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Probability Theory

A new approach to Kolmogorov equations in infinite dimensions and applications to stochastic generalized Burgers equations

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Abstract

We develop a new method to uniquely solve a large class of heat equations, so called Kolmogorov equations in infinitely many variables. The equations are analyzed in spaces of sequentially weakly continuous functions weighted by proper (Lyapunov type) functions. In this way, for the first time, the solutions are constructed everywhere without exceptional sets for equations with possibly non-locally Lipschitz drifts. Apart from general analytic interest, the main motivation is to apply this to uniquely solve martingale problems in the sense of Stroock–Varadhan given by stochastic partial differential equations from hydrodynamics, such as the stochastic Navier–Stokes equations. In this Note this is done in the case of the stochastic generalized Burgers equation. Uniqueness is shown in the sense of Markov flows. ***To cite this article: M. Röckner, Z. Sobol, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Une nouvelle approche aux équations de Kolmogorov en dimension infinie et des applications à l'équation stochastique de Burgers. Nous proposons une nouvelle méthode de résoudre une large classe d'équations de chaleur, c'est-à-dire, d'équations de Kolmogorov en dimension infinie. Nous considèrons ces équations dans les espaces des fonctions faiblement séquentiellement continues et subordonnées aux fonctions du type de Liapounoff appropriées. Nos résultats donnent la première construction d'une solution qui existe partout dans le cas de coefficients non lipschitziens. Ces études sont motivées par des applications aux problèmes de martingales au sens de Stroock–Varadhan pour les équations stochastiques aux dérivées partielles de l'hydrodynamique du type de Navier–Stokes. En particulier, l'équation stochastique de Burgers est analysée. L'unicité est établie au sens des flots markoviens. ***Pour citer cet article : M. Röckner, Z. Sobol, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

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On considére sur $X := L^2((0,1), \mathrm{d}r)$ où $\mathrm{d}r$ est la mesure de Lebesgue l'équation différentielle stochastique

$$\begin{aligned} &\mathrm{d}x_t(r) = \big(\Delta_r x_t(r) + \nabla_r \Psi\big(x_t(r)\big) + \Phi\big(r, x_t(r)\big)\big)\,\mathrm{d}t + \sqrt{A}\,\mathrm{d}w_t(r), \\ &x_0 = x \in X, \end{aligned} \tag{1}$$

où Δ_r dénote le Laplacien avec conditions au bord de Dirichlet sur $(0,1)$, $(w_t)_{t\geqslant 0}$ est un mouvement Brownien cylindrique sur X et A, Ψ et Φ vérifient les conditions suivantes :

(A) A est un opérateur linéaire symmétrique, positif et à trace de X dans X tel que $A_N := P_N A P_N$ soit un opérateur inversible représenté par une matrice diagonale dans E_N pour toute valeur $N \in \mathbb{N}$, E_N étant l'espace engendré par les N premières valeurs propres de Δ et P_N le projecteur orthogonal de X sur E_N. On pose

$$a_0 := \sup_{x \in H_0^1} \frac{(x, Ax)}{|x'|_2^2}.$$

(Ψ) $\Psi \in C^{1,1}(\mathbb{R})$ (Ψ est différentiable avec une dérivée localement Lipschitzienne) et il existe $C \in [0,\infty)$ et une fonction Borélienne et bornée $\omega : [0,\infty) \to [0,\infty)$ tendant vers zéro à l'infini telle que

$$|\Psi_{xx}|(x) \leqslant C + \sqrt{|x|}\,\omega\big(|x|\big) \quad \text{pour } \mathrm{d}x \text{ p.p. } x \in \mathbb{R}.$$

($\Phi 1$) Φ est Borel mesurable dans la première variable et continue dans la seconde et il existe $g \in L^{q_1}(0,1)$ avec $q_1 \in [2,\infty]$ et $q_2 \in [1,\infty)$ tels que

$$\big|\Phi(r,x)\big| \leqslant g(r)\big(1 + |x|^{q_2}\big) \quad \text{pour tout } r \in (0,1),\ x \in \mathbb{R}.$$

($\Phi 2$) Il existe $h_0, h_1 \in L^1_+(0,1)$, $|h_1|_1 < 2$, tel que pour presque tout $r \in (0,1)$

$$\Phi(r,x)\,\mathrm{sign}\,x \leqslant h_0(r) + h_1(r)|x| \quad \text{pour tout } x \in \mathbb{R}.$$

($\Phi 3$) Il existe $\rho_0 \in (0,1]$, $g_0 \in L^1_+(0,1)$, $g_1 \in L^{p_1}_+(0,1)$ avec $p_1 \in [2,\infty]$ et une fonction $\omega : [0,\infty) \to [0,\infty)$ vérifiant (Ψ) telle que, en posant $\sigma(r,x) := \frac{|x|}{\sqrt{r(1-r)}}$ ou a

$$\Phi(r,y) - \Phi(r,x) \leqslant \big[g_0(r) + g_1(r)\big|\sigma(r,x)\big|^{2-1/p_1}\omega\big(\sigma(r,x)\big)\big](y-x)$$

pour tout $x, y \in \mathbb{R}$, $0 \leqslant y - x \leqslant \rho_0$ et presque tout $r \in (0,1)$.

L'operatéur de Kolmogorov L dans (1) est défini par

$$\begin{aligned} &Lu(x) := \frac{1}{2}\mathrm{Tr}\big(AD^2u(x)\big) + \left(\Delta x + \frac{\mathrm{d}}{\mathrm{d}r}\Psi(x) + \Phi(\cdot,x), Du(x)\right), \\ &u \in \mathcal{D} := \big\{u = g \circ P_N \mid N \in \mathbb{N}, g \in C_b^2(E_N)\big\}. \end{aligned}$$

Notre objectif est d'abord de résoudre l'équation de Kolmogorov (1) c'est à dire le probléme de Cauchy en dimension infinie $u_t = Lu_t$ et obtenir les probabilités de transition d'un processus de Markov. Apres quoi nous montrons que ce processus de Markov admet des trajectoires faiblement continues et obtenons une solution de (1) au sens du problème de martingales de Stroock–Varadhan.

Théorème 0.1 (Resultat principal). *Supposons* (A), (Ψ) *et* ($\Phi 1$)–($\Phi 3$) *vérifiées. Soit* $\kappa_0 := \frac{2-|h_1|_1}{8a_0}$ (*avec* a_0 *comme dans* (A) *et* h_1 *comme dans* ($\Phi 2$)) *et* $p \in [2,\infty) \cap (q_2 - 3 + \frac{2}{q_1}, \infty)$ (*avec* q_1, q_2 *comme dans* ($\Phi 1$)). *Posons* $X_p := L^p((0,1), \mathrm{d}r)$ *et, pour* $\kappa > 0$, $\Theta_{p,\kappa}(x) := \mathrm{e}^{\kappa|x|_2^2}(1 + |x|_p^p)(1 + |x'|_2^2)$, $x \in X$.

(a) *Il existe un et un seul semigroupe* $(p_t)_{t>0}$ *de noyaux de probabilité sur* X_p *qui vérifie les conditions suivantes.*
   (i) *Pour tout* $u \in \mathcal{D}$ *et* $x \in X_p$*, la fonction* $t \mapsto p_t(|Lu|)(x)$ *est localement Lebesgue intégrable sur* $[0,\infty)$ *et* $p_t u(x) - u(x) = \int_0^t p_s(Lu)(x)\,\mathrm{d}s$.
   (ii) *Il existe* $\kappa \in (0,\kappa_0)$ *et* $\lambda > 0$ *tels que* $\int_0^\infty \mathrm{e}^{-\lambda s} p_s(\Theta_{p,\kappa})(x)\,\mathrm{d}s$ *soit fini pour tout* $x \in X_p$.
(b) *Il existe un et un seul processus de Markov conservatif* $\mathbb{M} := (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, (x_t)_{t\geqslant 0}, (\mathbb{P}_x)_{x\in X_p})$ *dans* X_p *tel que*
   (i) $\mathbb{M}$ *est solution du probléme de martingales pour* $(L,\mathcal{D})$ *c'st a dire que pour tout* $u \in \mathcal{D}$ *et tout* $x \in X_p$ *la fonction* $t \mapsto |Lu(x_t)|$ *est localement Lebesgue integrable sur* $[0,\infty)$, $\mathbb{P}_x$ *presque sûrement et* $m_t := u(x_t) - u(x) - \int_0^t Lu(x_s)\,\mathrm{d}s$, $t \geqslant 0$, *est une* $((\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P}_x)$ *martingale partant de* 0 *(cf.* [11]).
   (iii) *Il existe* $\kappa \in (0,\kappa_0)$ *et* $\lambda > 0$ *tels que* $\mathbb{E}_x[\int_0^\infty \mathrm{e}^{-\lambda_\kappa s}(\Theta_{p,\kappa})(x_s)\,\mathrm{d}s]$ *soit fini pour tout* $x \in X_p$.
(c) $\mathbb{M}$ *est un processus de Markov au sens fort à trajectoires continues pour la topologie faible de* X_p*, tel que* $(p_t)_{t\geqslant 0}$ *est son semigroupe associé.*
(d) *Pour* $t > 0$ *et* $u \in \mathcal{D}$*, la fonction* $p_t u$ *peut se prolonger de manière univoque en une fonction continue localement Lipschitzienne sur* X*, que l'on désignera encore par* $p_t f$*, telle que* $p_{s+t}u(x) \to p_s u(x)$ *quand* $t \to 0$ *pour tout* $s \in [0,\infty)$, $x \in X_p$ *et* $u \in \mathcal{D}$.

1. Introduction

Consider the following stochastic partial differential equation on $X := L^2(0,1) = L^2((0,1),\mathrm{d}r)$ (where $\mathrm{d}r$ denotes the Lebesgue measure)

$$\begin{aligned} &\mathrm{d}x_t(r) = \big(\Delta_r x_t(r) + \nabla_r \Psi\big(x_t(r)\big) + \Phi\big(r, x_t(r)\big)\big)\,\mathrm{d}t + \sqrt{A}\,\mathrm{d}w_t(r),\\ &x_0 = x \in X. \end{aligned} \tag{2}$$

Here Δ_r denotes the Dirichlet Laplacian on $(0,1)$, $(w_t)_{t\geqslant 0}$ is a cylindrical Brownian motion on X, $A : X \to X$ is a positive symmetric linear operator of trace class and $\Psi : \mathbb{R} \to \mathbb{R}$ and $\Phi : (0,1)\times\mathbb{R} \to \mathbb{R}$ are real-valued functions. This type of equation contains classical stochastic reaction diffusion equations as well as the well-studied stochastic Burgers equations (see, e.g., [3]) as special cases and are therefore called generalized stochastic Burgers equations.

The Kolmogorov operator L for (2) is defined as follows.

$$\begin{aligned} &Lu(x) := \frac{1}{2}\mathrm{Tr}\big(AD^2u(x)\big) + \big(\Delta x + \nabla\Psi(x) + \Phi(\cdot,x), Du(x)\big),\\ &u \in \mathcal{D} := \big\{u = g \circ P_N \mid N \in \mathbb{N},\ g \in C_b^2(E_N)\big\}. \end{aligned} \tag{3}$$

Here D and D^2 denote the first and second Frechèt derivatives, respectively, E_N is the space spanned by first N eigenfunctions of Δ and $P_N : X \to E_N$ is the orthogonal projection, $N \in \mathbb{N}$.

Our aim is to solve the Kolmogorov equations corresponding to (2), i.e., solve the partial differential equation in infinitely many variables given by $u_t = Lu$, gaining the transition probabilities of a Markov process. Then we show that this Markov process has weakly continuous sample paths and solves (2) in the sense of Stroock–Varadhan's martingale problem [11].

To this end we develop a new technique: Instead of analyzing the equation $u_t = Lu$ in spaces of bounded continuous functions or $L^p(\mu)$-spaces where μ is an infinitesimally invariant (or excessive) measure of L (as was done by many authors in previous papers, cf. [1,2,4–7,10]) we consider this equation in weighted spaces WC_V of weakly continuous functions on X. WC_V is equipped with the weighted norm $\|f\|_V := \sup_{\{V<\infty\}} V^{-1}|f|$, where

V is a Lyapunov function for L. To keep the maximum principle for L, we define WC_V similarly to the space $C_\infty(\mathbb{R}^d)$:

$$WC_V := \Big\{ f\colon \{V<\infty\} \to \mathbb{R} \mid f \text{ is continuous on each } \{V \leqslant R\},\ R\in\mathbb{R}, \\ \text{in the weak topology inherited from } X, \text{ and } \lim_{R\to\infty} \sup_{\{V\geqslant R\}} V^{-1}|f| = 0 \Big\}.$$

In this Note we consider V of the form $V_{p,\kappa}(x) := \mathrm{e}^{\kappa|x|_2^2}(1+|x|_p^p)$, $x\in X$, $p\geqslant 2$, $\kappa>0$. Obviously, $X_p := \{V_{p,\kappa}<\infty\}$ is nothing but $L^p((0,1),\mathrm{d}r)$ for all $\kappa>0$. Functions $\Theta_{p,\kappa}(x) := V_{p,\kappa}(x)(1+|x'|_2^2)$, $x\in X$, $p\geqslant 2$, $\kappa>0$, are of great importance in the construction because of the inequality $(\lambda_{p,\kappa}-L)V_{p,\kappa} \geqslant m_{p,\kappa}\Theta_{p,\kappa}$ for some $\lambda_{p,\kappa}, m_{p,\kappa}>0$.

To measure local Lipschitz continuity, we introduce a family of semi-norms $(\cdot)_{p,\kappa}$, $p\geqslant 2$, $\kappa>0$,

$$(f)_{p,\kappa} := \sup_{y_1,y_2\in X_p} \big(V_{p,\kappa}(y_1)\vee V_{p,\kappa}(y_2)\big)^{-1} \frac{|f(y_1)-f(y_2)|}{|y_1-y_2|_2} \quad (\leqslant\infty).$$

2. Hypotheses

Now we present our hypotheses on the coefficients in SPDE (2), respectively, the Kolmogorov operator (3), and the main result.

(A) A is a nonnegative symmetric linear operator from X to X of trace class such that $A_N := P_N A P_N$ is an invertible operator represented by a diagonal matix on E_N for all $N\in\mathbb{N}$. We set

$$a_0 := \sup_{x\in H_0^1} \frac{(x,Ax)}{|x'|_2^2},$$

where H_0^1 denotes the Sobolev space of order 1 in $L^2((0,1),\mathrm{d}r)$ with Dirichlet boundary conditions.

(Ψ) $\Psi\in C^{1,1}(\mathbb{R})$ (i.e., Ψ is differentiable with locally Lipschitz derivative) and there exist $C\in[0,\infty)$ and a bounded, Borel-measurable function $\omega\colon[0,\infty)\to[0,\infty)$ vanishing at infinity such that

$$|\Psi_{xx}|(x) \leqslant C + \sqrt{|x|}\,\omega\big(|x|\big) \quad \text{for } \mathrm{d}x\text{-a.e. } x\in\mathbb{R}.$$

$(\Phi 1)$ Φ is Borel-measurable in the first and continuous in the second variable and there exist $g\in L^{q_1}(0,1)$ with $q_1\in[2,\infty]$ and $q_2\in[1,\infty)$ such that

$$\big|\Phi(r,x)\big| \leqslant g(r)\big(1+|x|^{q_2}\big) \quad \text{for all } r\in(0,1),\ x\in\mathbb{R}.$$

$(\Phi 2)$ There exist $h_0,h_1\in L_+^1(0,1)$, $|h_1|_1<2$, such that for a.e. $r\in(0,1)$

$$\Phi(r,x)\operatorname{sign} x \leqslant h_0(r)+h_1(r)|x| \quad \text{for all } x\in\mathbb{R}.$$

$(\Phi 3)$ There exist $\rho_0\in(0,1]$, $g_0\in L_+^1(0,1)$, $g_1\in L_+^{p_1}(0,1)$ for some $p_1\in[2,\infty]$, and a function $\omega\colon[0,\infty)\to[0,\infty)$ as in (Ψ) such that, with $\sigma(r,x) := |x|/\sqrt{r(1-r)}$,

$$\Phi(r,y)-\Phi(r,x) \leqslant \Big[g_0(r)+g_1(r)\big|\sigma(r,x)\big|^{2-1/p_1}\omega\big(\sigma(r,x)\big)\Big](y-x)$$

for all $x,y\in\mathbb{R}$, $0\leqslant y-x\leqslant\rho_0$, a.a. $r\in(0,1)$.

3. Main result

Theorem 3.1. *Suppose* (A), (Ψ) *and* $(\Phi 1)$–$(\Phi 3)$ *hold. Let* $\kappa_0 := \frac{2-|h_1|_1}{8a_0}$ (*with* a_0 *as in* (A) *and* h_1 *as in* $(\Phi 2)$), *and let* $p \in [2,\infty) \cap (q_2 - 3 + \frac{2}{q_1}, \infty)$ (*with* q_1, q_2 *as in* $(\Phi 1)$).

(a) *There exists a unique semigroup* $(p_t)_{t>0}$ *of probability kernels on* X_p, *satisfying the following.*
  (i) *For all* $u \in \mathcal{D}$ *and* $x \in X_p$, *the function* $t \mapsto p_t(|Lu|)(x)$ *is locally Lebesgue integrable on* $[0,\infty)$ *and* $p_t u(x) - u(x) = \int_0^t p_s(Lu)(x)\,\mathrm{d}s$.
  (ii) *There exists* $\kappa \in (0,\kappa_0)$ *and* $\lambda > 0$ *such that* $\int_0^\infty \mathrm{e}^{-\lambda s} p_s(\Theta_{p,\kappa})(x)\,\mathrm{d}s$ *is finite for all* $x \in X_p$.

(b) *There exists a unique conservative Markov process* $\mathbb{M} := (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, (x_t)_{t\geqslant 0}, (\mathbb{P}_x)_{x\in X_p})$ *in* X_p *satisfying the following.*
  (i) $\mathbb{M}$ *solves the martingale problem for* $(L,\mathcal{D})$, *i.e., for all* $u \in \mathcal{D}$ *and all* $x \in X_p$ *the function* $t \mapsto |Lu(x_t)|$ *is locally Lebesgue integrable on* $[0,\infty)$ $\mathbb{P}_x$*-a.s. and* $m_t := u(x_t) - u(x) - \int_0^t Lu(x_s)\,\mathrm{d}s$, $t \geqslant 0$, *is an* $((\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P}_x)$*-martingale starting at* 0 (*cf.* [11]).
  (ii) *There exists* $\kappa \in (0,\kappa_0)$ *and* $\lambda > 0$ *such that* $\mathbb{E}_x[\int_0^\infty \mathrm{e}^{-\lambda_\kappa s}(\Theta_{p,\kappa})(x_s)\,\mathrm{d}s]$ *is finite for all* $x \in X_p$.

(c) $\mathbb{M}$ *is a strong Markov process with trajectories continuous in the weak topology of* X_p, *whose transition probability semigroup is given by* $(p_t)_{t>0}$.

(d) *For* $t \geqslant 0$, $x \in X_p$ *and* $u \in \mathcal{D}$ *let* $P_t u(x) := p_t u(x)$. *Then for all* $\kappa \in (0,\kappa_0)$, P_t *is extended to a* C_0*-semigroup of quasi-contractions on* $WC_{p,\kappa}$. *In particular,* $p_{s+t}u(x) \to p_s u(x)$ *as* $t \to 0$ *for all* $s \in [0,\infty)$, $x \in X_p$ *and* $u \in \mathcal{D}$.

(e) *For* $t > 0$ *and* $u \in \mathcal{D}$, *the function* $p_t u$ *uniquely extends to a locally Lipschitz continuous function on* X, *again denoted by* $p_t u$. *For* $q \geqslant 2$ *and* $\kappa \in (0,\kappa_0)$ *there exists* $\lambda_{q,\kappa} > 0$ (*independent of* t *and* u) *such that*

$$\|p_t u\|_{q,\kappa} \leqslant \mathrm{e}^{\lambda_{q,\kappa} t}\|u\|_{q,\kappa} \quad \text{and} \quad (p_t u)_{q,\kappa} \leqslant \mathrm{e}^{\lambda_{q,\kappa} t}(u)_{q,\kappa}.$$

Complete proofs are contained in [9].

Finally, we emphasize that our approach is quite general and also works for other SPDE, e.g., with the underlying one-dimensional domain $(0,1)$ replaced by an open set in $\mathbb{R}^d$ (see [8] for the case of the 2D stochastic Navier–Stokes equation).

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