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# Maximal smoothness of the anti-analytic part of a trigonometric null series ☆

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**Abstract**

We proved recently (C. R. Acad. Sci. Paris, Ser. I 336 (2003) 475–478) that the anti-analytic part of a trigonometric series, converging to zero almost everywhere, may belong to $L^2$ on the circle. Here we prove that it can even be $C^\infty$, and we characterize precisely the possible degree of smoothness in terms of the rate of decrease of the Fourier coefficients. This sharp condition might be viewed as a 'new quasi-analyticity'. ***To cite this article: G. Kozma, A. Olevskiĭ, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***


**Résumé**

**Régularité maximale de la partie anti-analytique d'une série trigonométrique nulle presque partout.** Nous avons montré récemment (C. R. Acad. Sci. Paris, Ser. I 336 (2003) 475–478) que la partie anti-analytique d'une série trigonométrique qui converge vers zéro presque partout peut appartenir à $L^2$ sur le cercle. Nous montrons ici qu'elle peut même appartenir à $C^\infty$, et nous donnons le meilleur degré de régularité possible en termes de rapidité de décroissance des coefficients de Fourier. Il s'agit d'une nouvelle sorte de quasi-analyticité. ***Pour citer cet article : G. Kozma, A. Olevskiĭ, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***


## Version française abrégée

Les résultats principaux sont les suivants (Théorèmes 1.2 et 1.3).
Soit $\omega:\mathbb{R}^+\to\mathbb{R}^+,\ \omega(t)/t$ concave et $\sum\frac{1}{\omega(n)}=\infty$. Il existe alors une série trigonométrique

$$\sum c(n)\,\mathrm{e}^{\mathrm{i}nt} \tag{1}$$

qui converge vers zéro presque partout, avec $c(0)\neq 0$ et

$$c(n)=\mathrm{O}\big(\exp\big(-\omega\big(\log|n|\big)\big)\big),\quad n<0. \tag{4}$$

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Inversement, si la série (1) converge vers zéro presque partout et vérifie (4) avec $\omega(t)/t$ croissant et $\sum \frac{1}{\omega(n)} < \infty$, tous les $c(n)$ sont nuls.

Si l'on remplace « presque partout » par « sur un ensemble de mesure positive », (4) est changé en la condition classique de quasi-analyticité.

La Note esquisse la preuve du premier énoncé. La clé est la constitution d'un ensemble de Cantor aléatoire (non symétrique, de mesure de Lebesgue nulle), et d'une fonction harmonique aléatoire dans le disque, dont les valeurs frontières sur le Cantor sont $+\infty$ et hors du Cantor $< \infty$. Cette construction et son utilisation exigent quelque soin.

## 1. Results

The classical Menshov example shows that a (nontrivial) trigonometric series

$$\sum c(n)\,\mathrm{e}^{\mathrm{i}nt} \tag{1}$$

may converge to zero almost everywhere (a.e.). Such a series is called a null series. This result was the origin of modern uniqueness theory in Fourier Analysis, see [1,4,5]. A null series can not be analytic, that is involve positive frequencies only. This follows from Abel and Privalov theorems. On the other hand, we proved recently [6] that the anti-analytic part can be small in the sense that

$$\sum_{n<0} \bigl|c(n)\bigr|^2 < \infty. \tag{2}$$

It turns out that a much stronger property is possible: the anti-analytic part can be infinitely smooth.

**Theorem 1.1.** *There exists a trigonometric series* (1) *convergent to zero a.e., such that*

$$c(n) = \mathrm{O}\left(\frac{1}{|n|^k}\right) \quad (n<0) \ \textit{for every } k = 1, 2, \ldots.$$

Moreover the following result is true:

**Theorem 1.2.** *Let $\omega$ be a function $\mathbb{R}^+ \to \mathbb{R}^+$, $\omega(t)/t$ concave and*

$$\sum \frac{1}{\omega(n)} = \infty. \tag{3}$$

*Then there exists a null-series such that the amplitudes in the negative spectrum satisfy the condition*:

$$c(n) = \mathrm{O}\bigl(\exp\bigl(-\omega\bigl(\log|n|\bigr)\bigr)\bigr), \quad n<0. \tag{4}$$

It is remarkable that the condition is sharp. The following uniqueness theorem is true.

**Theorem 1.3.** *If a series* (1) *converges to zero a.e., and the coefficients satisfy the condition* (4), *where $\omega(t)/t$ increase and*

$$\sum \frac{1}{\omega(n)} < \infty, \tag{5}$$

*then $c(n) = 0$ for all $n \in \mathbb{Z}$.*

So for series (1) converging a.e. on the circle, (4) and (5) appears as a sharp quasi-analyticity condition for the amplitudes of the negative spectrum, which ensures the uniqueness property. Those amplitudes of a null series may, for example, decrease as $n^{-\log\log n}$ but not as $n^{-(\log\log n)^2}$.

With respect to Theorem 1.3, it should be mentioned that if one replaces convergence a.e. with convergence on a set $E$ of positive measure, then a sharp uniqueness condition is the usual quasi-analyticity:

$$c(n) = \mathrm{O}\big(\exp\big(-\rho(n)\big)\big) \quad (n < 0), \qquad \sum \frac{\rho(n)}{n^2} = \infty. \tag{6}$$

This follows from Beurling theorem [2] extended by Borichev [3], which implies that a series (1), (6) converging on $E$ to zero is trivial. The sharpness follows from classical results, see [7]. In fact, in [3], the sum of the analytic part of (1) is understood (like in Privalov theorem) as a non-tangential boundary limit, which is assumed to exist on $E$. In this setting uniqueness holds under doubly exponentional growth condition of this part in the disc. Our Theorem 1.3 also admits such a version, but the growth conditions necessary are much stronger.

Below we give a sketch of the ideas involved in the proof of Theorem 1.1. Theorem 1.2 can be obtained basically by the same approach. We do not discuss here the proof of Theorem 1.3.

In the proof below we construct a probabilistically-skewed 'thick' Cantor set $K$ of measure zero and a random harmonic function $f$ on the disk with singularities on $K$. Taking $F = \exp(f + i\tilde{f})$ and denoting by $F^*$ the boundary value of $F$ on the circle, we shall show that $F^*$ is smooth and that the Taylor coefficients $\widehat{F}(n) \to 0$ with probability 1. Hence the coefficients $c(n) := \widehat{F}(n) - \widehat{F^*}(n)$ are the Fourier coefficients of a singular compactly supported distribution on $\mathbb{T}$ and $c(n) \to 0$, which gives, by [4, p. 54], that (1) converges to zero almost everywhere, as required.

It is interesting to compare the proof to the one used in [6]. There $f$ was the Poisson integral of a singular (non-stochastic) measure on $K$. This approach, however, cannot work here, even if $f$ is taken to be the sum of a singular measure and an $L^1$ function.

## 2. Construction

Let

$$\sigma_n := \frac{1}{2^n \log(n+2)}, \quad n \geqslant 1, \quad \sigma_0 = 1, \tag{7}$$

$$\tau_n := \frac{1}{12}(\sigma_{n-1} - 2\sigma_n) \approx \frac{1}{2^n n \log^2 n}, \tag{8}$$

where $X \approx Y$ stands, as usual, for $cX \leqslant Y \leqslant CX$, and where $c$ and $C$ stand, here and everywhere, for some absolute constants. Let $l \in C^\infty]0,1]$ be a function satisfying $l(x) = -\log^2 x$ for $x < 1/3$, $l(x) = -1$ for $x > 2/3$ and $l \leqslant -1$ everywhere.

Given $s \in [0,1]$ define functions on $\mathbb{R}$

$$l^{\pm}(x; s) := \begin{cases} l(x), & 0 < x \leqslant 1, \\ -1, & 1 < x \leqslant 2 \pm s, \\ l(3 \pm s - x), & 2 \pm s < x \leqslant 3 \pm s \end{cases}$$

and 0 otherwise.

Assume at the $n$th step of induction that we have $2^n$ intervals $I(n,k)$ of length $\sigma_n$ (intervals of rank $n$), and let $K_n := \bigcup_{k=0}^{2^n-1} I(n,k)$; assume also we have a function $f_n : [0,1] \to \mathbb{R}$ such that $f_n|_{I(n,k)} = M(n)$, i.e., some constant independent of $k$. $f_{n+1}$ would differ from $f_n$ only inside $K_n$. Examine therefore one $I = I(n,k)$. Divide $I$ into two equal parts, $I = I' \cup I''$. $f_{n+1}$ will now be defined on the sides of $I'$ using some $s = s(n+1, 2k)$

$$f_{n+1} := \begin{cases} n\sqrt{\log n} \cdot l^+(x/\tau_{n+1}; s) & \text{left side of } I', \\ n\sqrt{\log n} \cdot l^-(x/\tau_{n+1}; s) & \text{right side of } I' \end{cases} \tag{9}$$

which leaves a space of $\frac{1}{2}\sigma_n - 6\tau_{n+1} = \sigma_{n+1}$ in $I'$ undefined — this will be $I(n+1, 2k)$ and on it $f_{n+1}$ will be equal to $M_{n+1}$, which will be fixed from the condition

$$\int_{I'} f_{n+1} - f_n = 0 \tag{10}$$

and it is clear that $M_{n+1}$ does not depend on $s$. Repeat the construction inside $I''$ with $s = s(n+1, 2k+1)$. We remark that the factor $n\sqrt{\log n}$ in (9), or to be more precise, the fact that it is superlinear, is the one that guarantees that the final function $F$ is $C^\infty$.

For now the choice of the $s(n,k)$ is arbitrary. It is only for the last step, that we will take the $s(n,k)$ to be random (independent and uniformly distributed on $[0,1]$). Then we will prove that a null series with smooth anti-analytic part is generated for almost any choice of $s$-es.

## 3. Estimates

### 3.1. The maximum of $f_n$

The magnitude of the $\tau_n$ (8) together with (9) gives that the negative part of $f_n$ of rank $i$ has integral $\approx \log^{-3/2} i$, and hence a sum and (7) gives

$$M_n \approx \frac{n}{\sqrt{\log n}}. \tag{11}$$

Similarly, for any interval $I$ of rank $n-1$,

$$\int_I \left| f_n(x) - f_{n-1}(x) \right| \overset{(10)}{=} 2\int_I (f_n - f_{n-1})^- \approx \frac{1}{2^n \log^{3/2} n} \leqslant C2^{-n}. \tag{12}$$

We remark that the fact that $M_n$ is sublinear is the one that guarantees that our final $F$ will have $\widehat{F}(m) \to 0$. Hence the proof hinges around the following observation: even though $K$ has measure zero, it is sufficiently thick so that it would be possible to balance superlinear growth outside $K$ (the $n\sqrt{\log n}$ factor in (9)) with sublinear growth inside $K$. The proof of Theorem 1.2 explores this effect to its maximum.

### 3.2. The limit of the $f_n$

We identify $[0,1]$ with the circle $\{|z| = 1\}$, extend $f_n$ as harmonic functions into the disk $\mathbb{D}$ and denote the extensions by $f_n$ as well. We need to estimate $f_n$ and their derivatives $f_n^{(D)}$ (we mean tangential derivative, i.e., if $f = f(r\,\mathrm{e}^{2\pi \mathrm{i}\theta})$ then $f' := \frac{\mathrm{d}f}{\mathrm{d}\theta}$). Using (10), (12), integration by parts and standard estimates for the derivatives of the Poisson kernel one can prove:

$$\left| f_{n+1}^{(D)}(z) - f_n^{(D)}(z) \right| \leqslant \frac{C(D)}{2^n d(z, K_n)^{D+1}} \quad \forall z \in \overline{\mathbb{D}} \setminus K_n,\ \forall D \in \{0, 1, \ldots\}, \tag{13}$$

where $d(z,K)$ denotes the distance of the point $z$ from the set $K$. Denote by $\widetilde{f_n}$ the harmonic conjugate of $f_n$. Using the conjugate Poisson kernel we get the same estimate for $|\widetilde{f_{n+1}}^{(D)}(z) - \widetilde{f_n}^{(D)}(z)|$.

These two inequalities show that $f_n$ and $\widetilde{f_n}$ converge uniformly on compact subsets of $\overline{\mathbb{D}} \setminus K$. Denote their limits by $f$ and $\tilde{f}$ respectively — $\lim \widetilde{f_n}$ is clearly the conjugate of $\lim f_n$, which justifies the notation $\tilde{f}$.

The boundary values of $f$ are simple to estimate, as $f|_{[0,1]\setminus K_n} \equiv f_n|_{[0,1]\setminus K_n}$. Hence, directly from the definitions of $f_n$ and $l$ we get that $f$ has singularities on $K$ and on a countable set of points $Q$ — the boundaries and middles of all the intervals $I(n,k)$. Denote $K' := K \cup Q$. From (13) and properties of $f_n$ one can deduce that on $\mathbb{T}$,

$$\left|f^{(D)}(x)\right| \leqslant \frac{C(D)}{d(x,K')^{D+1}}. \tag{14}$$

This also holds for $\tilde{f}^{(D)}$, though it is necessary to first prove an analog of (14) for $\widetilde{f_n}$ uniformly in $n$ and take the limit as $n \to \infty$. The estimate for $\widetilde{f_n}$ follows in turn from the estimate for $f_n$ and estimates on the derivatives of the Hilbert kernel.

### 3.3. Smoothness

Define now $F = \exp(f + \mathrm{i}\tilde{f})$. We use the notation $F^*$ for the boundary value, considered as a function on $\mathbb{T}$, in order to distinguish it from the "true" limit value of $F$ on the boundary of the circle which is a distribution with a singular part supported on $K$. We note that $F$ is not in $H^\infty$ and therefore the coefficients $c(n) = \widehat{F}(n) - \widehat{F^*}(n)$ are non-trivial.

A rather straightforward calculation starting from (11) shows that

$$f(x) \leqslant -c \log \frac{1}{d(x,K')} \sqrt{\log\log \frac{1}{d(x,K')}} \quad \forall x \in \mathbb{T} \setminus K'. \tag{15}$$

Combining the fact that $f$ goes to $-\infty$ faster than $\log 1/d(z,K')$ with the rough estimates of (14) (and the corresponding inequality for $\tilde{f}$) one can prove that $F^* \in C^\infty([0,1])$.

## 4. Probability

Denote $F_n = \exp(f_n + \mathrm{i}\widetilde{f_n})$ for $n = n(m) = \lfloor C \log m \rfloor$. Then another relatively simple conclusion from (13) is that for some $C$ sufficiently large, the following inequality for Taylor coefficients holds

$$\left|\widehat{F_n}(m) - \widehat{F}(m)\right| = \left| \int\limits_{|z|=1-1/m} z^{-m-1}\big(F_n(z) - F(z)\big)\,\mathrm{d}z \right| \leqslant \frac{C}{m}. \tag{16}$$

We shall not give many details for the probabilistic argument. In general it uses a fourth moment calculation. Define therefore, for every $0 \leqslant k < 2^n$,

$$\mathcal{I}_k = \int\limits_{I(n,k)} F_n(x)\,\mathrm{e}^{\mathrm{i}mx}\,\mathrm{d}x,$$

for which we have an absolute bound (from (11))

$$|\mathcal{I}_k| \leqslant \int\limits_{I(n,k)} \left|F_n(x)\right| \leqslant \sigma_n\,\mathrm{e}^{Cn/\sqrt{\log n}} =: \gamma_n = \gamma. \tag{17}$$

**Lemma 4.1.** *Let $0 \leqslant k_1, k_2, k_3, k_4 < 2^n$ and let $1 \leqslant r \leqslant n$, and assume that $I(n,k_i)$ belong to at least three different intervals of rank $r$. Then*

$$\mathbb{E}(\mathcal{I}_{k_1}\mathcal{I}_{k_2}\mathcal{I}_{k_3}\mathcal{I}_{k_4}) \leqslant \gamma^4 \frac{C \log^4 m}{m^2 \tau_r^3}.$$

Had we needed to estimate $\int \exp(f_n(x))\,\mathrm{e}^{\mathrm{i}mx}$ the lemma would have been standard, since $f$ has a local structure and by conditioning on the location of the intervals of rank $r-1$ we would achieve independence between the various $\mathcal{I}_k$-s. However, $F$ contains also the $\tilde{f}$ component which is non-local. Still, it turns out that after the conditioning step we are left with a function of two variables which can be estimated by two (rather long) integrations by parts. We skip this calculation entirely.

Proceeding with the proof of the theorem, define

$$X = X_m = \sum_{k=0}^{2^n-1} \int_{I(n,k)} F_n(x)\,\mathrm{e}^{\mathrm{i}mx}\,\mathrm{d}x.$$

The difference $\widehat{F_n}(m) - X$ is the integral over the subset of $\mathbb{T}$ where $f_n = f$, and there $F_n$ is $C^\infty$ uniformly in $n$, and in particular this integral is $\leqslant C/m$. Therefore we need only bound $X$, and we shall estimate $\mathbb{E}X^4$. Let

$$E(k_1,k_2,k_3,k_4) := \mathbb{E}\prod \mathcal{I}_{k_i},$$

let $r(k_1,\dots,k_4)$ be the minimal $r$ such that $I(n,k_i)$ are contained in at least 3 different intervals of rank $r$. A simple calculation shows

$$\#\{(k_1,\dots,k_4) : r(k_1,\dots,k_4) = r\} \approx 2^{4n-2r}.$$

The estimate of the lemma is useless if $r$ is too large. Let $R$ be some number. For $r \geqslant R$ use the simple $|E(k_1,\dots,k_4)| \leqslant \gamma^4$ and $\gamma = 2^{-n}m^{\mathrm{o}(1)}$ to get

$$E_1 := \sum_{r(k_1,\dots,k_4)\geqslant R} E(k_1,\dots,k_4) \leqslant C\gamma^4 2^{4n-2R} \leqslant m^{\mathrm{o}(1)}2^{-2R}. \tag{18}$$

For smaller $r$, we use the lemma to get $E(k_1,\dots,k_4) \leqslant \gamma^4 m^{-2+\mathrm{o}(1)}\tau_r^{-3}$ and then, using $\tau_r = 2^{-r+\mathrm{o}(r)}$,

$$\begin{aligned} E_2 :&= \sum_{r(k_1,\dots,k_4)<R} E(k_1,\dots,k_4) \leqslant \gamma^4 2^{4n} m^{-2+\mathrm{o}(1)} \sum_{r=1}^{R} 2^{-2r}\tau_r^{-3} \\ &= m^{-2+\mathrm{o}(1)} \sum_{r=1}^{R} 2^{r+\mathrm{o}(r)} = m^{-2+\mathrm{o}(1)} 2^{R+\mathrm{o}(R)}. \end{aligned} \tag{19}$$

Picking $R = \lfloor \frac{2}{3}\log m \rfloor$ we get from (18) and (19) that $\mathbb{E}X_m^4 \leqslant m^{-4/3+\mathrm{o}(1)}$ and hence $\mathbb{E}\big(\sum X_m^4\big) < \infty$ and in particular $X_m^4 \to 0$ with probability 1. As remarked above, this shows that $\widehat{F_n}(m) \to 0$ and hence using (16) that $\widehat{F}(m) \to 0$.

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