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A composition formula for squares of Hermite polynomials and its generalizations

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Abstract

We prove a general formula which, with appropriately chosen parameters, gives a composition formula for squares of Gould–Hopper polynomials $g_n^2(x,h)$, and hence also for Hermite polynomials. Our main tool is the classical Mehler formula, but with imaginary arguments. ***To cite this article: P. Graczyk, A. Nowak, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Une formule de composition pour les carrés des polynômes d'Hermite, et sa généralisation. Nous démontrons une formule générale qui, avec des coefficients convenablement choisis, donne une formule de composition pour les carrés des polynômes de Gould–Hopper $g_n^2(x,h)$ et, par conséquent, pour les carrés des polynômes d'Hermite. Notre outil principal est la formule de Mehler classique avec l'argument imaginaire. ***Pour citer cet article : P. Graczyk, A. Nowak, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Let $n \geqslant 1$ be a natural number. In what follows we shall use fairly standard multi-index notation, with multi-indices $\widetilde{m} = (m_1, \ldots, m_n) \in \mathbb{N}^n$ (in the sequel multi-indices will always be distinguished by tildes). In particular

$$|\widetilde{m}| = m_1 + \cdots + m_n$$

is the length of $\widetilde{m}$,

$$\widetilde{m}/2 = (m_1/2, \ldots, m_n/2), \quad \widetilde{m}! = m_1! \cdots m_n!, \quad x^{\widetilde{m}} = x_1^{m_1} \cdots x_n^{m_n}, \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Moreover,

$$\bar{0} \leqslant \tilde{k} \leqslant \widetilde{m} \equiv 0 \leqslant k_i \leqslant m_i, \quad 1 \leqslant i \leqslant n,$$

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where $\bar{0}$ denotes the multidimensional zero $(0,\dots,0)\in\mathbb{N}^n$. Given $x\in\mathbb{R}^n$, by $|x|$ we denote its Euclidean norm in $\mathbb{R}^n$.

For $\tilde{k}\leqslant\widetilde{m}/2$ we define $E_{\widetilde{m},\tilde{k}}=E_{m_1,k_1}\cdots E_{m_n,k_n}$, where

$$E_{N,k}=2^{N-2k}\frac{N!}{k!(N-2k)!},\quad N=0,1,2,\dots,\ 0\leqslant k\leqslant\frac{N}{2}.$$

Note that $E_{N,k}$ is the absolute value of the coefficient standing at $(N-2k)$-th power in the N-th Hermite polynomial H_N, since

$$H_N(u)=\sum_{0\leqslant k\leqslant N/2}(-1)^k E_{N,k}u^{N-2k},\quad N=0,1,2,\dots.$$

The numbers $E_{N,k}$ appear also in certain generalizations of Hermite polynomials, for instance in the polynomials

$$g_N^2(2u,p)=\sum_{0\leqslant k\leqslant N/2}E_{N,k}\,p^k u^{N-2k},\quad u,p\in\mathbb{R},$$

considered by Gould and Hopper [1]. Noteworthy, $g_N^2(u,p)$ are contained in certain general classes of hypergeometric polynomials studied by Brafman, Gould and Hopper, and others, see [4, Chapter 1, Section 1.9] and references therein. Multidimensional polynomials $g^2_{\widetilde{m}}(x,p)$, $\widetilde{m}\in\mathbb{N}^n$, $x\in\mathbb{R}^n$, are defined in a standard manner to be the tensor products $g^2_{m_1}(x_1,p)\cdots g^2_{m_n}(x_n,p)$, and similarly for multidimensional Hermite polynomials $H_{\widetilde{m}}(x)$, $x\in\mathbb{R}^n$.

Our main result can be expressed as follows.

Theorem 1. *Let M be an arbitrary natural number and let $\{\beta_j\}_{0\leqslant j\leqslant M/2}$ be arbitrary complex numbers. Then, for all $x\in\mathbb{R}^n$, we have*

$$\sum_{|\widetilde{m}|=M}\frac{M!}{\widetilde{m}!}\Bigg(\sum_{\bar{0}\leqslant\tilde{k}\leqslant\widetilde{m}/2}E_{\widetilde{m},\tilde{k}}x^{\widetilde{m}-2\tilde{k}}\beta_{|\tilde{k}|}\Bigg)^2 =\sum_{0\leqslant j\leqslant M/2}\Bigg[\frac{2^{2j}}{j!}(-M)_{2j}\left(\frac{n-1}{2}\right)_j\Bigg(\sum_{0\leqslant k\leqslant M/2-j}E_{M-2j,k}|x|^{M-2j-2k}\beta_{k+j}\Bigg)^2\Bigg],\tag{1}$$

where $(\lambda)_k$ denotes the Pochhammer symbol (recall that $(\lambda)_k=1$ if $k=0$ and $(\lambda)_k=\lambda(\lambda+1)\cdots(\lambda+k-1)$ for $k=1,2,\dots$).

Proof. After expanding both sides, the identity (1) takes the form

$$\sum_{|\widetilde{m}|=M}\frac{M!}{\widetilde{m}!}\sum_{\substack{\bar{0}\leqslant\tilde{k}\leqslant\widetilde{m}/2\\ \bar{0}\leqslant\tilde{l}\leqslant\widetilde{m}/2}}E_{\widetilde{m},\tilde{k}}E_{\widetilde{m},\tilde{l}}x^{2\widetilde{m}-2\tilde{k}-2\tilde{l}}\beta_{|\tilde{k}|}\beta_{|\tilde{l}|} =\sum_{0\leqslant j\leqslant M/2}\frac{2^{2j}}{j!}(-M)_{2j}\left(\frac{n-1}{2}\right)_j\sum_{\substack{0\leqslant k\leqslant M/2-j\\ 0\leqslant l\leqslant M/2-j}}E_{M-2j,k}E_{M-2j,l}|x|^{2M-4j-2k-2l}\beta_{k+j}\beta_{l+j},\tag{2}$$

and therefore it suffices to show that the coefficients of $\beta_r\beta_s$, $0\leqslant r,s\leqslant M/2$, are equal on both sides.

To proceed, we note that classical Mehler's formula (cf. [4, Chapter 1, Section 1.11], for instance) holds also with imaginary arguments (which seems to be well-known, and may be proved by essentially the same reasoning as in case of real arguments, see the proof in [3]). Therefore, observing that

$$g_N^2(2u,p)=\left(\frac{\mathrm{i}}{\sqrt{p}}\right)^{-N}H_N\left(\frac{\mathrm{i}}{\sqrt{p}}u\right),\quad p\neq 0,$$

and using "imaginary" version of Mehler's formula, one gets

$$\sum_{N=0}^{\infty} g_N^2(2u,p)g_N^2(2v,q)\frac{z^N}{2^N N!} = \sum_{N=0}^{\infty} H_N\left(\frac{\mathrm{i}}{\sqrt{p}}u\right)H_N\left(\frac{\mathrm{i}}{\sqrt{q}}v\right)\frac{(-\sqrt{pq}z)^N}{2^N N!}$$
$$= \frac{1}{\sqrt{1-pqz^2}}\exp\left(\frac{z^2(qu^2+pv^2)+2zuv}{1-pqz^2}\right),$$

provided $|z| < |pq|^{-1/2}$. In particular, for $u = v$ we have

$$\sum_{N=0}^{\infty} g_N^2(2u,p)g_N^2(2u,q)\frac{z^N}{2^N N!} = \frac{1}{\sqrt{1-pqz^2}}\exp\left(u^2\frac{(p+q)z^2+2z}{1-pqz^2}\right).$$

Consequently,

$$\prod_{i=1}^{n}\left[\sum_{m_i=0}^{\infty} g_{m_i}^2(2x_i,p)g_{m_i}^2(2x_i,q)\frac{z^{m_i}}{2^{m_i}(m_i)!}\right]$$
$$= \left(1-pqz^2\right)^{-(n-1)/2}\frac{1}{\sqrt{1-pqz^2}}\exp\left(|x|^2\frac{(p+q)z^2+2z}{1-pqz^2}\right)$$
$$= \left(\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{n-1}{2}\right)_j (pq)^j z^{2j}\right)\left(\sum_{k=0}^{\infty} g_k^2\big(2|x|,p\big)g_k^2\big(2|x|,q\big)\frac{z^k}{2^k k!}\right),$$

since

$$(1-u)^{-(n-1)/2} = \sum_{j=0}^{\infty}\left(\frac{n-1}{2}\right)_j \frac{u^j}{j!}, \quad |u| < 1.$$

Further, comparing the coefficients of z^M we obtain

$$\sum_{|\widetilde{m}|=M}\frac{M!}{\widetilde{m}!}g_{\widetilde{m}}^2(2x,p)g_{\widetilde{m}}^2(2x,q)$$
$$= \sum_{0\leqslant j\leqslant M/2}\frac{2^{2j}}{j!}(-M)_{2j}\left(\frac{n-1}{2}\right)_j (pq)^j g_{M-2j}^2\big(2|x|,p\big)g_{M-2j}^2\big(2|x|,q\big), \tag{3}$$

which after expansions yields

$$\sum_{|\widetilde{m}|=M}\frac{M!}{\widetilde{m}!}\sum_{\substack{\bar{0}\leqslant\tilde{k}\leqslant\widetilde{m}/2\\ \bar{0}\leqslant\tilde{l}\leqslant\widetilde{m}/2}} E_{\widetilde{m},\tilde{k}}E_{\widetilde{m},\tilde{l}}x^{2\widetilde{m}-2\tilde{k}-2\tilde{l}}p^{|\tilde{k}|}q^{|\tilde{l}|}$$
$$= \sum_{0\leqslant j\leqslant M/2}\frac{2^{2j}}{j!}(-M)_{2j}\left(\frac{n-1}{2}\right)_j \sum_{\substack{0\leqslant k\leqslant M/2-j\\ 0\leqslant l\leqslant M/2-j}} E_{M-2j,k}E_{M-2j,l}|x|^{2M-4j-2k-2l}p^{k+j}q^{l+j}.$$

Since the above identity holds for all $p,q>0$, it follows that the coefficients of $p^r q^s$, $0\leqslant r,s\leqslant M/2$, are equal on both sides. This, in view of (2), finishes the proof. □

We note two important applications of Theorem 1.

Theorem 2. *For all $M\in\mathbb{N}$, $p\in\mathbb{R}$ and $x\in\mathbb{R}^n$ we have*

$$\sum_{|\widetilde{m}|=M} \frac{1}{\widetilde{m}!}\big[g^2_{\widetilde{m}}(x,p)\big]^2 = \sum_{0\leqslant j\leqslant M/2} \frac{(2p)^{2j}}{j!(M-2j)!}\left(\frac{n-1}{2}\right)_j \big[g^2_{M-2j}\big(|x|,p\big)\big]^2, \tag{4}$$

$$\sum_{|\widetilde{m}|=M} \frac{1}{\widetilde{m}!}\big[H_{\widetilde{m}}(x)\big]^2 = \sum_{0\leqslant j\leqslant M/2} \frac{2^{2j}}{j!(M-2j)!}\left(\frac{n-1}{2}\right)_j \big[H_{M-2j}\big(|x|\big)\big]^2. \tag{5}$$

The composition formula for squares of Gould–Hopper polynomials (4) is obtained by using Theorem 1 with $\beta_j = p^j$ (or taking $p = q$ in (3)). The identity for Hermite polynomials (5) follows by taking $\beta_j = (-1)^j$ in (1), or directly from (4), since $H_N(u) = g^2_N(2u, -1)$. Noteworthy, a bilinear version of (4) in p was derived and exploited in the proof above, see (3).

Let us also remark, that there is no bilinear version of (1). Nevertheless, by using the technique presented in the proof of Theorem 1, one may derive a bilinear extension of (4), and hence also of (5). In fact, the following holds.

Theorem 3. *For all $M \in \mathbb{N}$, $p \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$ we have*

$$\sum_{|\widetilde{m}|=M} \frac{1}{\widetilde{m}!} g^2_{\widetilde{m}}(x,p) g^2_{\widetilde{m}}(y,p) = \sum_{0\leqslant j\leqslant M/2} \frac{(2p)^{2j}}{j!(M-2j)!}\left(\frac{n-1}{2}\right)_j g^2_{M-2j}(X,p) g^2_{M-2j}(Y,p),$$

and

$$\sum_{|\widetilde{m}|=M} \frac{1}{\widetilde{m}!} H_{\widetilde{m}}(x) H_{\widetilde{m}}(y) = \sum_{0\leqslant j\leqslant M/2} \frac{2^{2j}}{j!(M-2j)!}\left(\frac{n-1}{2}\right)_j H_{M-2j}(X) H_{M-2j}(Y),$$

where

$$X = \frac{|x+y| + |x-y|}{2}, \qquad Y = \frac{|x+y| - |x-y|}{2}.$$

The necessity of deriving multivariate versions of (1) arises in harmonic analysis of orthogonal expansions, in connection with studying mutual dependence of higher order Riesz–Hermite and Riesz–Laguerre transforms, cf. [2]. However, we were not able to find in accessible literature even the simplest discussed consequence of Theorem 1, namely the composition formula for squares of Hermite polynomials (5).

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