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Bounded influence estimators for multivariate lognormal distributions

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Abstract

In this paper we consider the problem of robust estimation of some parameters related to a multivariate lognormal distribution. In this sense, we construct a class of estimators and discuss some of its properties, such as Fisher consistency, robustness and asymptotic normality. ***To cite this article: A. Toma, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Estimateurs à fonction d'influence bornée pour des lois lognormales multivariées. Dans cet article, nous considérons le problème de l'estimation robuste de certains paramètres relatifs à une distribution multivariée lognormale. Dans ce but, nous construisons une classe d'estimateurs et donnons certaines de leurs propriétés telles que la consistence au sens de Fisher, la robustesse et la normalité asymptotique. ***Pour citer cet article : A. Toma, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Version française abrégée

Nous considérons $Y = (Y^1, \ldots, Y^p)^t$ un vecteur aléatoire de dimension p avec une loi lognormale de paramètres μ et V. Notons $Y \sim \Lambda_p(\mu, V)$. Alors $X = (X^1, \ldots, X^p)^t = (\ln Y^1, \ldots, \ln Y^p)^t$ est un vecteur gaussien de dimension p de moyenne μ et de matrice de covariance V. La génèse et quelques propriétés d'une distribution multivariée lognormale sont données dans [6]. Iwase, Shimizu et Suzuki (cf. [5]), ont considéré les paramètres

$$\theta_{\alpha,B} = \mathrm{e}^{\alpha^t \mu + \mathrm{tr}\, BV}, \tag{1}$$

où α est un vecteur réel arbitraire à p dimensions et B est une matrice réelle d'ordre p. Le paramètre $\theta_{\alpha,B}$ peut exprimer plusieurs paramètres de $\Lambda_p(\mu, V)$, par exemple des moments produit ou le mode multivarié. Puisque les estimateurs classiques sont très sensibles à la présence d'outliers, des alternatives robustes doivent être cherchées. Le but de cet article est de construire des estimateurs robustes des paramètres $\theta_{\alpha,B}$.

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A partir de données lognormales de dimension p nous définissons des estimateurs pour les paramètres $\theta_{\alpha,B}$, en procédant de la façon suivante. On commence par appliquer une log-transformation aux données, puis on calcule des estimateurs robustes et consistents au sens de Fisher pour les données gaussiennes obtenues et enfin on transforme ces estimateurs pour la loi lognormale en utilisant la relation entre les paramètres des deux lois. Pour chaque choix d'estimateurs robustes consistents au sens de Fisher dans le cas normal multivarié à p dimensions, un autre estimateur robuste et consistent au sens de Fisher est obtenu. Dans le cas normal à p dimensions, plusieurs propositions robustes concernant le vecteur moyen et la matrice de covariance ont été faites, telles que les M-estimateurs (cf. [8]) et les S-estimateurs (cf. [2]).

Nous démontrons ici la consistence des estimateurs introduits pour la loi lognormale multivariée. Nous obtenons la consistence au sens de Fisher de ces estimateurs et nous en déduisons les fonctions d'influence correspondantes, qui apparaissent bornées. Nous considérons aussi le cas particulier où les estimateurs pour les paramètres de la distribution normale à p dimensions sont affines équivariants. Nous développons deux exemples, à savoir lorsque $\theta_{\alpha,B}$ exprime respectivement les moments produit et le mode bivarié. Aussi nous donnons un résultat concernant le point de rupture de nos estimateurs. Enfin, en supposant la normalité asymptotique des estimateurs dans le cas normal multivarié de dimension p et leur indépendence asymptotique, nous obtenons la normalité asymptotique de nos estimateurs.

1. Introduction and notations

We consider $Y = (Y^1, \ldots, Y^p)^t$ a p-dimensional lognormal random vector with parameters μ and V and denote $Y \sim \Lambda_p(\mu, V)$. Then $X = (X^1, \ldots, X^p)^t = (\ln Y^1, \ldots, \ln Y^p)^t$ has the p-variate normal distribution with the mean vector μ and the covariance matrix V. A genesis and some properties of the multivariate lognormal distribution, were discussed in [6]. Iwase, Shimizu and Suzuki (see [5]), treated the parameter

$$\theta_{\alpha,B} = \mathrm{e}^{\alpha^t \mu + \operatorname{tr} BV}, \tag{2}$$

where α is an arbitrary p-dimensional real vector and B is an arbitrary real matrix of order p. The parameter (2) can express some parameters of $\Lambda_p(\mu, V)$, for example the product moments or the multivariate mode. Since the classical estimators are very sensitive to the presence of outliers, robust alternatives need to be looked for. The aim of this paper is to construct robust estimators for the parameter $\theta_{\alpha,B}$. Note that for mean and covariance matrix of multivariate lognormal distribution, B-robust estimators were introduced in [11]. Here we extend those results for this general parameter $\theta_{\alpha,B}$ and also give a result regarding the breakdown properties of the proposed estimators.

In the following we consider those estimators $T_n(X_1, X_2, \ldots, X_n)$ with the property that there exists a functional T defined on a convex set D_T of distributions and valued in the parameters space Θ, such that, if $X_1, X_2, \ldots, X_n$ are i.i.d. random vectors with the distribution $G = G_\theta \in D_T$, then $T_n(X_1, X_2, \ldots, X_n) \xrightarrow{P} T(G)$. The notation G is the same for a probability distribution and its corresponding cumulative distribution function.

The functional T is called Fisher consistent if $T(G_\theta) = \theta$, for all $\theta \in \Theta$.

The influence function of the functional T in G measures the effect on T of adding a small mass at x and is defined as

$$IF(x; T, G) = \lim_{\varepsilon \to 0} \frac{T(\widetilde{G}_{\varepsilon x}) - T(G)}{\varepsilon}, \tag{3}$$

where $\widetilde{G}_{\varepsilon x} = (1 - \varepsilon)G + \varepsilon \delta_x$ and δ_x is the Dirac distribution.

The gross error sensitivity measures aproximately, the maximum contribution to the estimation error that can be produced by a single outlier and is defined as $\sup_x \|IF(x; T, G)\|$. Whenever the gross error sensitivity is finite, the estimator associated with the functional T is called B-robust (for details see [4]).

The robustness of an estimator could be also measured by means of the finite sample breakdown point (see [3]). The breakdown point of an estimator T_n at a collection $X=(X_1,\dots,X_n)$ is defined as the smallest fraction $\frac{m}{n}$ of outliers that can take the estimate over all bounds:

$$\varepsilon_n^*(T_n,X)=\min\left\{\frac{m}{n}\colon \sup_{\widetilde{X}_m}\left\|T_n(X)-T_n\big(\widetilde{X}_m\big)\right\|=\infty\right\}, \tag{4}$$

where the supremum is taken over all possible corrupted collections $\widetilde{X}_m$ obtained from X by replacing m points by arbitrary values.

We will use the Euclidean norm for p-vectors and the norm $\|A\|=(\sum_{i,j=1}^n|a_{ij}|^2)^{1/2}$ for matrices.

Throughout the paper, F denotes the p-variate normal distribution $N_p(\mu,V)$ where $\mu\in\mathbb{R}^p$ and V is a $p\times p$ symmetric positive definite matrix and F' denotes the p-variate lognormal distribution $\Lambda_p(\mu,V)$. We will consider those estimators t_n and C_n of μ and V, respectively, which are B-robust and Fisher consistent and will denote by t and C the corresponding statistical functionals. As a particular case we will consider the one when t_n and C_n are affine equivariant meaning that $t(AX+b)=At(X)+b$ and $C(AX+b)=AC(X)A^t$, for any $b\in\mathbb{R}^p$ and any $p\times p$ nonsingular matrix A (here the notation $T(Z)$ instead of $T(G)$ means that $Z\sim G$).

2. The results

In the following, for any $p\times p$ symmetric matrix $A=(a_{ij})$, let $\operatorname{uvec}A$ denote the $p(p+1)/2$ dimensional column vector $(a_{11},a_{22},\dots,a_{pp},a_{12},a_{13},\dots,a_{p-1,p})^t$ formed from the elements in the upper triangular half of A, including the diagonal elements. Let $f_{\alpha,B}$ be the function defined on $\mathbb{R}^{p+p(p+1)/2}$ and $\mathbb{R}$ valued,

$$f_{\alpha,B}\begin{pmatrix}x\\ \operatorname{uvec}X\end{pmatrix}=\mathrm{e}^{\alpha^t x+\operatorname{tr}BX}, \tag{5}$$

where $x=(x_1,\dots,x_p)^t$ and $X=(x_{ij})$ is a $p\times p$ symmetric matrix.

Let $Y_1,\dots,Y_n$ be a sample drawn from F'. We define $\theta_{\alpha,B,n}(Y_1,\dots,Y_n)$ the estimator of $\theta_{\alpha,B}$, by

$$\theta_{\alpha,B,n}(Y_1,\dots,Y_n)=f_{\alpha,B}\begin{pmatrix}t_n(X_1,\dots,X_n)\\ \operatorname{uvec}C_n(X_1,\dots,X_n)\end{pmatrix}, \tag{6}$$

where $X_k^l=\ln(Y_k^l)$ for all $k=1,\dots,n$ and $l=1,\dots,p$, X_k^l,Y_k^l being the components of the random vectors X_k and Y_k, respectively.

The problem of robust estimation in the case of the multivariate normal distribution has been widely studied. In this sense an overview of some existing estimators of multivariate location and covariance can be found in [9] (see also [2,8] and [10]).

Theorem 2.1. *Assume that the random vectors $Y_1,\dots,Y_n$ are i.i.d. from the distribution F'. Then $\theta_{\alpha,B,n}\xrightarrow{P}\theta_{\alpha,B}(F')$ where*

$$\theta_{\alpha,B}(F')=f_{\alpha,B}\begin{pmatrix}t(F'\circ u)\\ \operatorname{uvec}C(F'\circ u)\end{pmatrix}=\mathrm{e}^{\alpha^t t(F'\circ u)+\operatorname{tr}BC(F'\circ u)}, \tag{7}$$

u is the function defined on $\mathbb{R}^p$ and $(\mathbb{R}_+^*)^p$ valued, $u(x_1,\dots,x_p)=(\mathrm{e}^{x_1},\dots,\mathrm{e}^{x_p})$ and $(F'\circ u)(x_1,\dots,x_n)=F'(u(x_1,\dots,x_n))$.

Proof. From the consistency of the estimators t_n and C_n we obtain the consistency of $((t_n)^t,(\operatorname{uvec}C_n)^t)^t$ and then the continuity of the function $f_{\alpha,B}$ will imply the consistency of $\theta_{\alpha,B,n}$. The asymptotical value is obtained noting that $F=F'\circ u$. $\square$

We observe the fact that the statistical functional corresponding to the estimator $\theta_{\alpha,B,n}$ is

$$\theta_{\alpha,B}(G) = f_{\alpha,B}\begin{pmatrix} t(G\circ u) \\ \operatorname{uvec} C\,(G\circ u)\end{pmatrix} = \mathrm{e}^{\alpha^t t(G\circ u)+\operatorname{tr} BC(G\circ u)}. \tag{8}$$

Corollary 2.2. *The functional $\theta_{\alpha,B}(\cdot)$ is Fisher consistent for the parameter $\theta_{\alpha,B}$ at the distribution F', that is $\theta_{\alpha,B}(F') = \theta_{\alpha,B}$.*

Proof. The Fisher consistency of the functionals t and C imply the Fisher consistency of the functional $\theta_{\alpha,B}(\cdot)$ by using the relation (2). □

In the following, for every $x \in \mathbb{R}^p$ and A a $p \times p$ symmetric positive definite matrix, $\|x\|_A^2$ is the notation for $x^t Ax$. We recall that for any affine equivariant location estimator with corresponding functional t possesing an influence function, there exists a function γ_t defined on $[0,\infty[$ and $\mathbb{R}$ valued such that

$$IF(z;t,F) = \gamma_t\big(\|z-\mu\|^2_{V^{-1}}\big)(z-\mu), \tag{9}$$

where F is the distribution $N_p(\mu, V)$ (see [11]).

In Lemma 1 from [1], we see that for any affine equivariant scatter estimator C_n with corresponding functional C possesing an influence function, there exist two functions α_C and β_C defined on $[0,\infty[$ and $\mathbb{R}$ valued such that

$$IF(z;C,F) = \alpha_C\big(\|z-\mu\|^2_{V^{-1}}\big)(z-\mu)(z-\mu)^t - \beta_C\big(\|z-\mu\|^2_{V^{-1}}\big)V, \tag{10}$$

where F is the distribution $N_p(\mu, V)$.

Theorem 2.3. *The influence function of the functional $\theta_{\alpha,B}(\cdot)$ at the distribution F' is given by*

$$IF(x;\theta_{\alpha,B},F') = \theta_{\alpha,B}\big[\alpha^t IF(z;t,F) + \operatorname{tr}\big\{BIF(z;C,F)\big\}\big], \tag{11}$$

where $z = (z_1,\dots,z_p)^t$, $z_i = \ln x_i$ for all $i = 1,\dots,p$.

Proof. We first note that $\widetilde{F}'_{\varepsilon x}\circ u = \widetilde{F}_{\varepsilon z}$, where $z = (z_1,\dots,z_p)$, $z_i = \ln x_i$ for all $i = 1,\dots,p$ and u is the function from Theorem 2.1. Using the definition of the influence function and the Fisher consistency of the functionals, we get

$$\begin{aligned}
IF(x;\theta_{\alpha,B},F') &= \lim_{\varepsilon\to 0}\frac{\theta_{\alpha,B}(\widetilde{F}'_{\varepsilon x}) - \theta_{\alpha,B}(F')}{\varepsilon}\\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[f_{\alpha,B}\begin{pmatrix} t(\widetilde{F}_{\varepsilon z}) \\ \operatorname{uvec} C(\widetilde{F}_{\varepsilon z})\end{pmatrix} - f_{\alpha,B}\begin{pmatrix} t(F) \\ \operatorname{uvec} C(F)\end{pmatrix}\right]\\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\big[\mathrm{e}^{\sum_{i=1}^p \alpha_i t(\widetilde{F}_{\varepsilon z})_i + \sum_{i,j=1}^p b_{ij} C(\widetilde{F}_{\varepsilon z})_{ij}} - \mathrm{e}^{\sum_{i=1}^p \alpha_i t(F)_i + \sum_{i,j=1}^p b_{ij} C(F)_{ij}}\big]\\
&= \lim_{\varepsilon\to 0}\frac{\theta_{\alpha,B}}{\varepsilon}\big[\mathrm{e}^{\sum_{i=1}^p \alpha_i [t(\widetilde{F}_{\varepsilon z})_i - t(F)_i] + \sum_{i,j=1}^p b_{ij}[C(\widetilde{F}_{\varepsilon z})_{ij} - C(F)_{ij}]} - 1\big]\\
&= \theta_{\alpha,B}\left[\sum_{i=1}^p \alpha_i IF(z;t,F)_i + \sum_{i,j=1}^p b_{ij} IF(z;C,F)_{ij}\right]\\
&= \theta_{\alpha,B}\big[\alpha^t IF(z;t,F) + \operatorname{tr}\big\{BIF(z;C,F)\big\}\big]. \qquad \square
\end{aligned}$$

Using the B-robustness of the estimators t_n and C_n, Theorem 2.3 leads to the following result:

Corollary 2.4. *The estimator $\theta_{\alpha,B,n}$ is B-robust.*

Substituting (9) and (10) in expression (11) from Theorem 2.3 yields the following result:

Corollary 2.5. *Let t_n and C_n be affine equivariant estimators of μ and V, respectively. Then*

$$IF(x;\theta_{\alpha,B},F') = \theta_{\alpha,B}\big[\gamma_t\big(\|z-\mu\|^2_{V^{-1}}\big)\alpha^t(z-\mu) \tag{12}$$

$$+\,\alpha_C\big(\|z-\mu\|^2_{V^{-1}}\big)\operatorname{tr}\big\{B(z-\mu)(z-\mu)^t\big\} - \beta_C\big(\|z-\mu\|^2_{V^{-1}}\big)\operatorname{tr}\{BV\}\big], \tag{13}$$

where $z=(z_1,\dots,z_p)^t$, $z_i=\ln x_i$ for all $i=1,\dots,p$.

Remark 1. Note that for $p=2$, the mode of the bivariate distribution $\Lambda_2(\mu,V)$, with $\mu=(\mu_1,\mu_2)^t$ and $V=(v_{ij})$ a symmetric positive definite matrix of order 2, is given by $(m_1,m_2)^t = (\mathrm{e}^{\mu_1-v_{11}-v_{12}},\mathrm{e}^{\mu_2-v_{22}-v_{12}})^t$. Because of the symmetry of the subscripts, we treat here only m_1 which can be expressed by (2) setting $\alpha=(1,0)^t$ and

$$B = -\begin{pmatrix} 1 & 1/2 \\ 1/2 & 0 \end{pmatrix}. \tag{14}$$

In this case we can rewrite (11) as $m_1[IF(z;t,F)_1 - IF(z;C,F)_{11} - IF(z;C,F)_{12}]$.

Remark 2. Let $s=(s_1,\dots,s_p)^t$ be an arbitrary p-dimensional real vector and r be an arbitrary real number. We consider the parameter $\theta(s_1,\dots,s_p;r) = E[\prod_{i=1}^p (Y^i)^{s_i}]^r$. We have

$$E\left[\prod_{i=1}^p \big(Y^i\big)^{s_i}\right]^r = E\big[\mathrm{e}^{s^tX}\big]^r = \mathrm{e}^{rs^t\mu + rs^tVs/2} = \mathrm{e}^{(rs^t)\mu + \operatorname{tr}\{Vrss^t/2\}}. \tag{15}$$

Setting $\alpha = rs$ and $B = rss^t/2$ in (6) we obtain a B-robust estimator of $\theta(s_1,\dots,s_p;r)$. By Theorem 2.3, the influence function of the corresponding functional is

$$\theta(s_1,\dots,s_p;r)\left[rs^tIF(z;t,F) + \frac{r}{2}\operatorname{tr}\big\{ss^tIF(z;C,F)\big\}\right]. \tag{16}$$

Theorem 2.6. *Let $Y=(Y_1,\dots,Y_n)$ be a sample of p-variate obsevations each of them having the components positive numbers. Then*

$$\varepsilon_n^*(\theta_{\alpha,B,n},Y) \geqslant \min\big\{\varepsilon_n^*(t_n,X),\varepsilon_n^*(C_n,X)\big\}, \tag{17}$$

where $X=(X_1,\dots,X_n)$ with $X_1,\dots,X_n$ obtained by transformations as in the definition of the introduced estimators.

Proof. Let $g_{\alpha,B}$ be the function defined on $\mathbb{R}^p\times \mathrm{SPD}(p)$ and $\mathbb{R}$ valued, $g_{\alpha,B}(a,A) = \mathrm{e}^{\alpha^ta+\operatorname{tr}BA}$, $\mathrm{SPD}(p)$ being the set of all $p\times p$ symmetric and positive definite matrices. On $\mathbb{R}^p\times\mathrm{SPD}(p)$ we consider the norm $\|(a,A)\| = \max\{\|a\|,\|A\|\}$. Let Y and X be as above. The estimator $\theta_{\alpha,B,n}(Y)=\theta_{\alpha,B,n}(Y_1,\dots,Y_n)$ could be written as $g_{\alpha,B}(t_n(X),C_n(X))$. Replace at most $m=\min\{\varepsilon_n^*(t_n,X),\varepsilon_n^*(C_n,X)\}-1$ points of X and denote $\widetilde{X}_m$ the new corrupted collection. Because $\frac{m}{n}<\varepsilon_n^*(t_n,X)$ and $\frac{m}{n}<\varepsilon_n^*(C_n,X)$, we obtain $\sup_{\widetilde{X}_m}\|t_n(X)-t_n(\widetilde{X}_m)\|<\infty$ and $\sup_{\widetilde{X}_m}\|C_n(X)-C_n(\widetilde{X}_m)\|<\infty$. It follows that there exists a constant k which only depends on X, such that $\|t_n(X)-t_n(\widetilde{X}_m)\|<k$ and $\|C_n(X)-C_n(\widetilde{X}_m)\|<k$ for all $\widetilde{X}_m$. The function $g_{\alpha,B}$ is Lipschitzian and therefore there exists $K>0$ such that

$$\big\|\theta_{\alpha,B,n}(Y) - \theta_{\alpha,B,n}\big(\widetilde{Y}_m\big)\big\| = \big\|g_{\alpha,B}\big(t_n(X),C_n(X)\big) - g_{\alpha,B}\big(t_n\big(\widetilde{X}_m\big),C_n\big(\widetilde{X}_m\big)\big)\big\| < Kk, \tag{18}$$

for all $\widetilde{Y}_m$. We deduce that $\sup_{\widetilde{Y}_m}\|\theta_{\alpha,B,n}(Y)-\theta_{\alpha,B,n}(\widetilde{Y}_m)\|$ is finite and obtain the announced inequality. □

Theorem 2.7. *Suppose that t_n and C_n are asymptotically independent, $\sqrt{n}(t_n - \mu)$ has a limiting multinormal distribution with zero mean and asymptotic covariance matrix $ASV(t, F)$ and $\sqrt{n}\,\mathrm{uvec}(C_n - V)$ has a limiting multinormal distribution with zero mean and asymptotic covariance matrix $ASV(\mathrm{uvec}\,C, F)$. Then $\sqrt{n}(\theta_{\alpha,B,n} - \theta_{\alpha,B})$ has a limiting normal distribution with zero mean and asymptotic variance*

$$ASV(\theta_{\alpha,B}, F') = \theta^2_{\alpha,B}\big[\alpha^t ASV(t, F)\alpha + (\mathrm{uvec}\,B)^t ASV(\mathrm{uvec}\,C, F)(\mathrm{uvec}\,B)\big]. \tag{19}$$

Proof. By the hypothesis conditions we obtain that $(t_n^t, (\mathrm{uvec}\,C_n)^t)^t$ is asymptotically normal with mean $(\mu^t, (\mathrm{uvec}\,V)^t)^t$ and covariance matrix

$$\begin{pmatrix} ASV(t, F) & 0 \\ 0 & ASV(\mathrm{uvec}\,C, F) \end{pmatrix}. \tag{20}$$

Now we apply the well known delta method (see [7]) to $(t_n^t, (\mathrm{uvec}\,C_n)^t)^t$ and find that $\theta_{\alpha,B,n}$ is asymptotically normal with the mean $\theta_{\alpha,B}$ and variance

$$ASV(\theta_{\alpha,B}, F') = \theta^2_{\alpha,B}\big[\alpha^t ASV(t, F)\alpha + (\mathrm{uvec}\,B)^t ASV(\mathrm{uvec}\,C, F)(\mathrm{uvec}\,B)\big]. \qquad \square$$

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References

[1] C. Croux, G. Haesbroeck, Principal component analysis based on robust estimators of the covariance or correlation matrix: influence functions and efficiencies, Biometrika 87 (2000) 603–618.
[2] L. Davies, Asymptotic behavior of S-estimators of multivariate location parameters and dispersion matrices, Ann. Statist. 15 (1987) 1269–1292.
[3] D.L. Donoho, P.J. Huber, The notion of breakdown point, in: P.J. Bickel, K.A. Doksum, J.L. Hodges (Eds.), A Festschrift for Erich L. Lechmann, Wadsworth, Belmont, CA, 1983.
[4] F.R. Hampel, E.M. Ronchetti, P.J. Rousseuw, W.A. Stahel, Robust Statistics: The Approach Based on Influence Function, Wiley, 1986.
[5] K. Iwase, K. Shimizu, M. Suzuki, On UMVU estimators for the multivariate lognormal distribution and their variances, Comm. Statist. Theory Methods 11 (1982) 687–697.
[6] R.M. Jones, K.S. Miller, On the multivariate lognormal distribution, J. Industrial Math. Soc. 16 (1966) 63–76.
[7] K.V. Mardia, J.T. Kent, J.M. Bibby, Multivariate Analysis, Academic Press, 1994.
[8] R.A. Maronna, Robust M-estimators of multivariate location and scatter, Ann. Statist. 4 (1976) 51–67.
[9] R.A. Maronna, V. Yohai, Robust estimation of multivariate location and scatter, in: S. Kots, C. Read, D. Banks (Eds.), Encyclopedia of Statistical Sciencies, Wiley, 1998.
[10] P.J. Rousseeuw, Multivariate estimation with high breakdown point, in: W. Grossmann, G. Pflug, I. Vincze, W. Wertz (Eds.), Math. Statist. Appl., vol. B, Reidel, Dodrecht, 1985, pp. 283–297.
[11] A. Toma, Robust estimators for the parameters of multivariate lognormal distributions, Comm. Statist. Theory Methods 32 (2003) 1405–1417.