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Number Theory

Mass formula for supersingular Drinfeld modules

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Abstract

We generalize Gekeler's mass formula for supersingular Drinfeld modules from rational function fields to arbitrary global function fields. The proof is based on a calculation of Tamagawa numbers. ***To cite this article: C.-F. Yu, J. Yu, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Une formule de masse pour les modules de Drinfeld supersinguliers. Nous démontrons une « formule de masse » pour les modules de Drinfeld supersinguliers. Cette formule généralise celle obtenue par Gekeler dans le cas de $\mathbb{F}_q[T]$. La démonstration repose sur un calcul de nombres de Tamagawa. ***Pour citer cet article : C.-F. Yu, J. Yu, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

1. Introduction

The classical mass formula of Deuring–Eichler comes from a 1–1 correspondence between isomorphism classes of supersingular elliptic curves in characteristic p and left ideal classes in a maximal order of a definite quaternion algebra over $\mathbb{Q}$ ramified at p. This correspondence allows us to gain deeper understanding on both the supersingular elliptic curves and the definite quaternion algebras over $\mathbb{Q}$ ramified at a single prime. It is not surprising that an analogous situation exists for global function fields. Here Drinfeld modules of arbitrary rank r play the role of elliptic curves, and on the algebraic side one considers central division algebras of dimension r^2 over a ground function field K which ramify at precise two places v, ∞ with invariants $1/r$, $-1/r$, respectively.

Gekeler in [3,2,5] obtained the mass formula, for the case K being the rational function field and r arbitrary, as well as for the general quaternion case: that is $r = 2$, and K arbitrary. The aim of this paper is to complete the picture, generalizing Gekeler's mass formula to both arbitrary r and arbitrary K. Instead of working on moduli schemes of Drinfeld modules, we reduce the general case to the rational function field case by computing and comparing Tamagawa measures for the multiplicative group scheme arising from the central division algebra in question.

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2. Statement of results

Let K be a global function field with constant field $\mathbb{F}_q$. Let ∞ be a place of K, referred to as the place at infinity. Let A be the subring of functions regular everywhere outside ∞. We fix a finite place v_0 of K and are interested in Drinfeld A-modules over A-fields of finite characteristic v_0.

Let τ denote the endomorphism $x \mapsto x^q$ of $\mathbb{G}_a$. Let $i : A \to L$ be an A-field of characteristic v_0 and $\phi : A \to L\{\tau\}$ be a Drinfeld module over L, where $L\{\tau\}$ is the non-commutative polynomial ring generated by τ. Given a non-zero ideal $\mathfrak{a} \subset A$, the $\mathfrak{a}$-torsion of the module ϕ consists of $\phi[\mathfrak{a}](\bar{L}) = \{\alpha \in \bar{L} \mid \phi(a)(\alpha) = 0,\ \forall a \in \mathfrak{a}\}$. There are positive integers r and h such that as finite A-modules,

$$\phi[\mathfrak{a}](\bar{L}) \simeq (A/\mathfrak{a})^r, \qquad \text{if } \mathfrak{p}_0 \text{ does not divide } \mathfrak{a}; \tag{1}$$

$$\phi[\mathfrak{p}_0](\bar{L}) \simeq (A/\mathfrak{p}_0)^{r-h}, \tag{2}$$

where $\mathfrak{p}_0$ is the prime ideal of A corresponding to the place v_0. The integers r and h are called the rank and height of the Drinfeld module ϕ, respectively. The height h may range from 1 to r. When $h = r$, ϕ is called *supersingular*.

Let $\Lambda(r, v_0)(L)$ denote the set of isomorphism classes of rank r supersingular Drinfeld modules over an algebraically closed field L of A-characteristic v_0. It is known that $\Lambda(r, v_0)(L)$ is finite and all the members are defined over a certain finite field. We write $\Lambda(r, v_0)$ for $\Lambda(r, v_0)(\overline{k(v_0)})$, where $k(v_0) = A/\mathfrak{p}_0$ is the residue field at the place v_0. Define $\operatorname{mass}(\Lambda(r, v_0)) := \sum_{\phi \in \Lambda(r,v_0)} \frac{1}{\#\operatorname{Aut}(\phi)}$ to be the mass of $\Lambda(r, v_0)$. The main result in this paper is

Theorem 2.1. *Let notations be as above. One has*

$$\operatorname{mass}\bigl(\Lambda(r, v_0)\bigr) = \frac{\#\operatorname{Pic}(A)}{q-1} \prod_{i=1}^{r-1} \zeta_K^{\infty, v_0}(-i).$$

Here $\zeta_K^{\infty,v_0}(s)$ is the ζ-function of the scheme $\operatorname{Spec} A \setminus \mathfrak{p}_0$:

$$\zeta_K^{\infty,v_0}(s) := \prod_{v \neq \infty, v_0} \bigl(1 - N(v)^{-s}\bigr)^{-1} = \zeta_K(s)\bigl(1 - N(\infty)^{-s}\bigr)\bigl(1 - N(v_0)^{-s}\bigr).$$

3. Supersingular Drinfeld modules

In this section, we recall some properties of supersingular Drinfeld modules, due to Drinfeld [1] and mainly to Gekeler [4]. We keep the notations in the previous section, and let k be a fixed algebraic closure of $k(\mathfrak{p}_0)$. If $\phi \in \Lambda(r, v_0)$, then there is a canonical formal $A_{\mathfrak{p}_0}$-module structure on $\phi[\mathfrak{p}_0^\infty]$, viewed as $A_{\mathfrak{p}_0}$-divisible group.

Theorem 3.1 (Drinfeld). *Up to isomorphism, there is a unique* 1*-dimensional formal $A_{\mathfrak{p}_0}$-module of height r over k. The endomorphism ring of $\phi[\mathfrak{p}_0^\infty]$ is the maximal order of the central division algebra over K_{v_0} with invariant* $1/r$.

Theorem 3.2 (Gekeler). *Let $\phi, \phi' \in \Lambda(r, v_0)$.*

1. $[\operatorname{End}(\phi) \otimes_A K\colon K] = r^2$.
2. *ϕ and ϕ' are isogenous.*
3. *The relative Frobenius morphism π_0 for ϕ over a sufficiently large finite field is in A.*
4. *The natural map* $\operatorname{Hom}_k(\phi, \phi') \otimes A_{\mathfrak{p}} \to \operatorname{Hom}_{A_{\mathfrak{p}}}(T_{\mathfrak{p}}(\phi), T_{\mathfrak{p}}(\phi'))$ *is bijective for $\mathfrak{p} \neq \mathfrak{p}_0$, where $T_{\mathfrak{p}}(\phi)$ is the $\mathfrak{p}$-adic Tate module of ϕ.*
5. *The natural map* $\operatorname{Hom}_k(\phi, \phi') \otimes A_{\mathfrak{p}_0} \to \operatorname{Hom}_{\mathrm{FM}}(\phi[\mathfrak{p}_0^\infty], \phi'[\mathfrak{p}_0^\infty])$ *is bijective, where the right-hand side is the set of homomorphisms of formal $A_{\mathfrak{p}_0}$-modules over k.*

(1)–(3) are proved in [4]. (4) and (5) are immediate consequences of (1) and (2).

Put $O_D := \mathrm{End}(\phi)$ and $D := \mathrm{End}(\phi) \otimes_A K$. It follows from Theorems 3.1 and 3.2 that D is the central division algebra over K of degree r^2 ramified exactly at ∞, v_0, with invariants $-1/r$, $1/r$, respectively, and that O_D is a maximal order of D.

Let G' be the group scheme of the multiplicative group of O_D over A. For each commutative A-algebra R, the group of R-points of G is $G'(R) = (O_D \otimes R)^\times$.

Corollary 3.3. *There is a natural bijection between $\Lambda(r, v_0)$ and the double coset space $G'(K)\backslash G'(\mathbb{A}_K^\infty)/G'(\hat{A})$, where $\mathbb{A}_K^\infty$ is the ring of finite adeles of K with respect to ∞ and $\hat{A}$ is the completion of A with respect to the ideal topology.*

This is a formulation of [4], Theorem 4.3 in adelic language. We briefly indicate the bijection. For $\phi' \in \Lambda(r, v_0)$, consider the map $\phi' \mapsto \mathrm{Hom}(\phi', \phi)$. Then as a formal consequence of Theorem 3.1, 3.2 and the fact that $T_{\mathfrak{p}}(\phi') \simeq T_{\mathfrak{p}}(\phi)$, this map induces a bijection between $\Lambda(r, v_0)$ and the set of isomorphism classes of left O_D-ideals in D.

We recall the definition of the mass of $G'(\hat{A})$. Let $\{c_1, c_2, \ldots, c_h\}$ be a (complete) set of representatives of the double coset space $G'(K)\backslash G'(\mathbb{A}_K^\infty)/G'(\hat{A})$, and let $\Gamma_{c_i} := G'(K) \cap c_i G'(\hat{A}) c_i^{-1}$. The discrete subgroup Γ_{c_i} is contained in the maximal open compact subgroup of $G'(K_\infty)$, hence is finite. Then the mass of $G'(\hat{A})$ is $\mathrm{mass}(G'(\hat{A})) := \sum_{i=1}^{h} (1/\#\Gamma_{c_i})$.

Let ϕ_c be the Drinfeld module corresponding to the double coset in $G'(K)\backslash G'(\mathbb{A}_K^\infty)/G'(\hat{A})$ represented by c, then $\mathrm{Aut}(\phi_c) \simeq \Gamma_c$, loc. cit. (also cf. [7], Lemma 2.8).

Corollary 3.4. $\mathrm{mass}(\Lambda(r, v_0)) = \mathrm{mass}(G'(\hat{A}))$.

4. Mass formula

Put $G = \mathrm{GL}_r$, $G_1 = \mathrm{SL}_r$, and G'_1 the norm one subgroup of G', viewed as group schemes over A. First we have

$$\mathrm{mass}\big(G'(\hat{A})\big) = \frac{\mathrm{vol}(G'(K)\backslash G'(\mathbb{A}_K^\infty))}{\mathrm{vol}(G'(\hat{A}))}, \tag{3}$$

for any Haar measure $\mathrm{d}g'$ on $G'(\mathbb{A}_K^\infty)$. The reduced norm gives an exact sequence $1 \to G'_1(\mathbb{A}_K) \to G'(\mathbb{A}_K) \to \mathbb{G}_{\mathrm{a}}(\mathbb{A}_K) \to 1$.

Choose a Haar measure $\mathrm{d}t$ on $\mathbb{G}_{\mathrm{a}}(\mathbb{A}_K)$, so it determines a Haar measure $\mathrm{d}g'_1$ on $G'_1(\mathbb{A}_K)$ with $\mathrm{d}g' = \mathrm{d}g'_1 \cdot \mathrm{d}t$. We have

$$\mathrm{vol}\big(G'(K)\backslash G'\big(\mathbb{A}_K^\infty\big)\big) = \mathrm{vol}\big(G'_1(K)\backslash G'_1\big(\mathbb{A}_K^\infty\big)\big) \cdot \mathrm{vol}\big(K^\times\backslash\big(\mathbb{A}_K^\infty\big)^\times\big), \tag{4}$$

$$\mathrm{vol}\big(G'(\hat{A})\big) = \mathrm{vol}\big(G'_1(\hat{A})\big) \cdot \mathrm{vol}\big(\hat{A}^\times\big). \tag{5}$$

From the exact sequence $1 \to \mathbb{F}_q^\times \to \hat{A}^\times \to K^\times\backslash(\mathbb{A}_K^\infty)^\times \to \mathrm{Pic}(A) \to 1$, one gets

$$\frac{\mathrm{vol}(K^\times\backslash(\mathbb{A}_K^\infty)^\times)}{\mathrm{vol}(\hat{A}^\times)} = \frac{\#\,\mathrm{Pic}(A)}{q-1}. \tag{6}$$

As $G'_1(K_\infty)$ is compact, $G'_1(K_\infty) = G'_1(O_\infty)$. So

$$\mathrm{mass}\big(G'(\hat{A})\big) = \frac{\#\,\mathrm{Pic}(A)}{q-1} \cdot \frac{\mathrm{vol}(G'_1(K)\backslash G'_1(\mathbb{A}_K))}{\prod_v \mathrm{vol}(G'_1(O_v))},$$

where v runs through all the places of K.

If $\omega'_{\mathbb{A}}$ is the Tamagawa measure on G'_1, we have

$$\operatorname{mass}\bigl(G'(\hat{A})\bigr) = \frac{\#\operatorname{Pic}(A)}{q-1} \cdot \omega'_{\mathbb{A}}(H')^{-1}, \quad H' = \prod_v G'_1(O_v), \tag{7}$$

because the Tamagawa number $\tau(G'_1)$ is equal to 1, cf. [6], Theorem 3.3.1.

Let ω be an invariant differential form of top degree on G_1 and ω' be the pull back of ω via an inner isomorphism $\alpha : G'_1 \to G_1$. They give rise to the Tamagawa measures $\omega_{\mathbb{A}}$ and $\omega'_{\mathbb{A}}$ on G_1 and G'_1, respectively. Then

$$\omega'_{\mathbb{A}}(H') = \frac{\omega'_{v_0}(G'_1(O_{v_0})) \cdot \omega'_{\infty}(G'_1(O_{\infty}))}{\omega_{v_0}(G_1(O_{v_0})) \cdot \omega_{\infty}(G_1(O_{\infty}))} \cdot \omega_{\mathbb{A}}(H). \tag{8}$$

Here $H = \prod_v G_1(O_v)$. It is well known that

$$\omega_{\mathbb{A}}(H)^{-1} = q^{(g-1)\dim G_1} \prod_{i=1}^{r-1} \zeta_K(1+i), \tag{9}$$

where g is the genus of the function field K. The latter equals $\prod_{i=1}^{r-1} \zeta_K(-i)$ by the functional equation. It follows from (7)–(9) that (3) is expressed as

$$\operatorname{mass}\bigl(G'(\hat{A})\bigr) = \frac{\#\operatorname{Pic}(A)}{q-1} \cdot \prod_{i=1}^{r-1} \zeta_K(-i) \cdot \lambda_{v_0}\lambda_{\infty}, \quad \lambda_v = \frac{\omega_v(G_1(O_v))}{\omega'_v(G'_1(O_v))}. \tag{10}$$

Note that when K varies, λ_v depends on K_v but not on K. When $K = \mathbb{F}_q(T)$, Gekeler's mass formula states

$$\operatorname{mass}\bigl(G'(\hat{A})\bigr) = \frac{1}{q-1} \cdot \prod_{i=1}^{r-1} \zeta_K(-i) \cdot \prod_{i=1}^{r-1} \bigl(1 - N(v_0)^i\bigr)\bigl(1 - N(\infty)^i\bigr). \tag{11}$$

Comparing (10) with (11) by varying v_0, we get

$$\lambda_v = \prod_{i=1}^{r-1} \bigl(N(v)^i - 1\bigr). \tag{12}$$

Hence we have proved

Theorem 4.1.

$$\operatorname{mass}\bigl(G'(\hat{A})\bigr) = \frac{\#\operatorname{Pic}(A)}{q-1} \cdot \prod_{i=1}^{r-1} \zeta_K^{\infty, v_0}(-i).$$

By Corollary 3.4, Theorem 2.1 is proved.

References

[1] V. Drinfeld, Elliptic modules, Math. USSR-Sb. 23 (1976) 561–592.

[2] E.-U. Gekeler, Sur les classes d'idéaux des ordres de certains corps gauches, C. R. Acad. Sci. Paris Sér. I Math. 309 (1989) 577–580.

[3] E.-U. Gekeler, Sur la géométrie de certaines algèbres de quaternions, in: Sém. Théor. Nombres Bordeaux, vol. 2, 1990, pp. 143–153.

[4] E.-U. Gekeler, On finite Drinfeld modules, J. Algebra 141 (1991) 187–203.

[5] E.-U. Gekeler, On the arithmetic of some division algebras, Comment. Math. Helv. 67 (1992) 316–333.

[6] A. Weil, Adèles and Algebraic Groups. With appendices by M. Demazure and T. Ono, in: Progr. Math., vol. 23, Birkhäuser, Boston, MA, 1982.

[7] C.-F. Yu, On the mass formula of supersingular abelian varieties with real multiplications. J. Australian Math. Soc., in press.