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Absolute Lipschitz extendability

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Abstract

A metric space X is said to be *absolutely Lipschitz extendable* if every Lipschitz function f from X into *any* Banach space Z can be extended to *any* containing space $Y \supseteq X$, where the loss in the Lipschitz constant in the extension is independent of Y, Z, and f. We show that various classes of natural metric spaces are absolutely Lipschitz extendable. ***To cite this article: J.R. Lee, A. Naor, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Sur la propriété d'extension lipschitzienne absolue. On dit qu'un espace métrique X a la propriété d'*extension lipschitzienne absolue* si pour tout espace de Banach Z, toute fonction lipschitzienne f de X dans Z peut être étendue à *tout* espace métrique Y contenant X, avec une perte dans la constante de Lipschitz de l'extension qui ne dépend pas du choix de Y, Z et f. Nous montrons que plusieurs classes naturelles d'espaces métriques ont la propriété d'extension lipschitzienne absolue. ***Pour citer cet article : J.R. Lee, A. Naor, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Let (Y, d_Y), (Z, d_Z) be metric spaces, and for every $X \subseteq Y$, denote by $e(X, Y, Z)$ the infimum over all constants K such that every Lipschitz function $f : X \to Z$ can be extended to a function $\tilde{f} : Y \to Z$ satisfying $\|\tilde{f}\|_{\mathrm{Lip}} \leqslant K\|f\|_{\mathrm{Lip}}$. (If no such K exists, we set $e(X, Y, Z) = \infty$.) We also define $e(Y, Z) = \sup\{e(X, Y, Z)\colon X \subseteq Y\}$ and for every integer n, $e_n(Y, Z) = \sup\{e(X, Y, Z)\colon X \subseteq Y, |X| \leqslant n\}$.

Estimating $e(Y, Z)$ is a classical and fundamental problem that has attracted a lot of attention due to its intrinsic interest and applications to geometry and approximation theory. It is a classical fact that for every metric space Y, $e(Y, \ell_\infty) = 1$, and Kirszbraun's famous extension theorem [8] states that whenever H_1 and H_2 are Hilbert spaces, $e(H_1, H_2) = 1$. We refer to the books [2,16] for a detailed account of the case $e(Y, Z) = 1$ and list below three results which deal with the case $e(Y, Z) > 1$, when the target space Z is a Banach space. In what follows, C is a universal constant.

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T1. (Johnson, Lindenstrauss and Schechtman [7]) For every metric space Y and every Banach space Z, $e_n(Y, Z) \leqslant C \log n$.
T2. (Johnson, Lindenstrauss and Schechtman [7]) For every d-dimensional normed space Y and every Banach space Z, $e(Y, Z) \leqslant Cd$.
T3. (Matoušek [12]) For every metric tree T and every Banach space Z, $e(T, Z) \leqslant C$.

In this Note we observe a new phenomenon underlying these theorems which we refer to as *absolute extendability*—the notion that for some spaces X, Lipschitz functions f from X into *any* Banach space Z can be extended to *any* containing space $Y \supseteq X$, where the loss in the Lipschitz constant is independent of Y, Z, and f, and thus depends only on X. To this end, let us define, for a metric space X, the *absolute extendability constant* $ae(X)$ by

$$ae(X) = \sup\bigl\{e(X, Y, Z):\ Y \supseteq X, Z \text{ a Banach space}\bigr\}.$$

If $ae(X) < \infty$, we say that X is *absolutely extendable*. Additionally, for a family of metric spaces $\mathcal{M}$, let us define $ae(\mathcal{M}) = \sup_{X\in\mathcal{M}} ae(X)$ to be a uniform bound on the extendability of metrics in $\mathcal{M}$. As far as we are aware, the only previously known families of absolutely extendable metrics had such a property for a "trivial" reason; these are the cases when X is an absolute Lipschitz retract or when the family $\mathcal{M}$ consists of finite metrics of uniformly bounded cardinality (it is not too difficult to see that (T1) is true when $\log n$ is replaced by n).

In order to state our results, let us introduce some notation. Let $G = (V, E)$ be a countable graph with edge lengths in $[0, \infty]$. Denote by $\Sigma(G)$ the one-dimensional simplicial complex that arises from G by replacing every edge e of G by an interval whose length is equal to that of e. We now define the set of metrics supported on G, denoted $\langle G\rangle$, as the set of all subsets of $\Sigma(G)$ for all possible non-negative lengths on edges of G. For a family of graphs $\mathcal{F}$, let $\langle\mathcal{F}\rangle = \bigcup_{G\in\mathcal{F}}\langle G\rangle$. Finally, recall that the *doubling constant* of a metric space X, denoted $\lambda(X)$, is the infimum over all numbers λ such that every ball in X can be covered by λ balls of half the radius. When $\lambda(X) < \infty$, one says that X is *doubling*.

Theorem 1. *The following extension results hold true*:

(1) *For a family of finite graphs $\mathcal{F}$, $ae(\langle\mathcal{F}\rangle) < \infty$* if and only if *$\langle\mathcal{F}\rangle$ does not contain all finite metrics.*
(2) *If M is a two-dimensional Riemannian manifold of genus g, then for every $X \subseteq M$, $ae(X) \leqslant Cg$.*
(3) *For every metric space X, we have $ae(X) \leqslant C \log \lambda(X)$.*
(4) *For every n-point metric space X, $ae(X) \leqslant C \frac{\log n}{\log\log n}$.*

Observe that (2) implies that for every planar graph G, $ae(\langle G\rangle) \leqslant C$, which improves (T3). Additionally since any n-point metric space is isometrically embeddable in a compact two-dimensional Riemannian manifold of genus $\mathrm{O}(n^3)$, in (2) above $ae(M)$ must tend to infinity with the genus of M. Since $\log\lambda(X) = \mathrm{O}(\log n)$ for any n-point metric space X, and $\log\lambda(X) = \mathrm{O}(d)$ whenever X is a subset of some d-dimensional normed space, (3) unifies and generalizes (T1) and (T2). Finally, it is clear that (4) improves on (T1) by a factor of $\log\log n$.

In what follows we will sketch the main steps in the proof of Theorem 1. In particular, in the ensuing arguments we will ignore all measurability assumptions. We refer to our upcoming paper [10] for detailed proofs and additional results.

Let (Y, d) be a metric space and X a subspace of Y. For the purpose of proving extension results, we may assume that X is closed. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and fix $K > 0$. We shall say that a function $\Psi : \Omega \times Y \to [0, \infty)$ is a *K-gentle partition of unity with respect to X* if for every $x \in Y \setminus X$ $\int_\Omega \Psi(\omega, x)\,\mathrm{d}\mu(\omega) = 1$, for every $\omega \in \Omega$ and $x \in X$, $\Psi(\omega, x) = 0$, and there exists a mapping $\gamma : \Omega \to X$ such that for every $x, y \in Y$,

$$\int_\Omega d\bigl(\gamma(\omega), x\bigr) \cdot \bigl|\Psi(\omega, x) - \Psi(\omega, y)\bigr|\,\mathrm{d}\mu(\omega) \leqslant K d(x, y).$$

Let Z be a Banach space, and $f : X \to Z$ a Lipschitz function. We extend f to a function $\tilde{f} : Y \to Z$ by defining for $x \in Y \setminus X$, $\tilde{f}(x) = \int_\Omega f(\gamma(\omega))\Psi(\omega, x)\,\mathrm{d}\mu(\omega)$. It is not difficult to check that the K-gentle condition ensures that $\|\tilde{f}\|_{\mathrm{Lip}} \leqslant 3K\|f\|_{\mathrm{Lip}}$. All the statements in Theorem 1 actually produce K-gentle partitions of unity for the appropriate value of K.

Stochastic decomposition. We construct gentle partitions of unity by first producing an appropriate distribution over partitions of Y. We say that $(\Omega, \Pr, \{\Gamma^i(\cdot), \gamma^i(\cdot)\}_{i\in I})$ is a *stochastic decomposition of Y with respect to X* if I is some index set, $(\Omega, \Pr)$ is a probability space, for every $\omega \in \Omega$, $\{\Gamma^i(\omega)\}_{i\in I}$ is a partition of Y and for every $i \in I$, $\gamma^i : \Omega \to X$ is a function such that for all $\omega \in \Omega$, $d(\gamma^i(\omega), \Gamma^i(\omega)) \leqslant 2d(X, \Gamma^i(\omega))$. For $\Delta > 0$ the decomposition is said to be Δ-bounded if for every $\omega \in \Omega$ and $i \in I$, $\mathrm{diam}(\Gamma^i(\omega)) \leqslant \Delta$. A Δ-bounded decomposition is called (ε, δ)-padded if for every $x \in Y$ such that $d(x, X) < \varepsilon\Delta$, $\Pr(\exists i \in I \text{ s.t. } d(x, X \setminus \Gamma^i(\omega)) \geqslant \varepsilon\Delta) \geqslant \delta$.

Since we are interested in bounding the absolute extendability constant of a metric space X, we need to impose intrinsic geometric restrictions on X which ensure that *every* super-space $Y \supseteq X$ admits an appropriate stochastic decomposition with respect to X. This is a achieved via the following partition extension lemma.

Lemma 2 (Partition extension). *Let (Y, d) be a metric space and X a closed subspace of Y. If X admits an (ε, δ)-padded Δ-bounded stochastic decomposition* (*with respect to itself*)*, then Y admits an $(\frac{\varepsilon}{16}, \delta)$-padded $(1 + \frac{\varepsilon}{2})\Delta$-bounded stochastic decomposition with respect to X.*

To prove Lemma 2 we argue as follows. Let $\{(\Gamma^i(\cdot), \gamma^i(\cdot), \Omega, \mu)\}_{i\in I}$ be an (ε, δ)-padded Δ-bounded stochastic decomposition of X with respect to itself. For every point $x \in Y$, let $t_x \in X$ be such that $d(x, t_x) \leqslant 2d(x, X)$. Now, for every $\omega \in \Omega$ and $i \in I$, consider the set

$$\widehat{\Gamma}^i(\omega) = \Gamma^i(\omega) \cup \big\{x \in Y \colon\ t_x \in \Gamma^i(\omega) \text{ and } d(x, t_x) \leqslant \varepsilon\Delta/4\big\}.$$

Finally, for any point $x \in Y \setminus \bigcup_{i\in I} \widehat{\Gamma}^i(\omega)$, place x in a singleton set $\{x\}$. It is not difficult to check that this yields the required decomposition of Y with respect to X.

We pass from padded decompositions to gentle partitions of unity as follows. Let X be a closed subset of Y such that for every $n \in \mathbb{Z}$, Y admits an (ε, δ)-padded 2^n-bounded stochastic decomposition with respect to X. We claim that then Y also admits a $\frac{C}{\varepsilon\delta}$-gentle partition of unity with respect to X.

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be any 2-Lipschitz map with $\mathrm{supp}(\varphi) \subset [\frac{1}{2}, 4]$ and $\varphi \equiv 1$ on $[1, 2]$. Additionally, let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be such that $g \equiv 0$ on $[0, 1]$, $g \equiv 1$ on $[2, \infty)$ and $g(x) = x - 1$ on $[1, 2]$. For every $n \in \mathbb{Z}$ let $(\Omega_n, \Pr_n, \{\Gamma_n^i(\cdot), \gamma_n^i(\cdot)\}_{i\in I})$ be a 2^n-bounded stochastic decomposition of Y with respect to X, and denote by (Ω, μ) be the disjoint union of $\{I \times \Omega_n\}_{n\in\mathbb{Z}}$ (where the measure on I is the counting measure). For every $x \in Y$ and $\omega \in \Omega_n$ there is a unique $i \in I$ for which $x \in \Gamma_n^i(\omega)$, and we denote $\pi_\omega^n(x) = d(x, Y \setminus \Gamma_n^i(\omega))$. For every $n \in \mathbb{Z}$, $\omega \in \Omega_n$, $i \in I$ and $x \in Y$ set:

$$\Psi(i, \omega, x) = \frac{1}{S(x)} \cdot g\left(\frac{\pi_\omega^n(x)}{\varepsilon 2^{n-1}}\right) \cdot \varphi\left(\frac{d(x, X)}{\varepsilon 2^{n-3}}\right) \cdot \mathbf{1}_{\Gamma_n^i(\omega)}(x) \quad \text{and} \quad \gamma(i, \omega) = \gamma_n^i(\omega), \tag{1}$$

where $S(x)$ is a normalization factor ensuring that $\int_\Omega \Psi(i, \omega, x)\,\mathrm{d}\mu(i, \omega) = 1$. It is possible to show that this construction yields the required gentle partition of unity; we refer to [10] for the details.

The notion of a padded decomposition is motivated by recent advances in combinatorics and theoretical computer science. Often in theoretical computer science, one needs to analyze data with an inherent metric structure, and constructing well behaved stochastic decompositions has proved itself to be extremely useful in various algorithmic applications. Variants of this approach have appeared in numerous contexts; see for instance [11,9,1]. The structural results of [9,14] imply that for every integer r, if G is a graph which does not admit the complete graph K_r as a minor, then for some constant $c_r > 0$, any $X \in \langle G\rangle$ admits a Δ-bounded $(c_r, 1/2)$-padded stochastic decomposition with respect to itself (for every $\Delta > 0$). Part (2) of Theorem 1 follows from this decomposition and a characterization (in terms of excluded minors) of graphs that can be realized on a surface of bounded genus with no edge-crossings (see, e.g., [13]).

Additionally, for a family of graphs $\mathcal{F}$, let $mc(\mathcal{F})$ denote its closure under taking minors, i.e., the maximal minor-closed family containing $\mathcal{F}$. A deep theorem of Robertson and Seymour [15] states that if $mc(\mathcal{F})$ does not contain *all* finite graphs, then there exists $r \in \mathbb{N}$ such that all the graphs in $\mathcal{F}$ exclude a K_r minor. Since contraction/deletion of an edge corresponds to weighting by $0/\infty$, respectively, it follows that $\langle \mathcal{F} \rangle = \langle mc(\mathcal{F}) \rangle$. Thus if $\langle \mathcal{F} \rangle$ does not contain all finite metrics, then certainly $mc(\mathcal{F})$ does not contain all finite graphs. From the above reasoning, it follows that $ae(\langle \mathcal{F} \rangle) < \infty$, proving part (1) of Theorem 1. Part (3) of Theorem 1 is based on a variant of a construction from [6] showing that if (X, d) is doubling then it admits a Δ-bounded $(c/\log \lambda(X), 1/2)$-padded stochastic decomposition with respect to itself, for each $\Delta > 0$. Finally, part (4) of Theorem 1 follows from the decomposition of [3] and the improved analysis of [4,5]. We refer to the upcoming paper [10] for a detailed account of these constructions and variants thereof, as well for additional extension theorems based on different notions of stochastic metric decomposition. In particular, in [10] we discuss results analogous to Theorem 1 in which the target space is not a Banach space, e.g., CAT(0) and related spaces.

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