G) \geqslant \frac{1}{N} \cdot \frac{\mathrm{Log}\, 2}{\max(d-k,0)+3}$ *où d est l'indice de Γ dans G et k est le nombre d'éléments de S. En particulier, on a* $\mathrm{Ent}_{\mathrm{alg}}(G) \geqslant \frac{1}{N} \cdot \frac{\mathrm{Log}\, 2}{\max(d,3)}$ *(on a* $\mathrm{Ent}_{\mathrm{alg}}(G) \geqslant \frac{1}{N} \cdot \frac{\mathrm{Log}\, 2}{\max(d-1,3)}$ *si G est, de plus, sans torsion).*

La preuve du Théorème 0.1 (i) et (ii) est inspirée par le Théorème 2.6 et la Remarque 2.7 de [1].

On remarquera que tout groupe fondamental G d'une variété compacte localement symétrique de courbure strictement négative vérifie les hypothèses du Théorème 0.1 (voir [1], Lemme 1.6). Si on se restreint aux groupes hyperboliques, on déduit du Théorème 0.1(iii) le résultat de finitude suivant :

Corollaire 0.2. *Soit $h(\alpha,N,d)$ l'ensemble des groupes marqués α-hyperboliques (G,S), qui sont des extensions finies (d'indice $\leqslant d$) d'au moins un groupe N-non abélien satisfaisant la propriété de transitivité de la commutation. Alors, pour tous les éléments $(G,S) \in h(\alpha,N,d)$ sauf un nombre fini (à isomorphisme marqué près), on a* $\mathrm{Ent}_S(G) \geqslant \frac{1}{N} \cdot \frac{\mathrm{Log}\, 2}{3}$.

Posons $a_+ = \max(a,0)$, une application géométrique du Théorème 0.1(iii), est le :

Théorème 0.3. *Soient N et d des entiers naturels arbitraires. Soit $\chi(N,d)$ l'ensemble des variétés compactes X qui satisfont les propriétés suivantes* :

(i) *$\pi_1(X)$ contient un sous-groupe N-non abélien d'indice $\leqslant d$ qui satisfait la propriété de transitivité de la commutation* ;
(ii) *il existe une métrique g sur X telle que* :

$$\mathrm{diam}(g) \cdot \mathrm{Ent}_{\mathrm{vol}}(g) < \frac{1}{N} \cdot \frac{\mathrm{Log}\, 2}{6}.$$

Alors l'ensemble $\chi(N,d)$ contient un nombre fini (inférieur à $(d-3)_+\, 2^{(d-1)(d-2)^3}$) de 1-types d'homotopie. De plus le premier nombre de Betti de toute variété $X \in \chi(N,d)$ satisfait $b_1(X) \leqslant d-1$.

1. Introduction

Let G be a finitely generated group endowed with a finite generating set S. In the following, we shall always assume S to be symmetric, i.e. the identity $e \notin S$ and $s \in S$ implies $s^{-1} \in S$. The *entropy* of (G, S) is the limit $\mathrm{Ent}_S(G) = \lim_{n\to\infty}(1/n)\,\mathrm{Log}\, f(n)$ where $f(n)$ denotes the number of elements of G in the ball $B_{(G,d_S)}(e, n)$, i.e. the ball of radius n centered at the identity $e \in G$ with respect to the word metric d_S of (G, S). The *algebraic entropy* $\mathrm{Ent}_{\mathrm{alg}}(G)$ is defined as $\inf_S \mathrm{Ent}_S(G)$, for S running over all finite generating sets of G. The group G has *exponential growth* if $\mathrm{Ent}_S(G) > 0$ for some (and hence for any) finite generating set S; it has *uniform exponential growth* if $\mathrm{Ent}_{\mathrm{alg}}(G) > 0$. In 1981 Gromov formulated (see [4], Remark 5.12) the following conjecture:

Every group of exponential growth has uniform exponential growth.

This conjecture is justified by the fact, proved by Sambusetti [6], that there exist groups G such that $\mathrm{Ent}_{\mathrm{alg}}(G) < \mathrm{Ent}_S(G)$ for every generating set S. Koubi [5] positively solved this conjecture for word hyperbolic groups G, exhibiting a strictly positive constant C_G (depending on the group G) such that $\mathrm{Ent}_{\mathrm{alg}}(G) \geqslant C_G$. More recently, Wilson [8] gave examples of groups having exponential but not uniform exponential growth. This justifies the following questions: what classes of groups have the property that the algebraic entropy is attained for some generating set? is bounded from below by some universal constant? Our main purpose in this paper is to prove finiteness results on groups and their presentations when their entropy (with respect to their generating sets) is bounded above by some universal constant (see Corollary 2.3, Theorem 2.5, Corollary 2.6, Theorem 2.7, Corollary 2.8). From these results we shall derive lower bounds for the algebraic entropy of our groups and determine cases where it is attained. Our results apply in particular to word hyperbolic groups G which contain a δ-thick subgroup Γ of finite index d (a group Γ is called *δ-thick* if Γ is isomorphic to the fundamental group of a compact manifold X with sectional curvature $K \leqslant -1$ and injectivity radius $\mathrm{InjRad}(X) \geqslant \delta$). A particularly interesting class of examples is provided by the following fact (see [1], Lemma 1.6): for any $\delta > 0$, the fundamental group of a compact locally symmetric manifold M of non-compact type and rank 1 contains a δ-thick subgroup of finite index. More generally, our results apply to any group G containing a *N-non-abelian* subgroup Γ.

Definition 1.1. Let $N > 0$ be an integer. A finitely generated group Γ is said to be *N-non-abelian* if every normal abelian subgroup of Γ is reduced to the identity $\{e\}$ and if it has the following property: for every pair of elements $\gamma_1, \gamma_2 \in \Gamma$ which do not commute, at least one of the pairs $\{\gamma_1^N, \gamma_2^N\}$ or $\{\gamma_1^N, \gamma_2^{-N}\}$ generate a free semi-group in Γ.

The main examples of N-non-abelian groups are given by the δ-thick groups and their (possibly infinite index) non-abelian subgroups (see Proposition 1.14 of [1]). Moreover, these groups have also the *property of transitivity of commutation* (i.e. for all $\gamma_1, \gamma_2, \gamma_3$ different from the identity, if γ_1 commutes with γ_2 and if γ_2 commutes with γ_3, then γ_1 commutes with γ_3); other examples are given by the free products of such groups with each other, or with abelian torsion-free groups.

2. Finiteness results

The main argument in the proofs of the finiteness results is the following theorem:

Theorem 2.1. *Let N be a positive integer. Let G be a finitely generated group and let Γ be a N-non-abelian subgroup which has the property of transitivity of commutation. Then*

(i) *for every generating set S of G, we have* $\mathrm{Ent}_S(G) > 0$;

(ii) *for every generating set S of G, there exists $g \in G$ such that (if $\Gamma^* = \Gamma \setminus \{e\}$)*

$$d_S(e, g\Gamma^* g^{-1}) \geqslant \frac{1}{N} \cdot \frac{\mathrm{Log}\,2}{\mathrm{Ent}_S(G)} - 1;$$

(iii) *if Γ has finite index d in G, then every generating set S of G such that $\mathrm{Ent}_S(G) < \frac{1}{N} \cdot \frac{\mathrm{Log}\,2}{3}$ contains at most $d-1$ elements;*

(iv) *for every generating set S, we have $\mathrm{Ent}_S(G) \geqslant \frac{1}{N} \cdot \frac{\mathrm{Log}\,2}{\max(d-k,0)+3}$ where d is the index of Γ in G and k is the number of elements of S. In particular, one has $\mathrm{Ent}_{\mathrm{alg}}(G) \geqslant \frac{1}{N} \cdot \frac{\mathrm{Log}\,2}{\max(d,3)}$ (one has $\mathrm{Ent}_{\mathrm{alg}}(G) \geqslant \frac{1}{N} \cdot \frac{\mathrm{Log}\,2}{\max(d-1,3)}$ if, moreover, G is torsion-free).*

The proof of Theorem 2.1 (i) and (ii) is based on revisited versions of Théorème 2.6 and Remarque 2.7 of [1].

Remark 1.

(a) Notice that, in the above inequality (ii), the group G is allowed to be either a finite or infinite extension of the subgroup Γ. In the case when Γ is normal, we get from (ii) $d_S(e, \Gamma^*) \geqslant \frac{1}{N} \cdot \frac{\mathrm{Log}\,2}{\mathrm{Ent}_S(G)} - 1$.

(b) One should compare the above estimate (iv) on the algebraic entropy $\mathrm{Ent}_{\mathrm{alg}}(G)$ with the more general result of Shalen and Wagreich ([7]). They show that, for a given finitely generated group G and a subgroup Γ of finite index d in G, one has $\mathrm{Ent}_{\mathrm{alg}}(G) \geqslant \frac{1}{2d-1}\,\mathrm{Ent}_{\mathrm{alg}}(\Gamma)$. Here, (iv) provides a more explicit lower bound for the entropy of G with a better dependence on the index d.

Let us now temporarily assume the group G to be word hyperbolic. Let us define $a_+ = \max(a, 0)$. Then the following finiteness result (for presentations) holds true:

Lemma 2.2. *For every $\alpha > 0$ and every $p \in \mathbb{N}$, for every torsion-free non-elementary word hyperbolic group G, there exist at most $([p/2]-1)_+ \, 2^{(p^{(12\alpha+9)})}$ (symmetric) generating sets S of G (modulo the relation $S \sim \Sigma$, which means that there is an isomorphism $G \to G$ sending S onto Σ) with no more than p elements, such that the group G is α-hyperbolic with respect to the algebraic distance associated to S.*

Remark 2. From our assumptions, the set of generating sets mentioned above is empty if $p \leqslant 3$.

By combining Theorem 2.1(iii) and Lemma 2.2 we obtain our first finiteness result:

Corollary 2.3. *Let G be a torsion-free finite extension of index $\leqslant d$ of a N-non-abelian group Γ such that Γ satisfies the property of transitivity of commutation. Then the set of equivalence classes (for the relation $\sim$ defined above in Lemma 2.2) of generating sets S of G such that*

(a) *G is α-hyperbolic with respect to S;*
(b) $\mathrm{Ent}_S(G) < \frac{1}{N} \cdot \frac{\mathrm{Log}\,2}{3}$

is finite and contains at most $([(d-3)/2])_+ \, 2^{((d-1)^{(12\alpha+9)})}$ elements.

Remark 3. If $d \leqslant 4$, assumption (b) is never satisfied (by Theorem 2.1(iv)).

In the above Lemma 2.2 and Corollary 2.3, the assumption, for the group G, to be torsion-free is not really needed (compare with Theorem 2.5 and Corollary 2.6). In Lemma 2.2 and Corollary 2.3 we were considering different generating sets on a fixed group G. In what follows, we shall consider different groups endowed with different generating sets. This is the reason why we shall now work in the category of marked groups (see, for example, [2]):

Definition 2.4. A *marked group* is a pair (G, S) consisting of a finitely generated group G and a generating set S of G. Two marked groups (G_1, S_1) and (G_2, S_2) are said to be *marked-isomorphic* if there is an isomorphism from G_1 to G_2 which maps S_1 onto S_2.

A marked group (G, S) is said to be *α-hyperbolic* if the group G is α-hyperbolic with respect to the algebraic distance associated to S.

We define $g(N, d, N_0)$ as the set of marked groups (G, S) satisfying the following properties:

(i) G contains a N-non-abelian subgroup Γ (of index $\leqslant d$) such that the commutation is transitive on Γ^*,
(ii) there exists a presentation $\langle S|R\rangle$ of G (where S is the given generating set) such that the length of every element $r \in R$ (for the word metric) is less or equal to N_0.

Notice that for every $N \geqslant 1$, $d \geqslant 1$, $N_0 \geqslant 0$, the set $g(N, d, N_0)$ is not empty.

Theorem 2.5. *For arbitrary integers N, d, $N_0 \in \mathbb{N}^*$, the number of elements (G, S) of $g(N, d, N_0)$ which satisfy $\mathrm{Ent}_S(G) < \frac{1}{N} \cdot \frac{\mathrm{Log}\,2}{3}$ is (up to marked isomorphisms) bounded above by $(d-3)_+ 2^{(d-1)(d-2)^{N_0}}$. Consequently, either $\mathrm{Ent}_S(G) \geqslant \frac{1}{N} \cdot \frac{\mathrm{Log}\,2}{3}$ for any $(G, S) \in g(N, d, N_0)$ or $\inf_{(G,S)\in g(N,d,N_0)} \mathrm{Ent}_S(G)$ is attained for some $(G_0, S_0) \in g(N, d, N_0)$.*

The above Theorem 2.5 and the a priori bound on the length of relations in word hyperbolic groups (see [3], Proposition 17, p. 76) imply the following:

Corollary 2.6. *Let $h(\alpha, N, d)$ be the set of marked α-hyperbolic groups (G, S), which are finite extensions (of index $\leqslant d$) of at least one N-non-abelian group satisfying the property of transitivity of commutation. Then, for all but finitely many pairs $(G, S) \in h(\alpha, N, d)$ (up to marked isomorphism), we have* $\mathrm{Ent}_S(G) \geqslant \frac{1}{N} \cdot \frac{\mathrm{Log}\,2}{3}$.

Notice that for every $N \geqslant 1$, $d \geqslant 1$, $\alpha \geqslant 0$, the set $h(\alpha, N, d)$ is not empty.

The following is a geometric application of the previous results:

Theorem 2.7. *Let N and d be arbitrary positive integers. Let $\chi(N, d)$ be the set of compact manifolds X which satisfy the following properties*:

(i) *$\pi_1(X)$ contains a N-non-abelian subgroup of index $\leqslant d$ which satisfies the property of transitivity of commutation*;
(ii) *there exists a metric g on X such that*:

$$\mathrm{diam}(g) \cdot \mathrm{Ent}_{\mathrm{vol}}(g) < \frac{1}{N} \cdot \frac{\mathrm{Log}\,2}{6}.$$

Then the set $\chi(N, d)$ contains a finite number (smaller than $(d-3)_+ 2^{(d-1)(d-2)^3}$) of 1-homotopy types. Moreover the first Betti number of any manifold $X \in \chi(N, d)$ satisfies $b_1(X) \leqslant d - 1$.

Corollary 2.8. *Consider the set $\bar{\chi}(\delta, d)$ of compact manifolds X which satisfy, for arbitrary values of $\delta \in (0, +\infty)$ and $d \in \mathbb{N}^*$, the following properties*:

(i) *X admits some finite covering X' (of index $\leqslant d$) which can be endowed with some metric g' whose sectional curvature $K_{g'}$ and injectivity radius* $\mathrm{InjRad}(g')$ *satisfy $K_{g'} \leqslant -1$ and* $\mathrm{InjRad}(g') \geqslant \delta$;

(ii) *X admits some Riemannian negatively curved metric g such that*

$$\mathrm{diam}(g) \cdot \mathrm{Ent}_{\mathrm{vol}}(g) < \frac{\delta}{4+\delta} \cdot \frac{\mathrm{Log}\, 2}{6}.$$

Then the set $\bar{\chi}(\delta, d)$ contains a finite number of homotopy types, which is bounded by $(d-4)_+ 2^{(d-1)(d-2)^3}$ and, for every element $X \in \bar{\chi}(\delta, d)$, we have $b_1(X) \leqslant [\frac{d-1}{2}]$.

Remark 4.

(a) The metrics g' and g can be chosen independently.

(b) Observe that the statement of Corollary 2.8 still holds true when g is not assumed to be negatively curved. In this case, we get that the number of 1-homotopy types and the first Betti number are respectively bounded by $(d-3)_+ 2^{(d-1)(d-2)^3}$ and $(d-1)$.

(c) The assumption (ii) of Corollary 2.8 does not depend on the dimension. This means that, in all dimensions n (except for a finite number) and for every negatively curved manifold (X^n, g) satisfying the assumption (i) we have:

$$\mathrm{diam}\big(X^n, g\big) \cdot \mathrm{Ent}_{\mathrm{vol}}\big(X^n, g\big) \geqslant \frac{\delta}{4+\delta} \cdot \frac{\mathrm{Log}\, 2}{6}.$$

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