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Algebraic Geometry

Exceptional curves on rational surfaces having $K^2 \geqslant 0$

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Abstract

We characterize the rational surfaces X which have a finite number of (-1)-curves under the assumption that $-K_X$ is nef, where K_X is a canonical divisor on X, and has self-intersection zero. We prove also that if $-K_X$ is not nef and has self-intersection zero, then X has a finite number of (-1)-curves. ***To cite this article: M. Lahyane, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Courbes exceptionnelles sur les surfaces rationnelles ayant $K^2 \geqslant 0$. Nous caractérisons les surfaces rationnelles X qui ont un nombre fini de (-1)-courbes sous les conditions que $-K_X$ soit nef, K_X étant un diviseur canonique sur X, et que K_X^2 soit égal à zero. Nous prouvons aussi que si $-K_X$ n'est pas nef et de carré nul, alors X a un nombre fini de (-1)-courbes. ***Pour citer cet article : M. Lahyane, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Version française abrégée

On appellera ici surface toute variété analytique complexe compacte, lisse de dimension complexe deux. Les surfaces sont divisées en deux classes en fonction de l'ensemble des courbes qu'on peut tracer sur elles : il y a celles qui n'ont qu'un nombre fini de courbes, comme certaines surfaces non algébriques, et celles qui sont riches en courbes, par exemple les surfaces projectives.

On s'intéresse ici aux (-1)-courbes (i.e., les courbes rationnelles lisses de carré -1) et on se demande quels types de surfaces peuvent porter un nombre infini de (-1)-courbes. La réponse est qu'il n'y a que les surfaces rationnelles, qui peuvent avoir un nombre infini de (-1)-courbes.

Dans cette Note, on donne une caractérisation des surfaces rationnelles qui ont un nombre fini de (-1)-courbes sous les deux conditions que $-K_X$ soit numériquement effectif et de carré nul (la condition d'effectivité numérique signifie que le nombre d'intersection du diviseur K_X avec n'importe quel diviseur effectif sur X est inférieur ou

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égal à zero, K_X étant un diviseur canonique sur X ; en abrégé, on dit que $-K_X$ est nef), cf. Théorème 0.3 ci-dessous. On prouve aussi qu'une surface rationnelle qui a un diviseur anti-canonique non nef et de carré égal à zero a toujours un nombre fini de (-1)-courbes (cf. Théorème 0.1 ci-dessous). D'autre part, la finitude de l'ensemble des (-1)-courbes sur une surface rationnelle est assurée dès que la surface considérée a un diviseur canonique de carré strictement positif (un tel fait est évident). Soit X une surface projective lisse rationnelle telle que le carré de K_X soit égal à zéro, où K_X représente un diviseur canonique sur X. Si $-K_X$ n'est pas nef (i.e., il existe une courbe entière C sur X telle que le nombre d'intersection de $-K_X$ et de C soit strictement inférieur à zéro), alors on prouve le résultat suivant :

Théorème 0.1. *Soit X une surface projective lisse rationnelle ayant $-K_X$ non nef et de carré égal à zéro. Alors, X a seulement un nombre fini de (-1)-courbes.*

Une conséquence immédiate est :

Corollaire 0.2. *Soit X une surface projective lisse rationnelle ayant $-K_X$ non nef et de carré égal à zéro. Alors, le cône des courbes de X est polyédrique.*

Supposons maintenant, jusqu'à la fin de cette section, que $-K_X$ est nef (i.e., le nombre d'intersection de K_X avec n'importe quel diviseur effectif sur X est inférieur ou égal à zero, ici K_X désigne un diviseur canonique sur X) et de carré égal à zéro. Il est facile de voir que X est un éclatement en neuf points (qui peuvent être infiniment proches) du plan projectif complexe.

Nagata [10] a montré que si les 9 points sont en position générale, alors X a une infinité de (-1)-courbes.

Harbourne [2, Théorème (3.1), ou Proposition (3.2)(1)] a étudié le cas où les points se trouvent sur une cubique irréductible réduite. En particulier, il montre que X a un nombre fini de (-1)-courbes si et seulement si l'ensemble des (-2)-courbes sur X engendre un sous-module maximal du groupe de Néron–Severi $NS(X)$ de X. Ici, une (-2)-courbe sur X est une courbe rationnelle lisse de carré égal à -2.

Persson et Miranda [9] ont étudié le cas où la position des 9 points donne une surface elliptique rationnelle munie d'une section. Ils ont classifié toutes les surfaces de ce type qui ont un nombre fini de (-1)-courbes et les ont appelés *surfaces rationnelles elliptiques jacobiennes extrêmales*. Pour chaque cas, ils ont donné le nombre de (-1)-courbes qu'on peut y trouver.

On a été informé que Harbourne et Miranda [5] ont étudié le cas où la surface elliptique rationnelle n'est pas une jacobienne. Particulièrement, ils ont déterminé le nombre des (-1)-courbes par le biais des fonctions génératrices.

Nous utiliserons les notations suivantes : $\sim$ désignera l'équivalence linéaire des diviseurs sur X ;

$[D]$ l'ensemble des diviseurs D' sur X tels que $D' \sim D$;

$\mathrm{Div}(X)$ le groupe des diviseurs sur X ;

$NS(X)$ le groupe quotient $\frac{\mathrm{Div}(X)}{\sim}$ de $\mathrm{Div}(X)$ par $\sim$ (les équivalences linéaire, numérique et algébrique sont les mêmes sur $\mathrm{Div}(X)$ du fait que la surface X est rationnelle) ;

$D \cdot D'$ est le nombre d'intersection du diviseur D avec le diviseur D', en particulier le carré de D est $D^2 = D \cdot D$;

$\overline{D}$ l'élément associé au diviseur D dans le produit tensoriel du groupe $NS(X)$ avec le corps des nombres rationnels au dessus de l'anneau des entiers.

Suivant [9], nous définirons une surface projective rationnelle lisse ayant un nombre fini de (-1)-courbes comme étant *une surface rationnelle extrêmale*. Notre résultat principal donne une classification des surfaces rationnelles extrêmales :

Théorème 0.3. *Soit X une surface projective lisse rationnelle telle que $-K_X$ soit nef et de carré égal à zéro. Alors on a équivalence entre* :

(1) *X est extrêmale.*
(2) *X satisfait les deux conditions ci-dessous* :

(a) *le rang de la matrice* $(C_i \cdot C_j)_{i,j=1,\ldots,r}$ *est égal à* 8*, où* $\{C_i;\ i = 1,\ldots,r\}$ *est l'ensemble fini des* (-2)*-courbes sur* X [*une* (-2)*-courbe est une courbe rationnelle lisse de carré* -2] ;
(b) *Il existe* r *nombres rationnels strictement positifs* a_i*,* $i = 1,\ldots,r$*, tels que* $-\overline{K}_X = \sum_{i=1}^{i=r} a_i \overline{C}_i$*.*

Comme application, on a le résultat suivant :

Corollaire 0.4. *Soit* X *comme dans le Théorème* 0.3 *et qui vérifie l'une des deux conditions équivalentes. Alors, le cône des courbes de* X *est polyédrique.*

Notre démonstration des Théorèmes 0.1 et 0.3 repose sur les Propositions 2.2, 2.4, 3.2, 3.3, 4.1 et 4.2 ci-dessous.

1. Introduction

By a surface we mean here a compact complex analytic manifold of complex dimension two. We are interested by the problem of knowing which surfaces have a finite number of curves, and we can distinguish two classes: those which have only a finite number of curves, e.g., some non-algebraic surfaces; and those which are rich in curves, e.g., the projective ones. We restrict ourselves to curves which are smooth, rational and of self-intersection -1 (such curves are called (-1)-curves) and ask which surfaces may have an infinite number of (-1)-curves. The answer is that only the rational surfaces may have an infinite number of (-1)-curves. In this Note, we give a characterization of rational surfaces which have a finite number of (-1)-curves under the assumption that an anti-canonical divisor of the surface is nef and of self-intersection zero (cf. Theorem 1.3 below). Also, we prove that a rational surface having an anti-canonical divisor not nef and of self-intersection zero has always a finite number of (-1)-curves (cf. Theorem 1.1 below). On the other hand, it is straightforward that a rational surface has a finite number of (-1)-curves if the self-intersection of its canonical divisor is greater than zero. Let X be a smooth projective rational surface such that the self-intersection of K_X is zero, where K_X is a canonical divisor on X. If $-K_X$ is not nef (i.e., there exists an integral curve C on X such that the intersection number of $-K_X$ with C is less than zero), then we prove the following:

Theorem 1.1. *Let* X *be a smooth projective rational surface having* $-K_X$ *of self-intersection zero, but not nef. Then* X *has only a finite number of* (-1)*-curves on it.*

An immediate consequence is:

Corollary 1.2. *Let* X *be a smooth projective rational surface having* $-K_X$ *of self-intersection zero, but not nef. Then the cone of curves of* X *is polyhedral.*

Assume now, until the end of this section, that $-K_X$ is nef (i.e., the intersection number of K_X with any effective divisor on X is less than or equal to zero, where K_X is a canonical divisor on X) and of self-intersection zero. It is easy to see that X is a blow-up of the projective plane at 9 points (possibly infinitely close).

Nagata [10] proved that if the 9 points are in general position, then X has an infinite number of (-1)-curves.

Harbourne [2, Theorem (3.1), or Proposition (3.2)(1)] studied the case when the points are on a reduced irreducible cubic. In particular, he proved that X has a finite number of (-1)-curves if and only if the set of all (-2)-curves on X span a maximal submodule of the Néron–Severi group $NS(X)$ of X. Here a (-2)-curve on X is a smooth rational curve of self-intersection -2.

Persson and Miranda [9] studied the case when the position of the 9 points gives a rational elliptic surface with a section. They classified all such surfaces which have a finite number of (-1)-curves and called them *extremal jacobian elliptic rational surfaces*. For each case, they gave the number of (-1)-curves.

We were informed that Harbourne and Miranda have studied the non-Jacobian case in [5]. Among other things, they computed the number of (-1)-curves by means of generating functions.

We will use the following notations: $\sim$ the linear equivalence of divisors on X;

$[D]$ the set of divisors D' on X such that $D' \sim D$;

$\mathrm{Div}(X)$ the group of divisors on X;

$NS(X)$ the group quotient $\frac{\mathrm{Div}(X)}{\sim}$ of $\mathrm{Div}(X)$ by $\sim$ (the linear, algebraic and numerical equivalences are the same on $\mathrm{Div}(X)$ since X is a rational surface);

$D \cdot D'$ will denote the intersection number of the divisor D with the divisor D', in particular the self-intersection of D is $D^2 = D \cdot D$;

$\overline{D}$ the associated element to the divisor D in the tensor product of the group $NS(X)$ with the field of rational numbers over the ring of integers.

Following [9], we will call a smooth rational projective surface having a finite number of (-1)-curves on it *an extremal rational surface*. Our main result gives a classification of extremal rational surfaces:

Theorem 1.3. *Let X be a smooth projective rational surface having $-K_X$ nef and of self-intersection zero. Then the following are equivalent*:

(1) *X is extremal.*
(2) *X satisfies the two conditions below*:
   (a) *the rank of the matrix $(C_i \cdot C_j)_{i,j=1,\dots,r}$ is equal to 8, where $\{C_i;\ i = 1,\dots,r\}$ is the finite set of (-2)-curves on X [a (-2)-curve is a smooth rational curve of self-intersection -2];*
   (b) *There exist r strictly positive rational numbers a_i, $i = 1,\dots,r$, such that $-\overline{K}_X = \sum_{i=1}^{i=r} a_i \overline{C}_i$.*

As an application, the following result holds:

Corollary 1.4. *Let X be as in Theorem 1.3 satisfying one of the two equivalent assertions. Then the cone of curves of X is polyhedral.*

2. Preliminary results

Let X be a smooth projective rational surface such that $K_X^2 = 0$, where K_X is a canonical divisor of X. We will assume that $-K_X$ is nef, i.e., $K_X \cdot D \leqslant 0$ for every effective divisor D on X. The standard notions, notation and results come from [1,3,4,6].

Definition 2.1. A (-2)-curve on X is a smooth rational curve of self-intersection -2.

It is easy to see that the set of (-2)-curves on X is finite, this follows immediately from:

Proposition 2.2. *If the (-2)-curves on X are linearly dependent over the field of rational numbers, then the surface X is an elliptic surface.*

Definition 2.3. A (-1)-divisor D on X is a divisor D on X such that $D^2 = -1 = D \cdot K_X$.

The following proposition characterizes all (-1)-divisors which are irreducible and reduced:

Proposition 2.4. *Let X be as above. A (-1)-divisor D on X is a (-1)-curve if and only if $D \cdot C_i \geqslant 0$ for each $i = 1,\dots,r$, where $\{C_i;\ i = 1,\dots,r\}$ is the finite set of (-2)-curves on X.*

3. Sketch of the Proof of Theorem 1.1

Recall that X is a smooth rational surface having a vanishing self-intersection of its canonical divisor. The fact that $-K_X$ is not nef is equivalent to the existence of a smooth rational curve of self-intersection $-n$, where n is larger than or equal to 3. See for instance [6]. We will prove that if X has a $(-n)$-curve, $n \geqslant 3$, then the number of (-1)-curves on X is finite. To this end, it is sufficient to consider the following two cases:

(i) X is a blowing-up at 8 points, possibly infinitely near, of $\mathbb{F}_e$, $\mathbb{F}_e = \mathbb{P}_{\mathbb{P}^1}(O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(e))$ is the Hirzebruch surface associated to the natural integer e, $e \geqslant 3$;
(ii) X is a blowing-up at nine points, possibly infinitely near, of the projective plane $\mathbb{P}^2$.

3.1. Case of a blow-up of a Hirzebruch surface

Using the following lemma:

Lemma 3.1. *Let X be the surface obtained by blowing up $\mathbb{F}_e$ at 8 points, possibly infinitely near, e is a natural integer, $e \geqslant 3$. Let S_1 be a section linearly equivalent to $S_0 + eF$, where S_0 (resp. F) is the minimal section (resp. a fibre) of $\mathbb{F}_e$ (S_0 is the section of self-intersection $-e$). Then the class $\mathcal{S}_1$ of the total transform of S_1 (resp. of F) is nef on X.*

Our result is:

Proposition 3.2. *Let X be a smooth rational surface with $K_X^2 = 0$. If X is a blowing-up of $\mathbb{F}_e$, $e \geqslant 3$, then the number of (-1)-curves on X is finite.*

3.2. Case of a blowing-up of the projective plane

In this case, one may get the following:

Proposition 3.3. *Let X be a smooth rational surface with $K_X^2 = 0$. If X is a blowing-up of the projective plane $\mathbb{P}^2$ and if X has a $(-n)$-curve $(n \geqslant 3)$, then the number of (-1)-curve on X is finite.*

4. Sketch of the proof of Theorem 1.3

Let s be the rank of the matrix $(C_i \cdot C_j)_{i,j=1,\ldots,r}$, where $\{C_i;\ i = 1, \ldots, r\}$ is the finite set of (-2)-curves on X.
If s is less than 8, we can find a divisor D such that the following conditions hold:

1. $D^2 < 0$,
2. $D \cdot K_X = 0$,
3. $D \cdot C_i = 0$, for each $i = 1, \ldots, r$.

Then fix a (-1)-curve E_0, and for each integer n, consider the (-1)-divisor E_n defined as follows:

$$E_n = E_0 + nD - \left(nE_0 \cdot D + \frac{n^2}{2} D^2\right)(-K_X).$$

By Proposition 2.4, E_n is a (-1)-curve. Two distinct integers n and m give two distinct (-1)-curves E_n and E_m; so X has an infinite number of (-1)-curves.

If s is equal to eight, then we distinguish between two cases:

First case: If the vector space over the field of rational numbers spanned by the set of (-2)-curves is not equal to the orthogonal complement of the canonical divisor, then according to the next Proposition 4.1, X will have an infinite number of (-1)-curves.

Proposition 4.1. *If the intersection form is negative definite on the space spanned by $\overline{C_i}$, where the C_i constitute a connected component of the set of all (-2)-curves on X, then the number of (-1)-curves on X is infinite.*

Second case: If the vector space over the field of rational numbers spanned by the set of (-2)-curves is equal to the orthogonal complement of the canonical divisor, then we distinguish between the cases where X is elliptic or not:

First subcase: If X is not elliptic, then according to Proposition 2.2, the (-2)-curves are linearly independent over the field of rational numbers; so the number of (-2)-curves on X is 9. Two cases arise: if the set of (-2)-curves is connected, then $-\overline{K}_X$ is a linear combination of strictly positive rational numbers of the family of (-2)-curves.

If the set of (-2)-curves is not connected, then $-\overline{K}_X$ is not a linear combination of strictly positive rational numbers of the family of (-2)-curves. An application of Proposition 4.1 to any connected component of the set of (-2)-curves for which the intersection form is negative definite gives the fact that X contains an infinite number of (-1)-curves. This case might not occur.

Second subcase: If X is elliptic, then $-\overline{K}_X$ is a linear combination of strictly positive rational numbers of all (-2)-curves. To finish the proof, we need the following proposition:

Proposition 4.2. *Assume that X satisfies the following two conditions*:

(a) *the rank of the matrix $(C_i \cdot C_j)_{i,j=1,\ldots,r}$ is equal to 8, where $\{C_i;\ i=1,\ldots,r\}$ is the finite set of (-2)-curves on X,*

(b) *there exist r strictly positive rational numbers a_i, $i=1,\ldots,r$, such that $-\overline{K}_X = \sum_{i=1}^{i=r} a_i \overline{C}_i$.*

Then X is extremal.

All the results of this Note are proved in detail in [7] and [8].

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