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C. R. Acad. Sci. Paris, Ser. I 338 (2004) 533–538
COMPTES RENDUS MATHEMATIQUE

Partial Differential Equations

Gradient bounds for solutions of semilinear parabolic equations without Bernstein's quadratic condition

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Received and accepted 22 December 2003

Presented by Haïm Brézis

Abstract

We establish gradient estimates for bounded solutions of semilinear parabolic equations, where the nonlinearity only satisfies one-sided quadratic upper growth assumptions, instead of the classical (two-sided) Bernstein's condition. This extends a recent work of Al. and Ar. Tersenov (Indiana Univ. Math. J. 50 (2001) 1899–1913), where results of this kind were obtained for radial solutions in a ball, by a different technique. ***To cite this article: J.-Ph. Bartier, Ph. Souplet, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Estimations du gradient des solutions d'équations paraboliques semi-linéaires sans la condition quadratique de Bernstein. Nous établissons des estimations du gradient pour les solutions bornées d'équations paraboliques semi-linéaires, où la nonlinéarité vérifie seulement des hypothèses unilatérales de croissance quadratique, au lieu des conditions de Bernstein (bilatérales) classiques. Nous étendons ainsi un travail récent de Al. et Ar. Tersenov (Indiana Univ. Math. J. 50 (2001) 1899–1913), où des résultats de ce type ont été obtenus pour les solutions radiales dans une boule, par une technique différente. ***Pour citer cet article : J.-Ph. Bartier, Ph. Souplet, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Version française abrégée

Considérons le problème de Cauchy–Dirichlet pour l'équation parabolique semi-linéaire

$$\begin{cases} u_t - \Delta u = F(u, \nabla u), \quad 0 < t < T,\ x \in \Omega, \\ u(t,x) = 0, \quad 0 < t < T,\ x \in \partial\Omega, \\ u(0,x) = u_0(x), \quad x \in \Omega, \end{cases} \tag{1}$$

avec F une fonction de classe C^1 et $u_0 \in W^{1,\infty}(\Omega)$. Ici, Ω est un domaine (éventuellement non borné) de $\mathbb{R}^N$ vérifiant une condition de sphère extérieure uniforme. Par une solution de (1), nous entendons une solution

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2003.12.030

classique $u \in C^{1,2}(Q_T) \cap C(\overline{Q_T})$, $Q_T = (0, T] \times \Omega$. Si Ω est non borné, nous supposons aussi que $u(t, x) \to 0$ lorsque $|x| \to \infty$, uniformément pour $t \in [0, T]$. Enfin, nous notons $F \leqslant \mathrm{O}(|p|^2)$ si $F(u, p) \leqslant C(u)(|p|^2 + 1)$ et $F \leqslant \mathrm{o}(|p|^2)$ si pour tout $\varepsilon > 0$, $F(u, p) \leqslant \varepsilon|p|^2 + C_\varepsilon(u)$, avec $C(u)$ et $C_\varepsilon(u)$ bornées pour u borné ; $F \leqslant \mathrm{o}(|p|)$ est défini similairement.

Il est bien connu [8] que si F vérifie la condition de croissance quadratique par rapport à ∇u :

$$|F| \leqslant \mathrm{O}\big(|p|^2\big), \tag{2}$$

connue sous le nom de condition de Bernstein (cf. [3]), alors une estimation a priori sur u entraîne une borne sur ∇u, d'où l'on peut déduire la resolubilité globale de (1). De plus, la condition (2) est en un sens optimale. En effet, si par exemple $F = |\nabla u|^q$, $q > 2$, alors il y a explosion du gradient pour certains u_0 : ∇u explose en temps fini, tandis que u reste bornée (voir [12] et cf. également [1] et les références citées dans [1,12] pour d'autres exemples).

Recemment, il a été montré dans [14] que si l'on se restreint aux solutions radiales $u = u(t, r)$, $r = |x|$, dans une boule Ω, la condition (2) peut être remplacée par une condition *unilatérale* de croissance quadratique. Plus précisément, en décomposant F comme $F = f + g$, il est démontré qu'une estimation a priori sur u entraîne une borne sur u_r à condition que $f = f(u, u_r)$ verifie (2) et que la « partie sur-quadratique dissipative » $g = g(u, u_r)$ soit décroissante en u et satisfasse $u_r\, g(0, u_r) \geqslant 0$. (Noter en particulier que si $g = g(u, |u_r|)$, ceci implique $g(0, u_r) = 0$.)

D'autre part, dans les domaines Ω quelconques, des estimations du gradient ont été obtenues pour les solutions positives de certaines équations ne satisfaisant pas (2), par exemple pour $u_t - \Delta u = u^p - |\nabla u|^q$, avec $p, q > 1$ (voir [10,13]).

Le but de cette Note est de donner un traitement plus général de ce problème pour l'Éq. (1). Nous établirons des estimations du gradient sous des conditions *unilatérales* de croissance quadratique sur F, *sans aucune hypothèse de symétrie et dans un domaine quelconque*.

Théorème 0.1. *Soit* $M > 0$. *Supposons que* $F = f + g$, *avec*

$$|f| \leqslant \mathrm{O}\big(|p|^2\big), \qquad |f_p| \leqslant \mathrm{O}\big(|p|\big), \qquad |f_u| \leqslant \mathrm{o}\big(|p|^2\big) \tag{3}$$

et

$$g(0, p) = 0, \qquad g_u \leqslant -\varepsilon|p \cdot g_p| \tag{4}$$

pour tout $|u| \leqslant M$, $p \in \mathbb{R}^N$, *avec* $\varepsilon = \varepsilon(M) > 0$. *Alors il existe* $C = C(M, T, F, \Omega) > 0$ *tel que pour toute solution de* (1) *vérifiant* $|u| \leqslant M$ *dans* $[0, T] \times \Omega$ *et* $|\nabla u_0| \leqslant M$ *dans* Ω, *on a* $|\nabla u| \leqslant C$ *dans* $[0, T] \times \Omega$.

Si nous nous restreignons aux solutions positives, nous obtenons une estimation du gradient sous des hypothèses différentes sur F.

Théorème 0.2. *La conclusion du Théorème* 0.1 *reste valable si* $u \geqslant 0$ *et si, au lieu de* (4), *nous supposons*

$$g(0, p) \leqslant 0, \qquad g_u \leqslant 0, \qquad p \cdot \frac{\partial}{\partial p}\left(\frac{g}{|p|}\right) \leqslant 0. \tag{5}$$

Notons que (3), (4) et (3), (5) expriment bien des conditions unilatérales de croissance par rapport à ∇u. Des exemples typiques où le Théorème 0.1 s'applique sont donnés par $F(u, \nabla u) = f(u, \nabla u) - |u|^{m-1}u|\nabla u|^r$, $m > 0$, $r \geqslant 2$ ou $F(u, \nabla u) = f(u, \nabla u) - (\mathrm{e}^u - 1)|\nabla u|^r$, $r \geqslant 2$, pour tout f vérifiant (3). Le Théorème 0.2 peut s'appliquer (pour les solutions positives) dans des cas où le Théorème 0.1 ne s'applique pas. Ceci a lieu par exemple lorsque $g = -|\nabla u|^r$, $r > 2$, ou $g = -|\nabla u|\mathrm{e}^{|\nabla u|}$. L'inverse peut également se produire si g a des oscillations par rapport à ∇u, par exemple pour $g = -|u|^{m-1}u|\nabla u|^r(2 + \sin(2r \log|\nabla u|))$, $m > 0$, $r \geqslant 2$.

Notre approche repose sur la méthode classique de Bernstein [3], développée dans [7,11,5,6], qui consiste à appliquer le principe du maximum à l'inconnue $|\nabla v|^2$, où $u = \phi(v)$. (En fait nous considèrerons ici une différence finie de v au lieu de $|\nabla v|^2$.) La principale nouveauté est que, par des choix appropriés de la fonction auxiliaire ϕ, il est possible d'éliminer la condition quadratique bilatérale (2).

1. Introduction and results

Let us consider the Dirichlet initial-boundary value problem for the semilinear parabolic equation

$$\begin{cases} u_t - \Delta u = F(u, \nabla u), & 0 < t < T,\ x \in \Omega, \\ u(t,x) = 0, & 0 < t < T,\ x \in \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases} \tag{1}$$

where F is a C^1 function, and $u_0 \in W^{1,\infty}(\Omega)$. Here Ω is a (possibly unbounded) domain of $\mathbb{R}^N$ satisfying a uniform exterior sphere condition. By a solution of (1), we mean a classical solution $u \in C^{1,2}(Q_T) \cap C(\overline{Q_T})$, $Q_T = (0,T] \times \Omega$. In case Ω is unbounded, we also assume that $u(t,x) \to 0$ as $|x| \to \infty$, uniformly for $t \in [0,T]$. In all the paper we write $F \leqslant \mathrm{O}(|p|^2)$ if $F(u,p) \leqslant C(u)(|p|^2 + 1)$ and $F \leqslant \mathrm{o}(|p|^2)$ if for all $\varepsilon > 0$, $F(u,p) \leqslant \varepsilon|p|^2 + C_\varepsilon(u)$, where $C(u)$ and $C_\varepsilon(u)$ remain bounded on bounded sets of u; $F \leqslant \mathrm{o}(|p|)$ is defined similarly.

It is well known (cf. [8], Theorem V.4.1 and Lemma VI.3.1) that if F satisfies the following (two-sided) quadratic growth condition with respect to ∇u:

$$|F| \leqslant \mathrm{O}\big(|p|^2\big), \tag{2}$$

usually referred to as Bernstein's condition (cf. [3]), then an a priori estimate on u implies a bound on ∇u, from which one can deduce the global classical solvability of (1). Moreover, it is known that (2) is in a sense optimal. Indeed, if for instance $F = |\nabla u|^q$, $q > 2$, then gradient blow-up occurs for suitable u_0, i.e.: ∇u blows up in finite time, whereas u remains bounded (see [12] and cf. also [1] and the references in [1,12] for other examples).

Recently, it was shown in [14] that if Ω is a ball, for radially symmetric solutions $u = u(t,r)$, $r = |x|$, (2) can be replaced by a *one-sided*, quadratic upper growth condition. Namely, decomposing F as $F = f + g$, it is proved that an a priori estimate on u implies a bound on u_r provided $f = f(u, u_r)$ verifies (2) and the 'dissipative super-quadratic part' $g = g(u, u_r)$ satisfies:

$$g \text{ is nonincreasing in } u \quad \text{and} \quad u_r\, g(0, u_r) \geqslant 0.$$

(Note in particular that if $g = g(u, |u_r|)$, this implies $g(0, u_r) = 0$.)

On the other hand, in general domains, gradient estimates have been proved for nonnegative bounded solutions of some particular equations not satisfying (2), for instance $u_t - \Delta u = u^p - |\nabla u|^q$, with $p, q > 1$ (see [10,13]).

Our purpose here is to give a somehow more general treatment of this problem for Eq. (1). Namely we shall prove gradient bounds under *one-sided*, quadratic upper growth conditions on F, *without any symmetry restrictions and in general domains*.

Theorem 1.1. *Let $M > 0$. Assume that $F = f + g$, where*

$$|f| \leqslant \mathrm{O}\big(|p|^2\big), \qquad |f_p| \leqslant \mathrm{O}\big(|p|\big), \qquad |f_u| \leqslant \mathrm{o}\big(|p|^2\big) \tag{3}$$

and

$$g(0,p) = 0, \qquad g_u \leqslant -\varepsilon|p \cdot g_p| \tag{4}$$

for all $|u| \leqslant M$, $p \in \mathbb{R}^N$, and some $\varepsilon = \varepsilon(M) > 0$. Then there exists $C = C(M,T,F,\Omega) > 0$ such that for any solution of (1) *satisfying*

$$|u| \leqslant M \quad \text{in } [0,T] \times \Omega \quad \text{and} \quad |\nabla u_0| \leqslant M \quad \text{in } \Omega,$$

it holds $|\nabla u| \leqslant C$ in $[0,T] \times \Omega$.

If we restrict to nonnegative solutions, we can obtain a gradient bound under different assumptions on F.

Theorem 1.2. *The conclusion of Theorem* 1.1 *remains in force if we assume* $u \geqslant 0$ *and if, instead of* (4), *we assume*

$$g(0,p) \leqslant 0, \qquad g_u \leqslant 0, \qquad p \cdot \frac{\partial}{\partial p}\left(\frac{g}{|p|}\right) \leqslant 0. \tag{5}$$

Note that (3), (4) and (3), (5) are indeed one-sided growth assumptions with respect to ∇u. In particular, (4) means that g has a monotonicity in u which in a sense compensates its variations in p. For instance, it is verified by any $g = -h(u)k(|\nabla u|)$, such that for some $C > 0$ and $\varepsilon = \varepsilon(M) > 0$, it holds

$$h(0) = 0, \quad h'(u) \geqslant \varepsilon\bigl|h(u)\bigr| \quad \text{for } |u| \leqslant M, \quad \text{and} \quad q\bigl|k'(q)\bigr| \leqslant Ck(q) \quad \text{for } q \geqslant 0.$$

As typical examples, Theorem 1.1 applies to

$$F(u,\nabla u) = f(u,\nabla u) - |u|^{m-1}u|\nabla u|^r, \quad m > 0,\ r \geqslant 2$$

or

$$F(u,\nabla u) = f(u,\nabla u) - \bigl(\mathrm{e}^u - 1\bigr)|\nabla u|^r, \quad r \geqslant 2,$$

for any f satisfying (3) (see Remark 2 below). As for Theorem 1.2, it may apply (for nonnegative solutions) in situations where Theorem 1.1 does not. This is the case for instance when $g = -|\nabla u|^r,\ r > 2$, or $g = -|\nabla u|\mathrm{e}^{|\nabla u|}$. Conversely, Theorem 1.1 may apply when Theorem 1.2 does not (even for nonnegative solutions), if g has some oscillations in ∇u. An example is given by

$$g = -|u|^{m-1}u|\nabla u|^r\bigl(2 + \sin\bigl(2r\log|\nabla u|\bigr)\bigr), \quad m > 0,\ r \geqslant 2.$$

Our basic approach relies on the classical Bernstein technique [3], as developped for instance in [7,11,5,6] to obtain gradient bounds under two-sided assumptions (for quasilinear equations as well). It consists in applying the maximum principle to the dependent variable $|\nabla v|^2$, where $u = \phi(v)$ (actually, we shall consider a finite difference of v instead of $|\nabla v|^2$). The main novelty is that, by careful choices of the auxiliary function ϕ, one can dispense with the usual two-sided quadratic condition (2).

Remark 1. More general quasilinear equations of the form $u_t - A(t,x,u,\nabla u)D^2u = F(t,x,u,\nabla u)$ can be considered. This will be developped in a forthcoming publication [2].

Remark 2. In Theorems 1.1 and 1.2, it is actually not necessary to assume F to be C^1 for $u = 0$ (provided F is continuous) and F_p need not exist for $p = 0$ (the hypotheses (3)–(5) being then understood for $0 < |u| \leqslant M$ and $p \in \mathbb{R}^N \setminus \{0\}$). Also, if we suppose in addition that ∇u is uniformly continuous in $\overline{Q_T}$, then it is enough to assume F to be C^1 for p large. Finally, the condition (4) can be weakened to $g(0,p) = 0$ and $g_u \leqslant \min(0, \varepsilon u p \cdot g_p)$. We refer to [2] for details about these facts.

Remark 3. Let us note that the proof of the gradient estimate in [8, Theorem V.4.1 and Lemma VI.3.1], under the two-sided assumption (2), relies on a different approach based on elaborate test-function techniques (see also [4,9] for related arguments). Although this method seems difficult to apply in our case, one advantage is that, unlike in [7,11,5,6], no growth assumption is needed on the derivatives F_u, F_p (which need not even exist). As for the proof from [14] in the radial framework, it is still different, based on Kruzhkov's idea of adding a new space variable.

2. Proofs

Fix a function ϕ of class C^3 on some compact interval J, with $\phi' > 0$ and $\phi(J) \supset [-M, M]$ (or $[0, M]$ in the case of Theorem 1.2). Fix also a constant $K > 0$ (ϕ and K will be chosen later on). Let $a \in \mathbb{R}^N$, $|a| = 1$, $h > 0$

and denote $d(x) = \operatorname{dist}(x, \partial\Omega)$ and $\Omega_h = \{x \in \Omega;\ d(x) > h\}$. Finally, set $u := \phi(v)$ and $w(t,x) = \mathrm{e}^{-Kt}(v(t,x+ha) - v(t,x))$, which is defined in $D_h = (0,T] \times \Omega_h$. The proof proceeds in two steps.

Claim 2.1. *We claim that, for a suitable choice of ϕ and K (depending only on M, F), w cannot achieve a positive maximum in D_h.*

We have $F(u, \nabla u) = u_t - \Delta u = \phi'(v)(v_t - \Delta v) - \phi''(v)|\nabla v|^2$, hence

$$v_t - \Delta v = \frac{F(\phi(v), \phi'(v)\nabla v)}{\phi'(v)} + \frac{\phi''(v)}{\phi'(v)}|\nabla v|^2.$$

Assume for contradiction that w achieves a positive maximum at some $(t_0, x_0) \in D_h$. Then, at this point, we have $w_t \geqslant 0$, $\Delta w \leqslant 0$ and $\nabla w = 0$, i.e., $p_0 := \nabla v(t_0, x_0) = \nabla v(t_0, x_0 + ha)$. It follows that $0 \leqslant \mathrm{e}^{Kt_0}(w_t - \Delta w) = G(v(t_0, x_0 + ha)) - G(v(t_0, x_0))$, where

$$G(z) := \frac{F(\phi(z), \phi'(z)p_0)}{\phi'(z)} + \frac{\phi''(z)}{\phi'(z)}|p_0|^2 - Kz.$$

Since $v(t_0, x_0 + ha) > v(t_0, x_0)$, we shall thus reach a contradiction if $G(z)$ is (stricly) decreasing. We compute

$$G'(z) = F_u + \frac{\phi''}{\phi'^2}(p \cdot F_p - F) + \frac{1}{\phi'^2}\left(\frac{\phi''}{\phi'}\right)'|p|^2 - K.$$

Here and in what follows, F, F_u and $p \cdot F_p$ are evaluated at $u = \phi(z)$ and $p = \phi'(z)p_0$.

Case 1. Assume (3), (4). In view of the fact that $ug(u,p) \leqslant 0$ due to (4), we look for a function ϕ such that $\phi(0) = 0$ and $z\phi''(z) \leqslant 0$. We take $\phi(z) = \mathrm{e}M \int_0^z \exp(-\delta\,\mathrm{e}^{\lambda s^2})\,\mathrm{d}s$, $z \in J := [-1, 1]$, where $\lambda > 0$ and $0 < \delta \leqslant \mathrm{e}^{-\lambda}$ will be chosen below. We compute

$$\phi' = \mathrm{e}M \exp\bigl(-\delta\,\mathrm{e}^{\lambda z^2}\bigr), \quad \phi'' = -2\delta\lambda z\,\mathrm{e}^{\lambda z^2}\phi', \quad \left(\frac{\phi''}{\phi'}\right)' = -2\delta\lambda\,\mathrm{e}^{\lambda z^2}\bigl(1 + 2\lambda z^2\bigr),$$

hence

$$G'(z) = F_u + \frac{2\delta\lambda\,\mathrm{e}^{\lambda z^2}}{\phi'}\left(z(F - p \cdot F_p) - \frac{1 + 2\lambda z^2}{\phi'}|p|^2\right) - K.$$

Note that $M \leqslant \phi' \leqslant \mathrm{e}M$. In particular, we have $[-M, M] \subset \phi(J)$. By (3), there exist a_0, $a_1 > 0$ and, for each $\eta > 0$, there exists $C_\eta > 0$, such that

$$|f_u| \leqslant \eta|p|^2 + C_\eta, \quad |f| + |p \cdot f_p| \leqslant a_0|p|^2 + a_1, \quad |u| \leqslant M, \quad p \in \mathbb{R}^N.$$

Take $\lambda = 2(a_0\mathrm{e}M)^2$. Since $ug(u,p) \leqslant 0$ and $z\phi(z) \geqslant 0$, we get

$$z(F - p \cdot F_p) \leqslant a_0|z||p|^2 + a_1 + zg - zp \cdot g_p \leqslant \frac{1 + 2\lambda z^2}{2\mathrm{e}M}|p|^2 + a_1 - zp \cdot g_p.$$

Choosing $\delta = \mathrm{e}^{-\lambda}\min(1, \varepsilon M/2\lambda)$, $\eta = \delta\lambda/\mathrm{e}^2M^2$, $K > C_\eta + 2\lambda a_1/M$ and using (4), we obtain that for all $z \in J$,

$$\begin{aligned} G'(z) &\leqslant \eta|p|^2 + C_\eta + g_u + \frac{2\delta\,\lambda\,\mathrm{e}^{\lambda z^2}}{\phi'}\left(-\frac{1}{2\mathrm{e}M}|p|^2 + a_1 - zp \cdot g_p\right) - K \\ &\leqslant \left(\eta - \frac{\delta\lambda}{\mathrm{e}^2M^2}\right)|p|^2 + \left(\frac{2\lambda a_1}{M} + C_\eta - K\right) + g_u + \varepsilon|p \cdot g_p| < 0. \end{aligned}$$

Case 2. Assume (3), (5). Since now $p \cdot g_p - g \leqslant 0$, we look for a function ϕ such that $\phi''(z) \geqslant 0$. We take $\phi(z) = \mathrm{e}M \int_0^z \exp(-\mathrm{e}^{-\lambda s})\,\mathrm{d}s$, $z \in J := [0, 1]$, where $\lambda > 0$ will be chosen below. We compute

$$\phi' = \mathrm{e}M \exp\bigl(-\mathrm{e}^{-\lambda z}\bigr), \quad \phi'' = \lambda\,\mathrm{e}^{-\lambda z}\phi', \quad \left(\frac{\phi''}{\phi'}\right)' = -\lambda^2\,\mathrm{e}^{-\lambda z}.$$

Note again that $M \leqslant \phi' \leqslant \mathrm{e}M$. In particular, we have $[0, M] \subset \phi(J)$. By (3), (5), there exist $a_0,\ a_1 > 0$ and, for each $\eta > 0$, there exists $C_\eta > 0$, such that

$$F_u \leqslant \eta|p|^2 + C_\eta, \quad p \cdot F_p - F = p \cdot f_p - f + |p|p \cdot \frac{\partial}{\partial p}\left(\frac{g}{|p|}\right) \leqslant a_0|p|^2 + a_1, \quad 0 \leqslant u \leqslant M,\ p \in \mathbb{R}^N.$$

Taking $\lambda = 2a_0\,\mathrm{e}M,\ \eta = 2a_0^2\,\mathrm{e}^{-\lambda},\ K > C_\eta + \lambda a_1/M$ and using $g_u \leqslant 0$, it follows that for all $z \in J$,

$$G'(z) = F_u + \frac{\lambda\,\mathrm{e}^{-\lambda z}}{\phi'}\left(p \cdot F_p - F - \frac{\lambda}{\phi'}|p|^2\right) - K \leqslant \eta|p|^2 + C_\eta + \frac{\lambda\,\mathrm{e}^{-\lambda z}}{\phi'}\left(\left(a_0 - \frac{\lambda}{\mathrm{e}M}\right)|p|^2 + a_1\right) - K$$
$$\leqslant \left(\eta - 2a_0^2\,\mathrm{e}^{-\lambda}\right)|p|^2 + C_\eta + \frac{\lambda a_1}{M} - K < 0.$$

Claim 2.2. *It holds $|u(t,x)| \leqslant Cd(x)$ for $d(x) \leqslant d_0$, where $C,\ d_0 > 0$ depend only on $M,\ F,\ \Omega$.*

This can be proved by a rather standard barrier argument (cf., e.g., [8, Lemma VI.3.1]). Therefore we just sketch the proof. Let $U \geqslant 0$ be the solution of $-\Delta U = 1$ for $1 < |x| < 2$, with $U = 0$ for $|x| = 1$ and $U = 1$ for $|x| = 2$. Next put $V(x) = \beta^{-1}\,\mathrm{Log}(1 + \mathrm{e}^{\beta M}U(x/\rho))$, with $\beta \geqslant 1,\ \rho > 0$. A calculation shows that $-\Delta V \geqslant \beta|\nabla V|^2 + C_1/(\rho^2\beta)$ for $\rho < |x| < 2\rho$. Since $0 \leqslant V \leqslant M + C_2$ and $F(u,p) \leqslant \mathrm{O}(|p|^2)$ for $u \geqslant 0$, by taking β large and ρ small we obtain that $-\Delta V \geqslant F(V, \nabla V)$. On the other hand, we have $V(x) \geqslant C_3(\beta\rho)^{-1}(|x| - \rho)$ for $\rho < |x| < 2\rho$, and $V > M$ for $|x| = 2\rho$. Using the uniform exterior sphere condition (taking ρ smaller if necessary), one can then compare u with a translate of the barrier V to deduce that $u(t,x) \leqslant Cd(x)$ for $d(x) \leqslant d_0$. If $u \geqslant 0$ (cf. Theorem 1.2), we are done. Under the assumptions of Theorem 1.1, the lower inequality follows by comparing with $-V$.

We can now conclude the proof of Theorems 1.1 and 1.2. Since $w(t,x) \to 0$ as $|x| \to \infty$, uniformly in t (in case Ω is unbounded), Claim 2.1 implies that the supremum of w in $\overline{D}_h$ is attained on the parabolic boundary of D_h (unless it is 0). Since $|\nabla u_0| \leqslant M$ and $\phi' \geqslant c_0 > 0$ in J, we have $w(0,x) \leqslant Ch$ in Ω_h and Claim 2.2 implies that $w(t,x) \leqslant Ch$ if $h \leqslant d_0/2$ and $d(x) = h$. Since a was any unit vector, by letting $h \to 0$ we deduce that $|\nabla v| \leqslant C\mathrm{e}^{KT}$ in $[0,T] \times \Omega$ and the conclusion follows. □

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