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Densities of some Poisson T-martingales and random covering numbers

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Abstract

The asymptotic behavior of the logarithm of the density of some **T**-martingales (in the sense of Kahane theory (Chinese Ann. Math. Ser. B 8 (1) (1987) 1–12)) is described in detail even in absence of statistical self-similarity. Poisson intensities of the form Lebesgue $\otimes\,\mu$ on $\mathbb{R}\times\mathbb{R}_+^*$ are involved in the construction of these martingales. We prove that there are three possible behaviors according to the fact that $\bar\alpha=\limsup_{\varepsilon\to 0}(-\log\varepsilon)^{-1}\int_{[\varepsilon,1)}\ell\,\mathrm{d}\mu(\ell)$ is zero, positive and finite, or infinite. This problem is closely related to the asymptotic behaviors of covering numbers in Poisson covering of the line and Dvoretzky covering of the circle. ***To cite this article: J. Barral, A. Fan, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***
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Résumé

Densités de certaines T-martingales poissonniennes et nombres de recouvrements aléatoires. Le comportement asymptotique du logarithme de la densité de certaines **T**-martingales (au sens de la théorie de Kahane (Chinese Ann. Math. Ser. B 8 (1) (1987) 1–12)) est décrit de façon précise même en l'absence d'auto-similarité en loi. La construction de ces martingales fait intervenir des intensités de Poisson de la forme Lebesgue $\otimes\,\mu$ sur $\mathbb{R}\times\mathbb{R}_+^*$. Nous montrons qu'il y a trois comportements possibles selon que $\bar\alpha=\limsup_{\varepsilon\to 0}(-\log\varepsilon)^{-1}\int_{[\varepsilon,1)}\ell\,\mathrm{d}\mu(\ell)$ est nul, strictement positif et fini ou infini. Cette question est intimement liée au comportements asymptotiques des nombres de recouvrements dans le recouvrement de Poisson pour la droite et le recouvrement de Dvoretzky pour le cercle. ***Pour citer cet article : J. Barral, A. Fan, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***
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1. Constructions of T-martingales

Let **T** be a locally compact metric space and $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space. Let $\big(\{Q_\varepsilon\},\{\mathcal{F}_\varepsilon\}\big)_{0<\varepsilon\leqslant 1}$ be a **T**-martingale in the sense that the filtration $\{\mathcal{F}_\varepsilon\}_{0<\varepsilon\leqslant 1}$ is nonincreasing, $Q_\varepsilon:\mathbf{T}\times\Omega\to\mathbb{R}_+$ are $\mathbf{T}\otimes\mathcal{F}$-measurable, and

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$$\mathbb{E}\big(Q_{\varepsilon'}(t)|\mathcal{F}_\varepsilon\big) = Q_\varepsilon(t) \quad (t \in \mathbf{T},\ 0 < \varepsilon' \leqslant \varepsilon \leqslant 1).$$

We suppose that $Q_\varepsilon(t)$ and $Q_\varepsilon(t')$ have the same probability distribution.

Such martingales appeared as densities of random measures considered by Mandelbrot and Kahane in [11, 9,10] and in subsequent works. All these works consider the vague limit $Q\sigma$ of $Q_\varepsilon\sigma$ as $\varepsilon \to 0$, where σ is a Radon measure on $\mathbf{T}$. One of the main problems is to determine the local dimension of $Q\sigma$, defined by $\underline{d}(Q\sigma, t) = \liminf_{r\to 0} \frac{\log Q\sigma(B(t,r))}{\log r}$. Under some conditions of statistical self-similarity of $Q\sigma$, the multifractal analysis of $Q\sigma$ concerning the level sets of $\underline{d}(Q\sigma, t)$ was performed [8,12,2,5] and $\underline{d}(Q\sigma, t)$ was linked, via the knowledge on the distribution of $Q\sigma$, to the asymptotic density

$$\underline{D}_Q(t) = \liminf_{r\to 0} \frac{\log Q_\varepsilon(t)}{\mathbb{E}\log Q_\varepsilon(t)}.$$

Without self-similarity of $Q\sigma$ there is no answer to the problem of multifractality of $Q\sigma$. We propose to directly study the natural asymptotic density $\underline{D}_Q(t)$. In this Note, we focus on two constructions.

Poissonian products of functions. They were introduced and studied in [5,3,6], also in [7] for a special case. Let $\nu = \lambda \otimes \mu$, where λ is the Lebesgue measure on $\mathbb{R}$ and μ is a locally finite Borel measure on $(0, 1]$.

Let $(B_k)_{k\geqslant 1}$ be a partition of $\mathbb{R} \times (0, 1]$ into Borel sets such that $0 < \nu(B_k) < \infty$. Let $\nu_{|B_k}$ denote the restriction of ν on B_k and choose a sequence $(M(k, n))_{n\geqslant 1}$ of B_k-valued random variables with common distribution $\nu(B_k)^{-1}\nu_{|B_k}$. Also choose a sequence $(N_k)_{k\geqslant 1}$ of Poisson random variables with parameter $\nu(B_k)$. Assume that all the previous random variables are mutually independent. The set $S = \bigcup_{k\geqslant 1}\{M(k, 1), \ldots, M(k, N_k)\}$ is a Poisson point process with intensity ν. Let ϕ be a Hölder continuous positive function defined on $[0, 1]$. Let W be a nonnegative integrable random variable and $(W(k, n))_{k,n\geqslant 1}$ be a sequence of independent copies of W, which are assumed to be independent of S. We will write $W_{s,\ell}$ for $W(k, n)$ if $M(k, n) = (s, \ell) \in S$.

For $q \in \mathbb{R}$, let $\psi(q) = -1 + \mathbb{E}(W^q)\int_0^1 \phi(s)^q\, \mathrm{d}s$. Define the $\mathbb{R}$-martingale

$$Q_\varepsilon(t) = \exp\big(-\psi(1)\nu\big(D_\varepsilon(t)\big)\big) \prod_{(s,\ell)\in S\cap D_\varepsilon(t)} W_{s,\ell}\phi\big(\ell^{-1}(t-s)\big) \quad (t \in \mathbb{R},\ 0 < \varepsilon \leqslant 1),$$

where $D_\varepsilon(t) = \{(s, \ell) \in \mathbb{R} \times \mathbb{R}_+,\ \ell \in [\varepsilon, 1),\ s \in (t - \ell, t)\}$.

Log-infinitely divisible cascades. A special statistically self-similar cascade was introduced in [1]. Fix $(m, s) \in \mathbb{R}^2$ and π a nonnegative Borel measure on $\mathbb{R}$ which satisfies $\int_{|u|\leqslant 1} u^2\pi(\mathrm{d}u) < \infty$ and $\pi((-1, 1)^c)) < \infty$. The measure π is the Lévy measure of a real valued infinitely divisible random variable X which has its characteristic function $\mathbb{E}(\mathrm{e}^{\mathrm{i}\xi X}) = \mathrm{e}^{\varphi_{\pi,m,s}(\xi)}$ with

$$\varphi_{\pi,m,s}(\xi) = \mathrm{i}m\xi - \frac{s^2}{2}\xi^2 + \int_{\mathbb{R}} \big(\mathrm{e}^{\mathrm{i}\xi u} - 1 - \mathrm{i}\xi \sin u\big)\pi(\mathrm{d}u).$$

With $m,\ s$ and π one can associate [13] a random function $P_{\pi,m,s}$ on $\mathcal{B}(\mathbb{R} \times (0, 1])$ (more precisely on Borel sets of finite ν measure) such that if $\nu(B) < \infty$, $\mathbb{E}(\mathrm{e}^{\mathrm{i}\xi P_{\pi,m,s}(B)}) = \mathrm{e}^{\varphi_{\pi,m,s}(\xi)\nu(B)}$ $(\forall\xi \in \mathbb{R})$, and if $A_1, \ldots, A_n$ are pairwise disjoint Borel subsets of B then $P(A_1), \ldots, P(A_n)$ are mutually independent. Let $I(q) = \int_{|u|\geqslant 1} \mathrm{e}^{qu}\pi(\mathrm{d}u)$. If $I(q) < \infty$ then define $\psi(q) = \varphi_{\pi,m,s}(-\mathrm{i}q)$. Assume that $I(1) < \infty$ and choose (m, s) such that $\varphi_{\pi,m,s}(-\mathrm{i}) = 0$. Then define the $\mathbb{R}$-martingale

$$Q_\varepsilon(t) = \mathrm{e}^{P_{\pi,m,s}(D_\varepsilon(t))}.$$

In the sequel we assume that $s = 0$ (without Gaussian component) and $\int_{|u|\leqslant 1} |u|\pi(\mathrm{d}u) < \infty$ (giving a control on the variations of $P_{\pi,m,0}(D_\varepsilon(t))$ in ε).

One recovers a special case of the first construction with $\phi = 1$ by taking π to be the probability law of $\log W$.

In this Note we announce some results obtained.

2. Asymptotic behavior of $\log Q_\varepsilon(t)$

For both previous constructions, we have $\mathbb{E}\log Q_\varepsilon(t) = (\psi'(0) - \psi(1))\nu(D_\varepsilon(t))$ whenever ψ is defined near 0. We assume that $\nu(D_\varepsilon(t)) \to \infty$ as $\varepsilon \to 0$ and consider the following level sets. For $\beta \in \mathbb{R}$ define

$$\underline{F}_\beta = \left\{t \in \mathbb{R}\colon \liminf_{\varepsilon\to 0} \frac{\log Q_\varepsilon(t)}{\nu(D_\varepsilon)} = \beta\right\}, \qquad \overline{F}_\beta = \left\{t \in \mathbb{R}\colon \limsup_{\varepsilon\to 0} \frac{\log Q_\varepsilon(t)}{\nu(D_\varepsilon)} = \beta\right\}, \qquad F_\beta = \underline{F}_\beta \cap \overline{F}_\beta.$$

Let

$$\bar\alpha = \limsup_{\varepsilon\to 0} \frac{\nu(D_\varepsilon(t))}{-\log\varepsilon} \quad \text{and} \quad \hat\alpha = \inf_{b\geqslant 2} \limsup_{n\to\infty} \sup_{m\geqslant 1} \frac{\nu(D_{b^{-(n+m)}}(t) \setminus D_{b^{-n}}(t))}{\log b^m}.$$

For $q \in \mathbb{R}$ such that $\psi(q)$ is defined, let $\theta(q) = \psi(q) - q\psi(1)$. Let J be the interior of the interval $\{q : \theta(q) < \infty\}$ (we have $(0,1) \subset J$). For $\alpha \geqslant 0$ and $q \in J$ define

$$\Lambda_\alpha(q) = 1 + \alpha\big(\theta(q) - q\theta'(q)\big).$$

Theorem 2.1 (Case $\bar\alpha = 0$). *Assume* $\limsup_{\varepsilon\to 0} \varepsilon\mu([\varepsilon, 1)) < \infty$. *Suppose* $\bar\alpha = 0$. *With probability one, for all* $q \in J$ *such that* $\Lambda_{\hat\alpha}(q) > 0$, *we have* $\dim F_{\theta'(q)} = \dim \underline{F}_{\theta'(q)} = \dim \overline{F}_{\theta'(q)} = 1$.

Theorem 2.2 (Case $0 < \bar\alpha < \infty$). *Assume that* $\limsup_{\varepsilon\to 0} \varepsilon\mu([\varepsilon, 1)) < \infty$. *Suppose* $0 < \bar\alpha < \infty$. *If* $J = \mathbb{R}$, *with probability one, for all* $q \in \mathbb{R}$ *such that* $\Lambda_{\hat\alpha}(q) > 0$, *we have* $\dim F_{\theta'(q)} = \Lambda_{\bar\alpha}(q)$ *and for all* $q \in \mathbb{R}$ *such that* $\Lambda_{\bar\alpha}(q) < 0$ *we have* $F_{\theta'(q)} = \emptyset$. *If, moreover,* $\bar\alpha$ *is defined by a limit* (*not just a limsup*), *the previous results hold for* $\underline{F}_{\theta'(q)}$ *and* $\overline{F}_{\theta'(q)}$ *instead of* $F_{\theta'(q)}$.

Theorem 2.3 (Case $\bar\alpha = +\infty$). *Assume* $\lim_{\varepsilon\to 0} \varepsilon\,\mu([\varepsilon, 1)) = +\infty$ *and* $0 \in J$. *Then, with probability one, we have* $\lim_{\varepsilon\to 0} \log Q_\varepsilon(t)/\nu(D_\varepsilon) = \theta'(0)$ $(\forall t \in \mathbb{R})$.

3. Dvoretzky covering numbers

We consider the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z} = [0, 1)$, a decreasing sequence $\{\ell_n\}_{n\geqslant 1}$ $(1 > \ell_n \downarrow 0)$ such that $\sum_{n=1}^\infty \ell_n = \infty$ and a sequence of i.i.d. random variables $\{\omega_n\}_{n\geqslant 1}$ of the uniform Lebesgue distribution. Denote by $I_n = \omega_n + (0, \ell_n)$ the open interval of length ℓ_n with left end point ω_n. Define, for $n \geqslant 1$, the nth *covering number* of $t \in \mathbb{T}$ by

$$N_n(t) = \mathrm{Card}\{1 \leqslant j \leqslant n\colon\ I_n \ni t\} = \sum_{k=1}^n 1_{(0,\ell_k)}(t - \omega_k).$$

Let $L_n = \sum_{k=1}^n \ell_k$. For any $\beta \geqslant 0$, we define the (random) sets

$$\underline{F}^D_\beta = \left\{t \in \mathbb{T}\colon \liminf_{n\to\infty} \frac{N_n(t)}{L_n} = \beta\right\}, \qquad \overline{F}^D_\beta = \left\{t \in \mathbb{T}\colon \limsup_{n\to\infty} \frac{N_n(t)}{L_n} = \beta\right\}, \qquad F^D_\beta = \underline{F}^D_\beta \cap \overline{F}^D_\beta.$$

Define

$$\bar\alpha^D = \limsup_{n\to\infty} \frac{\sum_{j=1}^n \ell_j}{-\log \ell_n}, \qquad \hat\alpha^D = \inf_{b\geqslant 2} \limsup_{n\to\infty} \sup_{m\geqslant 1} \frac{\sum_{\ell_j \in [b^{-(n+m)}, b^{-n}]} \ell_j}{\log b^m}$$

and

$$d_\alpha(\beta) = 1 + \alpha(\beta - 1 - \beta\log\beta), \quad \alpha, \beta \geqslant 0.$$

Theorem 3.1 (Case $\bar{\alpha}^D = 0$). *Assume* $\limsup_{n\to\infty} n\ell_n < \infty$. *Suppose* $\bar{\alpha}^D = 0$. *With probability one, for all* $\beta \geqslant 0$ *such that* $d_{\hat{\alpha}^D}(\beta) > 0$ *we have* $\dim F_\beta^D = \dim \underline{F}_\beta^D = \dim \overline{F}_\beta^D = 1$.

Theorem 3.2 (Case $0 < \bar{\alpha}^D < \infty$). *Assume* $\limsup_{n\to\infty} n\ell_n < \infty$. *Suppose* $0 < \bar{\alpha}^D < \infty$. *With probability one, for all* $\beta \geqslant 0$ *such that* $d_{\hat{\alpha}^D}(\beta) > 0$, *we have* $\dim F_\beta = d_{\bar{\alpha}^D}(\beta)$, *and* $F_\beta = \emptyset$ *for all* $\beta \geqslant 0$ *such that* $d_{\bar{\alpha}^D}(\beta) < 0$. *If, moreover,* $\bar{\alpha}^D$ *is defined by a limit* (*not just a limsup*), *the previous results hold for* $\underline{F}_\beta^D$ *and* $\overline{F}_\beta^D$ *instead of* F_β^D.

Theorem 3.3 (Case $\bar{\alpha}^D = +\infty$). *Assume* $\lim_{n\to\infty} n\ell_n = \infty$. *Then, with probability one,* $\lim_{n\to\infty} \frac{N_n(t)}{L_n} = 1$ $(\forall t \in \mathbb{T})$.

4. Comments

Theorems 3.1–3.3 are proved in [4]. They complete and improve [7] which deals with the case $\ell_n = \frac{\alpha}{n}$ and obtain the dimension of F_β^D for a fixed β almost surely. The covering number $N_n(t)$ is closely related to the Poisson covering number $\mathrm{Card}(S \cap D_\varepsilon(t))$ when $\mu = \sum_{n\geqslant 1} \delta_{\ell_n}$. It is easy to see that if $W \equiv 1$ and $\phi \equiv a$ we have $\log Q_\varepsilon(t) = \mathrm{Card}(S \cap D_\varepsilon(t)) \log a + (1-a)\nu(D_\varepsilon(t))$. Actually the method used in [4] can be adapted to study $\log Q_\varepsilon(t)$ for the general constructions presented in Section 1. One of our main tools is to define almost surely simultaneously (in q) the limit measures of $\frac{Q_\varepsilon(t)^q}{\mathbb{E}(Q_\varepsilon(t)^q)}\lambda$ (see [3,6]). These limit measures are carried by the sets in question. This yields a lower bound of the Hausdorff dimensions. The upper bounds are estimated through the definition of Hausdorff measures by using natural coverings. This involves uniform estimates similar to those obtained in the lower bound estimation.

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