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C. R. Acad. Sci. Paris, Ser. I 338 (2004) 581–586
Comptes Rendus Mathematique

Numerical Analysis

A level-set formulation of immersed boundary methods for fluid–structure interaction problems

Georges-Henri Cottet, Emmanuel Maitre

Laboratoire LMC-IMAG, Université Joseph Fourier, 38041 Grenoble cedex 9, France

Received 23 November 2003 ; accepted after revision 27 January 2004

Presented by Olivier Pironneau

Abstract

We propose a level set formulation of the immersed boundary method for fluid–structure problems in two and three dimensions. We prove that the resulting model verifies an energy estimate. ***To cite this article: G.-H. Cottet, E. Maitre, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Une formulation par ensembles de niveau des modèles de frontières immergées pour des problèmes d'interaction fluide–structure. On propose dans cette Note une formulation par ensembles de niveau de la méthode de frontière immergée, dans le cadre d'un problème de couplage fluide–structure en dimension deux ou trois. On démontre que le modèle obtenu vérifie une estimation d'énergie. ***Pour citer cet article : G.-H. Cottet, E. Maitre, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Version française abrégée

Les méthodes de frontière immergée ont été introduites par Peskin [5] en 1977. Leur but était de modéliser et simuler des systèmes fluide–structure dans le cas d'une structure constituée de fibres uni-dimensionnelles agissant sur le fluide par des forces élastiques. Mathématiquement, ces forces apparaissent en second membre des équations de Navier–Stokes comme des masses de Dirac à support sur les fibres. Numériquement, cette méthode de suivi d'interface nécessitait l'introduction de marqueurs Lagrangiens le long des fibres, accompagnée d'interpolation pour évaluer la vitesse de ces marqueurs sur un maillage fixe. Cette interpolation est une source d'erreur pouvant mener à des fibres numériquement perméables. D'autre part, en présence de grandes déformations, le suivi des marqueurs peut être délicat et des remaillages des fibres peuvent s'avérer nécessaires.

La méthode de capture d'interface par ensembles de niveau que nous proposons ici permet de pallier ces inconvénients car le déplacement de l'interface correspond à la résolution d'une équation d'advection sur maillage fixe. L'interface est calculée comme la ligne de niveau d'une fonction ϕ advectée par l'écoulement. Les forces

E-mail addresses: Georges-Henri.Cottet@imag.fr (G.-H. Cottet), Emmanuel.Maitre@imag.fr (E. Maitre).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2004.01.023

élastiques peuvent être calculées en observant que les dérivées de ϕ donnent directement accés à l'étirement de l'interface. Il est important de noter que cette observation s'applique aussi bien à des surfaces qu'à des courbes immergées, alors que la méthode originelle de Peskin se limite à des fibres monodimensionnelles.

Le modèle que nous obtenons est constitué des équations

$$\rho_\varepsilon(\phi)(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}) - \nu\Delta\mathbf{u} + \nabla p = \mathbf{F}_\varepsilon(x,t), \quad \operatorname{div}\mathbf{u} = 0, \quad \phi_t + \mathbf{u}\cdot\nabla\phi = 0,$$

où ρ_ε et $\mathbf{F}_\varepsilon$ sont définis dans (7) avec $\mathbf{F} = \mathbf{F}_{\boldsymbol{\tau}} + \mathbf{F}_{\mathbf{n}}$ définie dans (6). L'énergie élastique correspondante est de la forme $\int_\Omega E(|\nabla\phi|)\frac{1}{\varepsilon}\zeta(\frac{\phi}{\varepsilon})\,\mathrm{d}\mathbf{x}$. Nous montrons que les solutions régulières de ce modèle vérifient l'égalité d'énergie (9).

Dans le domaine du couplage fluide–structure, une estimation d'énergie est un gage de cohérence du modèle et de stabilité numérique. Les détails non présentés ici, ainsi que l'implémentation numérique de la méthode, feront l'objet de futures publications [2,3].

1. Introduction

The immersed boundary (IB) method was originally proposed by Peskin [5] in 1977. Its purpose was to model and simulate fluid–structure systems where the structures were made of one-dimensional fibers acting on the fluid in which they are immersed through elastic forces. Mathematically speaking, these forces appear in the right-hand side of the Navier–Stokes equations as Dirac masses with support on material curves. Later on (see [6] and the references therein) a density term, also modeled by Dirac masses was added to account for the mass of the fibers.

From the numerical point of view, this method belongs to the class of front tracking methods. It requires following Lagrangian markers along the fibers. These markers are used to compute the stretching of the fibers and the resulting forces. In a grid-based flow solver, interpolations are needed to evaluate markers velocity and to spread forces on nearby grid points.

As acknowledged by Peskin these interpolations are a possible source of errors that can lead to flow leaking through the fibers. Also, in severe deformations accurate computations of the stretching may require surgery techniques along the fibers.

The method we propose is along the lines of the level set method [4] and belongs to the class of front capturing methods. In this technique Lagrangian interfaces are not tracked explicitly but instead are envisioned as level sets of a function that is solution to an advection equation. In classical Fluid dynamics application of level set methods, the level set is the interface between two fluids. It supports a force which originates in the surface tension.

Although some links between immersed boundary and level set methods are outlined in [1], to our knowledge level set methods have not been designed or used to simulate fluid–structure systems. The goal of this Note is to derive a level set formulation for a fluid–structure system consisting of an immersed elastic membrane in an incompressible flow. Our derivation is based on the observation that a level set function describing an interface and passively advected by a flow gives direct access to the amount of stretching of the interface and thus provides the resulting elastic force. This allows us to rewrite the flow–structure system as a coupled system of conservation laws. Compared to the original Immersed boundary method, the advantage of this formulation is that it is more flexible in handling surface forces in 3D flows, or more complex fluid–structure coupling, and enables a better control of mass conservation, provided the discretization of the flow and level set equations have the desired conservation properties.

A noticeable feature of the method we propose is the possibility to derive energy estimates. In fluid–structure systems, energy estimates are the key tools to guarantee both the physical relevance of a given formulation and the stability of numerical discretizations.

In the following we first recall the classical immersed boundary setting, we then give our level set formulation, use this formulation to extend the method to immersed surfaces and derive energy estimates. Details on the implementation of the method and numerical illustrations for biological applications will be given elsewhere [2,3].

2. The immersed boundary method

We summarize here the IB method, as it is described in [5] (see also [6] and the references therein). Let Ω a domain filled with an incompressible flow with velocity $\mathbf{u}$ pressure p and density ρ. The Navier–Stokes equation is in Ω

$$\rho\big(\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}\big) - \nu\Delta\mathbf{u} + \nabla p = \mathbf{F}, \quad \nabla\cdot\mathbf{u} = 0, \tag{1}$$

where $\mathbf{F}$ is a volume force acting on the flow. For simplicity, we will assume in the sequel that Ω is a square or cubic box, with periodic boundary conditions and that the force results from an elastic immersed fiber described by a material curve $\Gamma(t)$ with a parametrization $\xi \to \mathbf{X}(t,\xi)$ satisfying $\mathbf{X}_t = \mathbf{u}(\mathbf{X},t)$. The elastic force $\mathbf{F}$ is given by the longitudinal stretching of the curve which is proportional to $|\mathbf{X}_\xi|$ (here and in the sequel we use subscripts to denote partial derivatives, when there is no ambiguity). In the following, we will choose for ξ the arc-length variable for the initial fiber, and assume for simplicity that the initial state of the fiber corresponds to rest (the general case can be recovered by straightforward normalization). With these assumptions, the stretching of the fiber is equal to $|\mathbf{X}_\xi| - 1$.

The value of the force $\mathbf{F}$ can be most simply obtained from an elastic potential. Following [5] the elastic potential and local density of force (with respect to the measure $\mathrm{d}\sigma = |\mathbf{X}_\xi|\,\mathrm{d}\xi$) imparted by the structure on the fluid is given by

$$\mathcal{E}(\mathbf{X}) = \int_0^L E\big(|\mathbf{X}_\xi|\big)\,\mathrm{d}\xi, \qquad f(\mathbf{X}) = \frac{1}{|\mathbf{X}_\xi|}\frac{\partial}{\partial\xi}\left(E'\big(|\mathbf{X}_\xi|\big)\frac{\mathbf{X}_\xi}{|\mathbf{X}_\xi|}\right). \tag{2}$$

The right-hand side for the Navier–Stokes equation (1) is finally given by

$$F(\mathbf{X}) = f(\mathbf{X})\delta_{\Gamma(t)}. \tag{3}$$

3. Level set formulation

We now consider the case of a two-dimensional interface $\Gamma(t)$ immersed in a three-dimensional flow. The interface is parameterized by a function $\mathbf{X}(t;\xi_1,\xi_2)$. Its normal is defined by the vector $\mathbf{X}_{\xi_1}\times\mathbf{X}_{\xi_2}$ and the surface element is given by $\mathrm{d}\sigma = \mathrm{d}\xi_1\,\mathrm{d}\xi_2|\mathbf{X}_{\xi_1}\times\mathbf{X}_{\xi_2}|$. As before, we assume that at time 0, $|\mathbf{X}_{\xi_1}\times\mathbf{X}_{\xi_2}| = 1$, and that the interface is at rest. The corresponding energy is given by

$$\mathcal{E}(\mathbf{X}) = \int E\big(|\mathbf{X}_{\xi_1}\times\mathbf{X}_{\xi_2}|\big)\,\mathrm{d}\boldsymbol{\xi}. \tag{4}$$

The starting point of the level set formulation [4] is to observe that any Lagrangian interface can be seen as the level set of a function ϕ that satisfies an advection equation:

$$\Gamma(t) = \big\{x\in\Omega,\ \phi(x,t) = 0\big\}, \quad \text{where } \begin{cases}\phi_t + \mathbf{u}\cdot\nabla\phi = 0 & \text{on } \Omega\times]0,T[,\\ \phi = \phi_0 & \text{on } \Omega\times\{0\}.\end{cases} \tag{5}$$

The initial value ϕ_0 of ϕ, is for example chosen as the signed distance function to the initial position of the interface $\Gamma(0)$. We choose it to be negative inside the membrane and positive outside. In this setting one can express the normal and the curvature to the interface as

$$n(x) = \frac{\nabla\phi}{|\nabla\phi|}, \qquad \kappa(x) = \operatorname{div}\frac{\nabla\phi}{|\nabla\phi|}.$$

The level set formulation of IB methods requires to express elastic forces in terms of the level set function. This can be done using the following result

Lemma 3.1. *The parametrization of Γ satisfies $\mathbf{X}_{\xi_1}\times\mathbf{X}_{\xi_2} = \alpha(\boldsymbol{\xi})\nabla\phi(\mathbf{X})$ where α is a scalar function.*

Proof. We have to show that $\mathbf{X}_{\xi_1} \times \mathbf{X}_{\xi_2}$ satisfies the same equation as $\nabla\phi$, that is: $(\mathbf{X}_{\xi_1} \times \mathbf{X}_{\xi_2})_t = -[\mathbf{Du}]^t \mathbf{X}_{\xi_1} \times \mathbf{X}_{\xi_2}$ where $[\mathbf{Du}]^t$ is the tensor $\partial u_j/\partial x_i$. Expanding the left-hand side above yields

$$(\mathbf{X}_{\xi_1})_t \times \mathbf{X}_{\xi_2} + \mathbf{X}_{\xi_1} \times (\mathbf{X}_{\xi_2})_t = [\mathbf{Du}]\mathbf{X}_{\xi_1} \times \mathbf{X}_{\xi_2} + \mathbf{X}_{\xi_1} \times [\mathbf{Du}]\mathbf{X}_{\xi_2}.$$

We then split $[\mathbf{Du}]$ into symmetric and antisymmetric parts: $[\mathbf{Du}] = \mathbf{S} + \mathbf{A}$. We observe that if S is a symmetric matrix with vanishing trace then $\mathbf{Sb} \times \mathbf{c} + \mathbf{b} \times \mathbf{Sc} = -\mathbf{S}(\mathbf{b} \times \mathbf{c})$. This is readily seen by expanding $\mathbf{b}$ an $\mathbf{c}$ in an orthogonal basis where $\mathbf{S}$ is diagonal, and using the fact that eigenvalues sum up to 0. As for the antisymmetric part, we rewrite classically $\mathbf{Ab} = \boldsymbol{\omega} \times \mathbf{b}$, where $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ is the vorticity, to obtain $\mathbf{Ab} \times \mathbf{c} + \mathbf{b} \times \mathbf{Ac} = \mathbf{A}(\mathbf{b} \times \mathbf{c})$ which proves our claim. □

As at time $t = 0$ $\nabla\phi$ and $\mathbf{X}_{\xi_1} \times \mathbf{X}_{\xi_2}$ have norm 1, we have $|\alpha(\boldsymbol{\xi})| \equiv 1$. To further simplify the discussion below, we will assume that $\alpha(\boldsymbol{\xi}) = 1$. We will denote in the sequel by $(\mathbf{n}, \boldsymbol{\tau}^1, \boldsymbol{\tau}^2)$ a local orthonormal basis on the surface Γ, with $\mathbf{n} = \nabla\phi/|\nabla\phi|$. We are now ready to express the elastic forces solely in terms of the gradient of the level set function ϕ.

Lemma 3.2. *Let* $\mathbf{X}, \mathbf{Y}$ *two* C^1 *surface parametrizations, and* ϕ *a* C^1 *function such that* $\mathbf{X}_{\xi_1} \times \mathbf{X}_{\xi_2} = \nabla\phi$*. Then* $\mathcal{E}(\mathbf{X} + \mathbf{Y}) - \mathcal{E}(\mathbf{X}) = -\int (\mathbf{F}_{\boldsymbol{\tau}} + \mathbf{F}_{\mathbf{n}}) \cdot \mathbf{Y}\, d\sigma + O(|\mathbf{Y}|^2)$ *where*

$$\mathbf{F}_{\boldsymbol{\tau}} = \big(\nabla_x\big(E'\big(|\nabla\phi|\big)\big) \cdot \boldsymbol{\tau}^1\big)\boldsymbol{\tau}^1 + \big(\nabla_x\big(E'\big(|\nabla\phi|\big)\big) \cdot \boldsymbol{\tau}^2\big)\boldsymbol{\tau}^2, \qquad \mathbf{F}_{\mathbf{n}} = -E'\big(|\nabla\phi|\big)\kappa\frac{\nabla\phi}{|\nabla\phi|}. \tag{6}$$

Proof. From (4) and using $\mathbf{X}_{\xi_1} \times \mathbf{X}_{\xi_2} = \nabla\phi$ one gets by elementary calculus and integration by parts

$$\begin{aligned}
\mathcal{E}(\mathbf{X} + \mathbf{Y}) - \mathcal{E}(\mathbf{X}) &\sim \int \big[(\mathbf{X}_{\xi_1}, \mathbf{Y}_{\xi_2}, \mathbf{n}) + (\mathbf{Y}_{\xi_1}, \mathbf{X}_{\xi_2}, \mathbf{n})\big] E'\big(|\nabla\phi|\big)\, d\boldsymbol{\xi} \\
&= \int \big[(\mathbf{n} \times \mathbf{X}_{\xi_1}) \cdot \mathbf{Y}_{\xi_2} + (\mathbf{X}_{\xi_2} \times \mathbf{n}) \cdot \mathbf{Y}_{\xi_1}\big] \cdot \mathbf{Y} E'\big(|\nabla\phi|\big)\, d\boldsymbol{\xi} \\
&= -\int \big[(\mathbf{n} \times \mathbf{X}_{\xi_1})_{\xi_2} + (\mathbf{X}_{\xi_2} \times \mathbf{n})_{\xi_1}\big] \cdot \mathbf{Y} E'\big(|\nabla\phi|\big)\, d\boldsymbol{\xi} \\
&\quad - \int (\mathbf{n} \times \mathbf{X}_{\xi_1}) \cdot \mathbf{Y} \frac{\partial}{\xi_2}\big(E'\big(|\nabla\phi|\big)\big) + (\mathbf{X}_{\xi_2} \times \mathbf{n}) \cdot \mathbf{Y} \frac{\partial}{\xi_1}\big(E'\big(|\nabla\phi|\big)\big)\, d\boldsymbol{\xi} \\
&= -\int \big[(\mathbf{n}_{\xi_2} \times \mathbf{X}_{\xi_1}) + (\mathbf{X}_{\xi_2} \times \mathbf{n}_{\xi_1})\big] \cdot \mathbf{Y} E'\big(|\nabla\phi|\big)\, d\boldsymbol{\xi} \\
&\quad - \int (\mathbf{n} \times \mathbf{X}_{\xi_1}) \cdot \mathbf{Y} \frac{\partial}{\xi_2}\big(E'\big(|\nabla\phi|\big)\big) + (\mathbf{X}_{\xi_2} \times \mathbf{n}) \cdot \mathbf{Y} \frac{\partial}{\xi_1}\big(E'\big(|\nabla\phi|\big)\big)\, d\boldsymbol{\xi}.
\end{aligned}$$

Developing the vectors $\mathbf{X}_{\xi_i}$ along the basis $\boldsymbol{\tau}^1$, $\boldsymbol{\tau}^2$ (see [2] for detailed calculations) allows us to write successively

$$\begin{aligned}
&\left(\frac{\partial\mathbf{n}}{\partial\boldsymbol{\tau}^1} \times \boldsymbol{\tau}^2 - \frac{\partial\mathbf{n}}{\partial\boldsymbol{\tau}^2} \times \boldsymbol{\tau}^1\right) = -|\nabla\phi|\left(\frac{\partial\mathbf{n}}{\partial\boldsymbol{\tau}^1} \times \boldsymbol{\tau}^2 - \frac{\partial\mathbf{n}}{\partial\boldsymbol{\tau}^2} \times \boldsymbol{\tau}^1\right) = \kappa\nabla\phi, \\
&(\mathbf{n} \times \mathbf{X}_{\xi_1}) E'_{\xi_2} + (\mathbf{X}_{\xi_2} \times \mathbf{n}) E'_{\xi_1} = \big[\big(\nabla E' \cdot \boldsymbol{\tau}^1\big)\boldsymbol{\tau}^1 + \big(\nabla E' \cdot \boldsymbol{\tau}^2\big)\boldsymbol{\tau}^2\big]|\nabla\phi|
\end{aligned}$$

from which the desired force values are easily obtained, as $d\sigma = |\mathbf{X}_{\xi_1} \times \mathbf{X}_{\xi_2}|\, d\boldsymbol{\xi}$. □

Note that, more intrinsically, $\mathbf{F}_{\boldsymbol{\tau}}$ can be written as the projection of $\nabla_x(E'(|\nabla\phi|))$ onto $\nabla\phi^{\perp}$ without reference to a local basis: $\mathbf{F}_{\boldsymbol{\tau}} = \nabla_x(E'(|\nabla\phi|)) - (\nabla_x(E'(|\nabla\phi|)) \cdot \frac{\nabla\phi}{|\nabla\phi|})\frac{\nabla\phi}{|\nabla\phi|}$.

The force is thus $\mathbf{F} = \mathbf{F}_{\boldsymbol{\tau}} + \mathbf{F}_{\mathbf{n}}$. To complete the level set formulation, it remains to regularize the Dirac mass in (3). We will also consider the case of an interface carrying mass. Let $\zeta : \mathbb{R} \to [0, 1]$ be a cut-off function. We define the regularized forces and density respectively by

$$\mathbf{F}_\varepsilon(x,t) = \mathbf{F}(x,t)\frac{1}{\varepsilon}\zeta\left(\frac{\phi}{\varepsilon}\right)|\nabla\phi| \quad \text{and} \quad \rho_\varepsilon(\phi) = \bar{\rho} + (\rho_f - \bar{\rho})\frac{1}{\varepsilon}\zeta\left(\frac{\phi}{\varepsilon}\right). \tag{7}$$

Note that the term $|\nabla\phi|$ in the force expression, coming from the regularization of the Dirac mass on the interface, is compensated by the stretching of the interface in the expression of the density. This reflects the fact that the mass of the interface does not depend on its stretching. It is actually a simple matter to verify that $\rho_\varepsilon(\phi)$, as it is defined above, satisfies the classical mass conservation equation. Our fluid–structure problem finally reduces to the following system:

Given initial data $(\phi_0, \mathbf{u}_0)$, find $(\phi, \mathbf{u}, p)$ such that

$$\rho_\varepsilon(\phi)(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}) - \nu\Delta\mathbf{u} + \nabla p = \mathbf{F}_\varepsilon(x,t), \quad \operatorname{div}\mathbf{u} = 0, \quad \phi_t + \mathbf{u}\cdot\nabla\phi = 0. \tag{8}$$

3.1. *Energy estimate*

It is natural to define the elastic energy in the level set formulation as $\int_\Omega E(|\nabla\phi|)\frac{1}{\varepsilon}\zeta(\frac{\phi}{\varepsilon})\,\mathrm{d}\mathbf{x}$. As a matter of fact, it is also possible to derive (7) directly from this expression of the energy (see [2]). For the problem (8) we can prove the following energy equality:

Proposition 3.3. *Let* $(\phi, \mathbf{u}, p)$ *a classical solution of* (8) *with periodic boundary conditions on* $\partial\Omega$. *Then*

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \rho_\varepsilon\big(\phi(x,T)\big)\mathbf{u}^2(x,T)\,\mathrm{d}x + \nu\int_0^T\int_\Omega |\nabla\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega E\big(|\nabla\phi|\big)\frac{1}{\varepsilon}\zeta\left(\frac{\phi}{\varepsilon}\right)\mathrm{d}x \\
&\quad = \frac{1}{2}\int_\Omega \rho_\varepsilon\big(\phi_0(x)\big)\mathbf{u}_0^2(x)\,\mathrm{d}x + \int_\Omega E\big(|\nabla\phi_0|\big)\frac{1}{\varepsilon}\zeta\left(\frac{\phi_0}{\varepsilon}\right)\mathrm{d}x.
\end{aligned} \tag{9}$$

Proof. We take $\mathbf{u}$ as a test function in the momentum equation. Using the transport equation on ϕ, the incompressibility condition and the periodic boundary condition on u, we get by standard calculations

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho_\varepsilon(\phi)\mathbf{u}^2\,\mathrm{d}x + \nu\int_\Omega |\nabla\mathbf{u}|^2\,\mathrm{d}x = \int_\Omega F_\varepsilon\cdot\mathbf{u}\,\mathrm{d}x.$$

To handle the forcing term, we split it into its tangential and normal parts using its intrinsic form:

$$\begin{aligned}
A &= \int_\Omega \left[\nabla\big(E'\big(|\nabla\phi|\big)\big) - \nabla\big(E'\big(|\nabla\phi|\big)\big)\cdot\frac{\nabla\phi}{|\nabla\phi|}\frac{\nabla\phi}{|\nabla\phi|}\right]\cdot\mathbf{u}\frac{1}{\varepsilon}\zeta\left(\frac{\phi}{\varepsilon}\right)|\nabla\phi|\,\mathrm{d}x \\
&= \int_\Omega \nabla\big(E'\big(|\nabla\phi|\big)\big)\cdot\left(|\nabla\phi|\mathbf{u} + \phi_t\frac{\nabla\phi}{|\nabla\phi|}\right)\frac{1}{\varepsilon}\zeta\left(\frac{\phi}{\varepsilon}\right)\mathrm{d}x, \\
B &= \int_\Omega -E'\big(|\nabla\phi|\big)\operatorname{div}\frac{\nabla\phi}{|\nabla\phi|}\nabla\phi\cdot\mathbf{u}\frac{1}{\varepsilon}\zeta\left(\frac{\phi}{\varepsilon}\right)\mathrm{d}x = \int_\Omega E'\big(|\nabla\phi|\big)\operatorname{div}\frac{\nabla\phi}{|\nabla\phi|}\phi_t\frac{1}{\varepsilon}\zeta\left(\frac{\phi}{\varepsilon}\right)\mathrm{d}x \\
&= \int_\Omega \operatorname{div}\left(E'\big(|\nabla\phi|\big)\frac{\nabla\phi}{|\nabla\phi|}\phi_t\frac{1}{\varepsilon}\zeta\left(\frac{\phi}{\varepsilon}\right)\right)\mathrm{d}x - \int_\Omega\left[E'\big(|\nabla\phi|\big)\nabla\phi_t\cdot\frac{\nabla\phi}{|\nabla\phi|}\right. \\
&\quad \left. + E''\big(|\nabla\phi|\big)\phi_t\nabla|\nabla\phi|\cdot\frac{\nabla\phi}{|\nabla\phi|}\right]\frac{1}{\varepsilon}\zeta\left(\frac{\phi}{\varepsilon}\right) + E'\big(|\nabla\phi|\big)\phi_t\frac{\nabla\phi}{|\nabla\phi|}\frac{1}{\varepsilon^2}\zeta'\left(\frac{\phi}{\varepsilon}\right)\nabla\phi\,\mathrm{d}x.
\end{aligned}$$

By periodicity, the first integral vanishes. Thus

$$B = -\int_\Omega E'\big(|\nabla\phi|\big)\frac{\partial}{\partial t}\left[|\nabla\phi|\frac{1}{\varepsilon}\zeta\left(\frac{\phi}{\varepsilon}\right)\right]\mathrm{d}x - \int_\Omega E''\big(|\nabla\phi|\big)\phi_t\nabla|\nabla\phi|\cdot\frac{\nabla\phi}{|\nabla\phi|}\frac{1}{\varepsilon}\zeta\left(\frac{\phi}{\varepsilon}\right)\mathrm{d}x$$
$$= -\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega E'\big(|\nabla\phi|\big)|\nabla\phi|\frac{1}{\varepsilon}\zeta\left(\frac{\phi}{\varepsilon}\right)\mathrm{d}x + \int_\Omega E''\big(|\nabla\phi|\big)|\nabla\phi|_t|\nabla\phi|\frac{1}{\varepsilon}\zeta\left(\frac{\phi}{\varepsilon}\right)\mathrm{d}x$$
$$- \int_\Omega E''\big(|\nabla\phi|\big)\phi_t\nabla|\nabla\phi|\cdot\frac{\nabla\phi}{|\nabla\phi|}\frac{1}{\varepsilon}\zeta\left(\frac{\phi}{\varepsilon}\right)\mathrm{d}x.$$

Therefore, in summing A and B, we are left with:

$$A + B = -\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega E'\big(|\nabla\phi|\big)|\nabla\phi|\frac{1}{\varepsilon}\zeta\left(\frac{\phi}{\varepsilon}\right)\mathrm{d}x + \int_\Omega E''\big(|\nabla\phi|\big)\big(|\nabla\phi|_t + \mathbf{u}\cdot\nabla|\nabla\phi|\big)|\nabla\phi|\frac{1}{\varepsilon}\zeta\left(\frac{\phi}{\varepsilon}\right)\mathrm{d}x.$$

Setting $\widetilde{E}(r) := \int E''(r)r\,\mathrm{d}r = E'(r)r - E(r)$, the second integral reads

$$\int_\Omega \big[\big(\widetilde{E}(|\nabla\phi|)\big)_t + \mathbf{u}\cdot\nabla\widetilde{E}\big(|\nabla\phi|\big)\big]\frac{1}{\varepsilon}\zeta\left(\frac{\phi}{\varepsilon}\right)\mathrm{d}x = \int_\Omega\left(\widetilde{E}\big(|\nabla\phi|\big)\frac{1}{\varepsilon}\zeta\left(\frac{\phi}{\varepsilon}\right)\right)_t + \mathbf{u}\cdot\nabla\left(\widetilde{E}\big(|\nabla\phi|\big)\frac{1}{\varepsilon}\zeta\left(\frac{\phi}{\varepsilon}\right)\right)\mathrm{d}x$$
$$- \int_\Omega \widetilde{E}\big(|\nabla\phi|\big)\left(\frac{1}{\varepsilon^2}\zeta'\left(\frac{\phi}{\varepsilon}\right)\right)\phi_t - \widetilde{E}\big(|\nabla\phi|\big)\left(\frac{1}{\varepsilon^2}\zeta'\left(\frac{\phi}{\varepsilon}\right)\right)\mathbf{u}\cdot\nabla\phi\,\mathrm{d}x.$$

From the transport equation, the last integral above is zero. Moreover $\int_\Omega \mathbf{u}\cdot\nabla\widetilde{E}(|\nabla\phi|)\frac{1}{\varepsilon}\zeta(\frac{\phi}{\varepsilon})\,\mathrm{d}x = 0$ from the incompressibility and the periodic boundary conditions. We have thus

$$A + B = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\big(-E'\big(|\nabla\phi|\big)|\nabla\phi| + \widetilde{E}\big(|\nabla\phi|\big)\big)\frac{1}{\varepsilon}\zeta\left(\frac{\phi}{\varepsilon}\right)\mathrm{d}x = -\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega E\big(|\nabla\phi|\big)\frac{1}{\varepsilon}\zeta\left(\frac{\phi}{\varepsilon}\right)\mathrm{d}x. \qquad \square$$

It is important to notice that in case ρ is not depending on ϕ, that is when the mass of the immersed boundary is neglected, the same energy equality would hold for the Stokes energy. Otherwise the inertial term in the Navier–Stokes equations cannot be omitted, and the density term has to be written under the form (8).

Note finally that the same results hold for homogeneous boundary conditions assuming that the interface stays at a distance from the boundary.

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