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Numerical Analysis

Asymptotic preserving scheme and numerical methods for radiative hydrodynamic models

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Abstract

In this Note, we present a scheme for nonlinear radiative systems which are compatible with diffusive asymptotics. The scheme is based on a splitting: firstly we use a relaxation step to change the problem into 2 identical systems of linear transport systems and, secondly, we use a so-called 'well balanced' scheme for each of the 2 systems. The main advantages of our scheme is that it is fully implicit and compatible with physical properties (positivity); it can be used with a nonconstant cross section and for nonuniform mesh. ***To cite this article: C. Buet, S. Cordier, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Analyse asymptotique et méthodes numériques pour les méthodes de moments en hydrodynamique radiative. Dans cette Note, nous présentons un schéma pour un modèle non linéaire de transfert radiatif, qui soit compatible avec la limite diffusion. Ce schéma est composé de deux étapes : une étape de relaxation qui transforme le système non linéaire en deux systèmes d'équation de transport linéaire identiques et un schéma « équilibre » pour chacun de ces systèmes. L'intérêt principal de notre schéma est d'être totalement implicite, de préserver les propriétés physiques (positivité) et d'être utilisable avec une section efficace variable et un maillage non uniforme. ***Pour citer cet article : C. Buet, S. Cordier, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***
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L'objectif de cette Note est de présenter une discrétisation d'un système hyperboliques de lois de conservation qui soit compatible avec le régime asymptotique de diffusion. On s'intéresse à un système de la forme (1) où ε est un petit paramètre.

En effet, lorsque $\varepsilon \to 0$, le système (1) se comporte comme une équation de diffusion dont le coefficient de diffusion est $1/3\sigma$, σ étant la section efficace. Le modèle limite dont les vitesses de propagation sont infinies pose

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de sérieux problèmes notamment au niveau numérique et de nombreux travaux ont été réalisées pour lever ces difficultés (limiteurs de flux, facteurs d'Eddington variables...). Cette problématique est très étroitement liée aux études sur les termes sources raides pour les systèmes hyperboliques, les méthodes de relaxation, les schémas dits 'asymptotic preserving'....

La méthode que nous présentons ici est constituée de deux étapes : la première consiste à transformer le système de deux équations non linéaires en un double système (7) de deux équations, linéaires et découplées connus sous le nom d'équations du télégraphe. Il s'agit d'une application des méthodes de relaxation [6] et cela conduit à doubler le nombre d'inconnues. Pour chacun des systèmes obtenus, on utilise ensuite la discrétisation proposée dans [4] que l'on généralise pour une section efficace non constante et un maillage non uniforme.

Le schéma ainsi obtenu est totalement implicite et il a toutes les propriétés requises : consistance lorsque $\Delta x \to 0$, comportement asymptotique lorsque $\varepsilon \to 0$, préservation du domaine invariant (ce qui revient dans les nouvelles variables à garantir la positivité des solutions). Un test numérique avec σ variant de 0 (au centre) à 100 (sur les bords du domaine) illustre cette discrétisation.

L'avantage de la méthode présentée ici est qu'elle peut être utilisée avec une section efficace σ variable, ce qui est très important en pratique car de tels problèmes de transfert radiatif sont couplés avec un modèle hydrodynamique qui va déterminer la valeur de σ. Celle-ci sera d'ordre 1 dans les zones denses ou opaques et pourra être très faible dans les zones dites transparentes. Pour pouvoir utiliser un schéma sans restriction sur les valeurs de σ, il est donc indispensable que la discrétisation ait un bon comportement y compris dans les zones transparentes.

Nous nous sommes également attachés à présenter une méthode utilisable avec un maillage non uniforme car les codes de calcul utilisent des techniques de rafinement de maillage automatique et il est donc important de pouvoir traiter de tels maillages. Par ailleurs, l'extension au cas multidimensionnel et à la prise en compte de terme correctifs, par exemple pour tenir compte d'effet relativiste ou des échanges d'énergie entre les photons et la matière, sera présentée dans [1].

1. Introduction

In this Note, we are interested in systems arising from radiative hydrodynamic problems of the following form

$$\varepsilon\partial_t U + \partial_x F(U) = \frac{1}{\varepsilon}R(U), \tag{1}$$

where $U = (\rho, j)$, $F(U) = (j, \rho h(j/\rho))$, $R(U) = (0, -\sigma j)$, $\sigma(x) > 0$ is the cross section, ε is a small parameter and h is an odd, positive, convex function. The function h represents, in the radiative transfert problem, the so called Eddington factor. There exists many such functions. Let us mention, for example (see [7] and the reference therein for other examples and more details)

$$h(u) = \frac{3 + 4u^2}{5 + 2\sqrt{4 - 3u^2}}.$$

This function satisfies the following properties:

$$h(0) = \frac{1}{3}, \quad u^2 \leqslant h(u) \leqslant 1. \tag{2}$$

Under hypothesis (2), it can be proven that the following properties are preserved under time evolution:

$$\rho \geqslant 0, \quad \|j\| \leqslant \rho. \tag{3}$$

This invariant property has a physical interpretation, since ρ and j are the two first moments of the distribution function $\rho = \int f\,\mathrm{d}\omega$, $j = \int f\omega\,\mathrm{d}\omega$. From the numerical point of view, this property means that the fluxes are the so-called limited fluxes.

In the limit as $\varepsilon \to 0$, a formal asymptotic limit implies that $j = \mathrm{O}(\varepsilon)$ due to the collision term and, at first order in ε, using the second equation of (1), we get

$$j = -\frac{\varepsilon}{\sigma}\partial_x\big(h(0)\rho\big). \tag{4}$$

Then, using $h(0) = 1/3$ and dropping this ansatz into the first equation, we obtain the following diffusion approximation

$$\partial_t \rho - h(0)\partial_x\left(\frac{1}{\sigma}\partial_x \rho\right) = 0. \tag{5}$$

Note that the solution for ρ of the limit heat equation (5) and j given by (4) does not satisfy automatically condition (3), because the gradient $\partial_x \rho$ can be arbitrarily large, e.g. if the initial data is discontinuous in ρ.

The goal of this Note is to present a scheme that is compatible with the limit $\varepsilon \to 0$ and with the invariant property (3) and is implicit. It is based on a time splitting, in two steps. The first step is based on a relaxation method and the second on well-balanced schemes [4].

Various methods have been proposed to get rid of these difficulties, such as variable Eddington factors for the so-called $P1$-approximation, or flux limiters for the diffusion approximation (see [7] and references therein). This is also related to a series of papers about asymptotic preserving schemes for kinetic problems, well balanced schemes, stiff source terms and relaxation methods, in the context of hyperbolic systems [4–6,8].

2. The asymptotic preserving scheme

Let us now describe one iteration which is decomposed into two steps, first a relaxation step which replaced the nonlinear system by two uncoupled linear systems and, second, the resolution of these systems using a well balanced scheme. This method permits the change of the nonlinear system into a system of two linear systems and makes it possible to use direct implicit solvers.

2.1. The relaxed scheme

Let us describe one iteration from $t = 0$ to $t = \Delta t$. We follow the method proposed in [6] which led to the introduction of an artificial vector valued variable (z, w) and to the expression of a linear system for the variables (ρ, z, w, j). The first step is the so called zero relaxation limit, and it can be interpreted as a projection step onto the equilibrium states:

$$z = j, \quad w = \rho h\left(\frac{j}{\rho}\right). \tag{6}$$

The second step consists in solving, during Δt, the transport part:

$$\begin{cases} \partial_t \rho + \dfrac{1}{\varepsilon}\partial_x z = 0,\\ \partial_t z + \dfrac{a}{\varepsilon}\partial_x \rho + \dfrac{\sigma}{\varepsilon^2} z = 0,\\ \partial_t w + \dfrac{a}{\varepsilon}\partial_x j = 0,\\ \partial_t j + \dfrac{1}{\varepsilon}\partial_x w + \dfrac{\sigma}{\varepsilon^2} j = 0. \end{cases} \tag{7}$$

In the relaxation part, the original variables (ρ and j) are unchanged. The coefficient a is constant in space but has to be chosen at each time step in order to recover the correct diffusion coefficient and to insure the stability condition (see Proposition 2.1 in [6]).

Let us emphasize that (7) is just two linear and identical systems, which are uncoupled, one for the quantities (ρ and z), the second for (w and a new variable $\bar{j} = aj$) which are in the form (1) but with $h \equiv a$.

Note that this system i.e. (1) with $h \equiv a$, once diagonalized, is the well-know Goldstein–Taylor or Telegraph equation with speed $\pm\sqrt{a}$:

$$\begin{cases} \partial_t u + \dfrac{\sqrt{a}}{\varepsilon}\partial_x u = \dfrac{\sigma}{2\varepsilon^2}(v-u), \\ \partial_t v - \dfrac{\sqrt{a}}{\varepsilon}\partial_x v = \dfrac{\sigma}{2\varepsilon^2}(u-v). \end{cases} \tag{8}$$

Setting $U = \sqrt{a}\rho + z + w + \sqrt{a}j$, $V = \sqrt{a}\rho - z + w - \sqrt{a}j$, $\overline{U} = \sqrt{a}\rho + z - w - \sqrt{a}j$, $\overline{V} = \sqrt{a}\rho - z - w + \sqrt{a}j$, (U, V) and $(\overline{U}, \overline{V})$ satisfy (8), and we show that for the transport part, the invariant domain (3) comes from the positivity of $U, V, \overline{U}, \overline{V}$ for a sufficiently large value of a, [1]. We prove that the choice

$$a = h\Big(\max_{x\in\mathbb{R}} \|u(x)\|\Big),$$

ensures the positivity of the initial data for $U, V, \overline{U}, \overline{V}$ and then of the solution $U, V, \overline{U}, \overline{V}$ of system (7). This choice also provides stability for the relaxed system. In the diffusive limit ($\varepsilon \to 0$), or for large time behaviour, we expect that $\max_{x\in\mathbb{R}} \|u(x)\| \to 0$ and therefore, a is close to $1/3$, i.e. we obtain the right asymptotic (5).

2.2. *The well-balanced scheme for the transport part*

We have now to discretize sytems (8) for (ρ, z) and similarly for $(w, \sqrt{a}j)$. We introduce a non-uniform mesh: we note by x_i, the center of the cell of size Δx_i with $i \in \mathbb{Z}$ and define $\Delta x_{i+\frac{1}{2}} = (\Delta x_i + \Delta x_{i+1})/2$. The proposed discretization is a so called well balanced scheme, which is a extension of the scheme described in [4].

$$\begin{cases} \dfrac{\mathrm{d}u_i}{\mathrm{d}t} + M_{i-\frac{1}{2}}\dfrac{\sqrt{a}}{\varepsilon\Delta x_i}(u_i - u_{i-1}) = M_{i-\frac{1}{2}}\dfrac{\Delta x_{i-\frac{1}{2}}}{\Delta x_i}\dfrac{\sigma_{i-\frac{1}{2}}}{2\varepsilon^2}(v_i - u_i), \\ \dfrac{\mathrm{d}v_i}{\mathrm{d}t} - M_{i+\frac{1}{2}}\dfrac{\sqrt{a}}{\varepsilon\Delta x_i}(v_{i+1} - v_i) = M_{i+\frac{1}{2}}\dfrac{\Delta x_{i+\frac{1}{2}}}{\Delta x_i}\dfrac{\sigma_{i+\frac{1}{2}}}{2\varepsilon^2}(u_i - v_i), \end{cases} \tag{9}$$

where the coefficient $M_{i+\frac{1}{2}}$ is defined by

$$M_{i+\frac{1}{2}} = \frac{2\sqrt{a}\varepsilon}{\sigma_{i+\frac{1}{2}}\Delta x_{i+\frac{1}{2}} + 2\sqrt{a}\varepsilon}, \tag{10}$$

and $\sigma_{i+\frac{1}{2}}$ is an arbitrary average of σ at the interface (e.g. arithmetic, harmonic...). The above scheme corresponds to that proposed in [4] for a uniform mesh, $\sigma = 2$ and a diffusion coefficient in the limit heat equation equal to $\frac{1}{2}$. Note that, in our case, the cross section is not assumed to be constant, which is the main interest from an applications point of view.

We can show that (9) is a monotone scheme and then (3) remains an invariant domain during the transport part. It is readily seen that, in the limit $\max_i(\Delta x_i) \to 0$, the coefficient $M_{i+\frac{1}{2}}$ tends to 1, and the consistency of scheme (9) with the continuous system (8) is satisfied, provided that, in the limit, the mesh is smooth enough, i.e. $\max_i(\frac{\Delta x_{i+1}}{\Delta x_i}) \to 0$.

The scheme (9) can be written in the original variables (ρ, z) and the same for $(w, \bar{j})$

$$\begin{cases} \dfrac{\mathrm{d}\rho_i}{\mathrm{d}t} + \dfrac{1}{\varepsilon\Delta x_i}(M_{i+\frac{1}{2}}z_{i+\frac{1}{2}} - M_{i-\frac{1}{2}}z_{i-\frac{1}{2}}) = 0, \\ \dfrac{\mathrm{d}z_i}{\mathrm{d}t} + \dfrac{a}{\varepsilon\Delta x_i}(M_{i+\frac{1}{2}}\rho_{i+\frac{1}{2}} - M_{i-\frac{1}{2}}\rho_{i-\frac{1}{2}}) = \dfrac{-\lambda_i}{2a\varepsilon^2}z_i + \dfrac{M_{i+\frac{1}{2}} - M_{i-\frac{1}{2}}}{\varepsilon\Delta x_i}(a\rho_i) \end{cases}$$

with

$$z_{i+\frac{1}{2}} = (z_i + z_{i+1} + \rho_{i+1} - \rho_i)/2, \qquad \rho_{i+\frac{1}{2}} = (\rho_i + \rho_{i+1} + z_{i+1} - z_i)/2,$$

and

$$\lambda_i = \frac{\Delta x_{i+\frac{1}{2}}}{\Delta x_i} M_{i+\frac{1}{2}} \sigma_{i+\frac{1}{2}} + \frac{\Delta x_{i-\frac{1}{2}}}{\Delta x_i} M_{i-\frac{1}{2}} \sigma_{i-\frac{1}{2}}.$$

Note that the formulae can be simplified for a uniform mesh and constant cross section. In this case, $\lambda = 2\sigma M$ and M is given by (10) with $\sigma_i = \sigma$ and the mesh size being constant; the second term of the right-hand side vanishes since $M_{i-\frac{1}{2}} = M_{i+\frac{1}{2}}$. Then, the proposed scheme reduces to a classical Godunov scheme

$$\varepsilon \frac{\mathrm{d}U_i}{\mathrm{d}t} + M(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}})/\Delta x = \frac{M}{\varepsilon} R(U_i), \tag{11}$$

where the fluxes at the interfaces are given by

$$F_{i+\frac{1}{2}} = \left[-a_i F_{i+1} + a_{i+1} F_i - a_i a_{i+1}(U_{i+1} - U_i)\right]/(a_{i+1} - a_i), \tag{12}$$

which are just the upwind fluxes for (8) with $a_{i+1} = -a_i = \sqrt{a}$ the characteristic speeds.

This form is useful because it is expressed in the original variable and this can be easily generalized to the nonlinear case. The discretization is a sum of a diffusive term and classical convective term. Thus, this discretization can be interpreted as a particular choice of adding a numerical viscosity term depending on ε.

Moreover, when $\varepsilon \to 0$, we show that the scheme is an approximation of the heat equation with a diffusion coefficient equal to $1/3$ for radiative hydrodynamic applications.

2.3. Time discretization

We claim that a fully implicit time discretization is suitable. Indeed, a partial implicit time discretization as described in [4] led to a prohibitive parabolic CFL (time step restriction) $\sigma \Delta t \leqslant (\Delta x)^2$ to ensure monotonicity. Note that the expected time step restriction with an explicit method for the transport part (CFL condition) is much more restrictive $\frac{\Delta t}{\Delta x} \leqslant \varepsilon$, and, similarly, the characteristic relaxation time is such that $\Delta t \sigma \leqslant \varepsilon^2$. The method proposed here allows us to use a fully implicit time discretization for the system of linear equations. Moreover, we prove that the properties of the invariant domain and asymptotic behaviour are the same as for time continuous discretization (9) (see [1] for more details).

3. Numerical results

We now illustrate our scheme on the following test. The domain is $[0, 2]$. The cross section is vanishing in the middle of the domain and is very large at the boundary.

$$\sigma(x) = 100(x-1)^4, \quad x \in]0, 2[.$$

The initial data is a characteristic function, for ρ with support in $[\frac{1}{2}, \frac{3}{2}]$. The initial flux j is equal to 0. The simulated time is $T = 0.1$ and the small parameter value takes the following values: $\varepsilon = 0$ (diffusion), $10^{-2,1,0}$. The mesh is uniform with either 100 or 1000 points and the time step is chosen such that $\Delta t/\Delta x = 0.05$ – see Fig. 1.

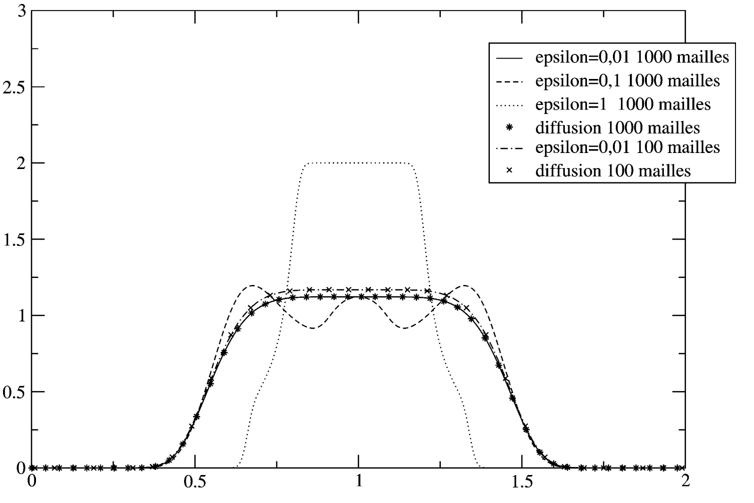

Fig. 1. Numerical example using test data.

Fig. 1. Exemple numérique avec des données d'essai.

4. Conclusions

It is also possible to extend this approach to multi-dimensional problem (including Adaptive Mesh Refinement). Lastly, it is also possible to take into account more complex source terms, such as the terms coming from relativistic effects, presented in [2], or the coupling with a hydrodynamical model due to energy transfert.

Let us also mention that the above method can be used for collisional kinetic model in diffusive regimes (like the Lorentz model presented in [3]) at least in the case of constant cross section and uniform mesh in space.

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