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Algebraic Geometry

Image of the Nash map in terms of wedges

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Abstract

M. Lejeune-Jalabert (Lecture Notes in Math., vol. 777, Springer-Verlag, 1980, pp. 303–336) proposed the following 'problem of wedges': let X be a surface over an algebraically closed field k of characteristic zero. Given a wedge $\phi : Spec\, k[[\xi, t]] \to X$, whose special arc lifts to the minimal resolution Y of X in an arc transversal to an irreducible component of the exceptional locus in a general point, does ϕ lift to Y? The main result in this Note is to extend this problem to a problem of wedges in a variety X of any dimension and to prove that, if the wedge problem is true for X, then the Nash problem is true for X. From this, necessary and sufficient conditions are given for an essential divisor to belong to the image of the Nash map, and we conclude that the Nash problem is true for sandwiched surface singularities. ***To cite this article: A.J. Reguera, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Image de l'application de Nash en termes de coins. M. Lejeune-Jalabert (Lecture Notes in Math., vol. 777, Springer-Verlag, 1980, pp. 303–336) a proposé le « problème de coins » suivant : soit X une surface sur un corps algébriquement clos de caractéristique zéro. Étant donné un coin $\phi : Spec\, k[[\xi, t]] \to X$, dont son arc spécial se relève à la résolution minimale Y de X en un arc transverse à une composante irréductible du lieu exceptionnel en un point général, ϕ se relève-t'il à Y ? Le résultat principal de cette Note est d'étendre ce problème à un problème de coins sur une variété X de dimension supérieure, et de démontrer que si le problème de coins est vrai pour X, alors le problème de Nash est vrai pour X. On en déduit des conditions nécessaires et suffisantes pour qu'un diviseur essentiel appartienne à l'image de l'application de Nash, et on conclut que le problème de Nash est vrai pour les singularités sandwich de surface. ***Pour citer cet article : A.J. Reguera, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Version française abrégée

Soit k un corps algébriquement clos, non dénombrable et de caractéristique zéro. Soit X une k-variété et S son lieu singulier. Un *diviseur essentiel sur* X est une valuation divisorielle ν du corps de fonctions $k(X)$ de X

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telle que, pour toute désingularisation $p : Y \to X$, le centre de ν dans Y est une composante irréductible du lieu exceptionnel de p.

Considérons le k-schéma (non de type fini) X_∞ des arcs sur X, et le fermé X_∞^S des arcs centrés en un point non spécifié de S. Étant donné un diviseur essentiel ν, soit E le centre de ν dans une désingularisation fixée $p : Y \to X$, et soit N_E l'adhérence de l'image, par l'application naturelle $Y_\infty \to X_\infty$, de l'ensemble des arcs sur Y centrés en un point non spécifié de E. L'ensemble N_E est un fermé irréductible de X_∞^S qui dépend seulement du diviseur essentiel ν, et qui détermine aussi ν. On a $X_\infty^S = \bigcup_E N_E$, où E varie parmi les centres des diviseurs essentiels dans une désingularisation fixée quelconque.

L'application de Nash est l'application $\mathcal{N}$ de l'ensemble des composantes irréductibles de X_∞^S dans l'ensemble des diviseurs essentiels de X, définie par $\mathcal{N}(N_E) := \nu$, où E est le centre de ν.

Étant donnée une extension de corps K de k, un K-coin sur X est un k-morphisme $\Phi : \mathit{Spec}\, K[[\xi, t]] \to X$. On appelle respectivement *arc générique* et *arc spécial de* Φ les arcs $\mathit{Spec}\, K((\xi))[[t]] \to X$ et $\mathit{Spec}\, K[[t]] \to X$ induits par les morphismes $K[[\xi, t]] \hookrightarrow K((\xi))[[t]]$ et $K[[\xi, t]] \hookrightarrow K[[t]]$, $\xi \mapsto 0$.

Le résultat principal de cette Note est :

Théorème 0.1. *Soit E_α un diviseur essentiel sur X. Soit z_α le point générique de N_{E_α}, et k_α son corps résiduel. Les conditions suivantes sont équivalentes* :

(i) *E_α appartient à l'image de l'application de Nash.*
(ii) *Pour toute désingularisation $p : Y \to X$ et pour toute extension de corps K de k_α, tout K-coin Φ sur X dont l'arc spécial est z_α et dont l'arc générique appartient à X_∞^S, se relève à Y.*
(ii′) *Il existe une désingularisation satisfaisant* (ii).

La partie fondamentale de la preuve est (ii′) $\Rightarrow$ (i). Celle-ci se réduit à un résultat de finitude dans l'espace des arcs : dans le Lemme 3.2, il est prouvé qu'il existe un ouvert affine W de X_∞ tel que $N_{E_\alpha} \cap W$ est un fermé non vide de W et dont l'idéal est finiment engendré. En particulier, l'anneau $\widehat{\mathcal{O}_{X_\infty, z_\alpha}}$ est noethérien, et il est alors possible d'appliquer le Lemme de Sélection de Courbe dans $\mathit{Spec}\, \widehat{\mathcal{O}_{X_\infty, z_\alpha}}$.

Comme conséquence du Théorème 2.1 et de [6], on obtient que *l'application de Nash est bijective pour les singularités sandwich de surface*.

1. Introduction

Let k be an uncountable algebraically closed field of characteristic zero, and let X be a k-variety (i.e., a reduced and irreducible scheme of finite type over k) with singular locus S. Let $\pi : X_\infty \to X$ be the canonical projection from the space of arcs X_∞ on X to X and let $j_n : X_\infty \to X_n$ be the projection from X_∞ to the space of n-jets X_n of X. For any closed subset C of X_∞ (resp. C_n of X_n) we will consider the schemes C, C_n with the reduced structure (see the introduction in [1]). Recall that $X_\infty = \lim_{\leftarrow} X_n$; that, for a field extension K of k, the K-points of X_∞ are in 1–1 correspondence with the arcs $\mathit{Spec}\, K[[t]] \to X$, and that, given $x \in X_\infty$ with residue field k_x, we have $\pi(x) = h_x(0)$ where $h_x : \mathit{Spec}\, k_x[[t]] \to X$ is the corresponding arc and 0 is the closed point of $\mathit{Spec}\, k_x[[t]]$ (see [1,4]).

The *Nash map* is a canonical map $\mathcal{N}$ from the set of irreducible components of $X_\infty^S := \pi^{-1}(S)$ into the set of essential components on a resolution of singularities Y of X. An *essential component* on Y is the center on Y of an essential divisor over X. An *essential divisor* over X is a divisorial valuation ν of the function field $k(X)$ of X centered in S such that the center of ν on any desingularization $p : Y \to X$ is an irreducible component of the exceptional locus $p^{-1}(S)$ on Y. Note that the set $\mathcal{E}_{Y/X}$ of essential components on Y is in 1–1 correspondence with the set of essential divisors $\mathcal{E}$ over X, hence the map $\mathcal{N}$ does not depend on Y.

We now outline the construction of the Nash map $\mathcal{N}$. Since k is a field of characteristic zero, by [4] Lemma 2.12, the arc h_z corresponding to the generic point of an irreducible component C of X_∞^S does not factor through S. Since p is proper and is an isomorphism outside S, there exists a unique arc $\tilde{h}_z$ on Y such that $h_z = p \circ \tilde{h}_z$. Its center $\tilde{h}_z(0)$ is the generic point of an essential component E on Y. The Nash map sends C to E; it is injective, but need not be surjective as shown by the 4-dimensional example given in [4].

In this Note, we give necessary and sufficient conditions for an essential divisor E to be in the image of the Nash map. We follow the strategy introduced in [5] which consists in proving that "the wedge problem implies the Nash problem".

2. Main result

We now introduce the necessary concepts to state our result.

(1) The generic point of the inverse image of an essential component E on a resolution of singularities Y of X under the canonical map $Y_\infty \to Y$ projects to the generic point of a closed subvariety N_E of X_∞^S by $Y_\infty \to X_\infty$. The variety N_E only depends on the divisorial valuation ν centered at E on Y, and we have $N_{\mathcal{N}(C)} = C$ for any irreducible component C of X_∞^S. Therefore

$$X_\infty^S = \bigcup_{E \in \mathcal{E}} N_E.$$

For any $x \in X_\infty \setminus S_\infty$, we will denote by $\tilde{h}_x : \mathit{Spec}\, k_x[[t]] \to Y$ the unique arc on Y lifting h_x. Two subspaces of N_E depending on Y will play a role: $N_E(Y) := \{x \in X_\infty \setminus S_\infty;\ \tilde{h}_x(0) \in E\}$, and $N_E^0(Y) := \{x \in N_E(Y);\ \tilde{h}_x$ intersects E transversally at a nonsingular point of $p^{-1}(S)_{\mathrm{red}}\}$. We will prove in Lemma 3.1 that both sets $N_E(Y)$ and $N_E^0(Y)$ are dense in N_E.

(2) For any field extension K of k, a morphism $\phi : \mathit{Spec}\, K[[\xi, t]] \to X$ is called a K-wedge on X; ϕ may also be viewed as a $K[[\xi]]$-point h_ϕ of X_∞, via the isomorphism $K[[\xi, t]] \cong K[[\xi]][[t]]$. The arcs defined by $h_\phi(0)$ and $h_\phi(\mathit{Spec}\, K((\xi))$ are called the special arc and the generic arc of ϕ, respectively.

Theorem 2.1. *Let E_α be an essential divisor over X. We will denote by z_α the generic point of $N_\alpha := N_{E_\alpha}$, and by k_α its residue field. The following conditions are equivalent*:

- (i) *E_α belongs to the image of the Nash map.*
- (ii) *For any resolution of singularities $p : Y \to X$ and for any field extension K of k_α, any K-wedge ϕ on X such that $h_\phi(0) = z_\alpha$ and $h_\phi(\mathit{Spec}\, K((\xi))) \in X_\infty^S$ lifts to Y.*
- (ii′) *There exists a resolution of singularities $p : Y \to X$ satisfying* (ii).

Remark 1. We may rephrase (ii) by saying that, for any p, any K-arc on the germ (X_∞^S, z_α) can be lifted uniquely to $(Y_\infty, \tilde{z}_\alpha)$ where $\tilde{z}_\alpha$ is the point of Y_∞ corresponding to $\tilde{h}_{z_\alpha}$.

Corollary 2.2. *The Nash map is bijective in the following cases*:

- (i) *Quasi-homogeneous surface singularities listed in* [5].
- (ii) *Sandwiched surface singularities* (*see* [6]).

If X is a surface and h_{z_α} is a smooth arc on X, then E_α belongs to the image of the Nash map (*see* [3]).

Idea of the proof of Theorem 2.1. (i) $\Rightarrow$ (ii). Let ϕ be a K-wedge as in (ii) and let h_ϕ be the corresponding arc on X_∞. Let η be the generic point of $\mathit{Spec}\, K[[\xi]]$. We have that $z_\alpha := h_\phi(0)$ is a specialization of $z := h_\phi(\eta)$.

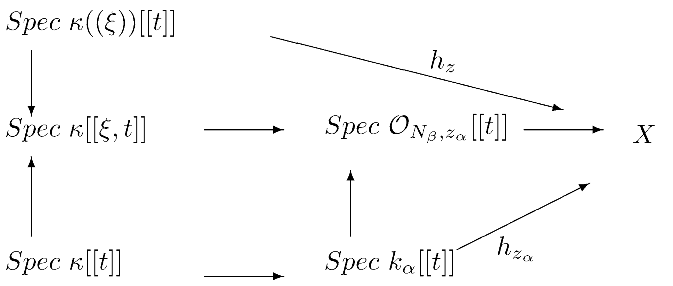

Fig. 1. Commutative diagram.

There exists β in $\mathcal{E}$ such that $z \in N_\beta$. Therefore $z_\alpha \in \overline{\{z\}} \subseteq N_\beta$. By (i), we have $N_\alpha = N_\beta$, hence $z = z_\alpha$ and the wedge lifts trivially to Y.

(ii) $\Rightarrow$ (ii′) is clear.

(ii′) $\Rightarrow$ (i). Assume that $N_\alpha \subset N_\beta$, $N_\alpha \neq N_\beta$, and let $p : Y \to X$ be a resolution of singularities satisfying (ii′). The natural inclusion $(N_\beta, z_\alpha) \subset X_\infty$ corresponds to a morphism $\mathit{Spec}\, \mathcal{O}_{N_\beta, z_\alpha}[[t]] \to X$. Here $\mathcal{O}_{N_\beta, z_\alpha}$ denotes the local ring of N_β (with its reduced structure) at the generic point z_α of N_α. We will build a commutative diagram as in Fig. 1 such that $z \in N_\beta \setminus N_\alpha$ and κ is a field extension of k_α. Hence $\tilde{h}_z(0)$ does not belong to the center of E_α on Y, thay we also denote by E_α. By (ii′), the wedge $\mathit{Spec}\, \kappa[[\xi, t]] \to X$ lifts to Y. This implies that the generic point of the *essential* component E_α on Y is a specialization of $\tilde{h}_z(0) \notin E_\alpha$, which is a contradiction. We will see in the next section that the diagram in Fig. 1 follows from Lemmas 3.1, 3.2 and 3.3. This construction is inspired by the sufficient condition to Nash problem in [9], Theorem 1.10 (see also the more recent work [8]).

3. Proof of (ii′) ⇒ (i)

We may assume that X is affine, let $X \subseteq \mathbb{A}^N_k$. The idea for the next result comes from [7], section “Correspondence of families to components”.

Lemma 3.1. *Let $p : Y \to X$ be a resolution of singularities, E_α an essential component on Y, and $\mathcal{P}$ the prime ideal of $\mathcal{O}_{X_\infty}$ defining N_α. Then,*

(i) *$N^0_\alpha(Y)$ is a nonempty open subset of N_α.*
(ii) *There exists $G_0 \in \mathcal{O}_{X_\infty} \setminus \mathcal{P}$ and a finitely generated ideal $\mathcal{I}$ of $\mathcal{O}_{X_\infty}$, such that the radical of $\mathcal{I}(\mathcal{O}_{X_\infty})_{G_0}$ is $\mathcal{P}(\mathcal{O}_{X_\infty})_{G_0}$.*

Proof. We may assume that E_α is a divisor of Y. Let U be an affine chart of Y such that $U \cap E_\alpha \neq \emptyset$ and E_α is defined in U by a single equation $\ell \in \mathcal{O}(U)$. Let z_α be the generic point of N_α, and ν_α the divisorial valuation of $k(X)$ centered on E_α. Let $f_i \in \mathcal{O}_X$, $0 \leqslant i \leqslant M$, be such that the birational map $X \cdots \to U$ is given by $y_i = f_i/f_0$, $1 \leqslant i \leqslant M$, and let $a \in \mathbb{N}$ and p a polynomial over k in $M+1$ variables such that

$$\ell\left(\frac{f_1}{f_0}, \dots, \frac{f_M}{f_0}\right) = \frac{p(f_0, \dots, f_M)}{f_0^a}.$$

Since $\tilde{h}^\sharp_{z_\alpha}(0)$ is the generic point of E_α, we have $b_i := \mathit{ord}_t\, h^\sharp_{z_\alpha}(f_i) < \infty$ for $0 \leqslant i \leqslant M$. Any $x \in X_\infty$ such that $\mathit{ord}_t\, h^\sharp_x(f_i) = b_i$ for $0 \leqslant i \leqslant M$ lifts to an arc $\tilde{h}_x$ on Y and, for such an x, the condition $\mathit{ord}_t\, h^\sharp_x(p(f_0, \dots, f_M)) = 1 + ab_0$ is equivalent to $\mathit{ord}_t\, \tilde{h}^\sharp_x(\ell) = 1$. Thus, $\Omega = \{x \in X_\infty \mid \mathit{ord}_t\, h^\sharp_x(f_i) = b_i,\ 0 \leqslant i \leqslant M,\ \mathit{ord}_t\, h^\sharp_x(p(f_0, \dots, f_M)) = 1 + ab_0\}$ is a nonempty open subset of N_α contained in $N^0_\alpha(Y)$. In fact,

$\Omega = D(G_0) \cap C$, for some $G_0 \in \mathcal{O}_{X_\infty} \setminus \mathcal{P}$, where $D(G_0) := (G_0 \neq 0)$ and C is the zero set of a finitely generated ideal $\mathcal{I}$ of $\mathcal{O}_{X_\infty}$ contained in $\mathcal{P}$. Hence $N_\alpha \cap D(G_0) = C \cap D(G_0)$, and (ii) follows from Nullstellensatz, since k is uncountable. From this argument applied to a finite cover of the set of nonsingular points of $p^{-1}(S)_{\mathrm{red}}$ in E_α by affine open subsets U, (i) follows. In particular, this implies that $b_i = \nu_\alpha(f_i)$. □

Let $\mathcal{O}_n := \mathcal{O}_{\overline{j_n(X_\infty)}}$, for $n \in \mathbb{N}$, and let $\mathcal{P}_n$ be the prime ideal of $\mathcal{O}_n$ defining $\overline{j_n(N_\alpha)}$. We have $\mathcal{O}_n \subseteq \mathcal{O}_{n+1}$, $\mathcal{O}_{X_\infty} = \bigcup_n \mathcal{O}_n$ and $\mathcal{P}_n = \mathcal{P} \cap \mathcal{O}_n$.

Lemma 3.2. *There exists $G \in \mathcal{O}_{X_\infty} \setminus \mathcal{P}$ such that the ideal $\mathcal{P}(\mathcal{O}_{X_\infty})_G$ is finitely generated.*

Proof. There is a closed subscheme X' of $\mathbb{A}_k^N$ such that $X' \supseteq X$, X' is a complete intersection and the arc h_{z_α} does not factor through the closure of $X' \setminus X$ (see [1] or [2]). Then X'_∞ and X_∞ are isomorphic in a neighbourhood of z_α. Hence, we may assume that X is a complete intersection. Following [1], proof of Lemma 4.1, it suffices to understand the case where X is a hypersurface. Suppose that it is defined by $f(\underline{x}) = 0$, where $\underline{x} = (x_1, \ldots, x_{d+1})$, and $e := \nu_\alpha(\frac{\partial f}{\partial x_1}) = ord_t\, h^\sharp_{z_\alpha}(\frac{\partial f}{\partial x_1}) \in \mathbb{N}$ is the ν_α-value of the Jacobian ideal of X. Let $h_\infty : \mathbb{A}^{d+1}_\infty \widehat{\times}\, Spec\, k[[t]] \to \mathbb{A}^{d+1}$ be the universal family, and $h^\sharp_\infty(f) = \sum_i \mathbf{F}_i t^i, h^\sharp_\infty(\frac{\partial f}{\partial x_j}) = \sum_i \mathbf{Q}_{j,i} t^i \in \mathcal{O}_{\mathbb{A}^{d+1}_\infty}[[t]]$, for $1 \leqslant j \leqslant d+1$. Then $\mathcal{O}_{X_\infty} = \mathcal{O}_{\mathbb{A}^{d+1}_\infty} / \sqrt{(\{\mathbf{F}_i\}_{i \geqslant 0})} = k[\underline{\mathbf{X}}_0, \underline{\mathbf{X}}_1, \ldots] / \sqrt{(\{\mathbf{F}_i\}_{i \geqslant 0})}$ where $\underline{\mathbf{X}}_i = (\mathbf{X}_{1,i}, \ldots, \mathbf{X}_{d+1,i})$ and $\mathbf{F}_i, \mathbf{Q}_{j,i} \in k[\underline{\mathbf{X}}_0, \ldots, \underline{\mathbf{X}}_i]$ for all j. For any $\mathbf{H} \in k[\underline{\mathbf{X}}_0, \underline{\mathbf{X}}_1, \ldots]$, we will denote by H its class in $\mathcal{O}_{X_\infty}$. Note that $Q_{j,i} \in \mathcal{O}_i$. From the argument of elimination, in [1], Lemma 4.1, based on Taylor's formula (see also [5]), it follows that, for $n \geqslant e$,

$$\mathbf{F}_{n+e+1} = \mathbf{H}_{n+e+1} + \sum_{j=1}^{d+1} \sum_{i=0}^{e} \mathbf{Q}_{j,i}\, \mathbf{X}_{j,n+e+1-i},$$

where $\mathbf{H}_{n+e+1} \in k[\underline{\mathbf{X}}_0, \ldots, \underline{\mathbf{X}}_n]$ is the coefficient in t^{n+e+1} of $f(\sum_{i=0}^n \underline{\mathbf{X}}_i t^i)$ (see (1′) in [1]). Let $F'_{n+e+1} := H_{n+e+1} + \sum_{j=1}^{d+1} Q_{j,e} X_{j,n+1} \in \mathcal{O}_{n+1}$. Then $0 = F'_{n+e+1} + \sum_{j=1}^{d+1} \sum_{i=0}^{e-1} Q_{j,i} X_{j,n+e+1-i}$ in $\mathcal{O}_{X_\infty}$, hence F'_{n+e+1} belongs to the finitely generated ideal $\mathcal{Q} \subseteq \mathcal{P}$ in $\mathcal{O}_{X_\infty}$ generated by $\{Q_{j,i}\}_{1 \leqslant j \leqslant d+1, 0 \leqslant i < e}$. Thus, $(\mathcal{P}_n + (F'_{n+e+1}))\mathcal{O}_{n+1} \subseteq \mathcal{P}_{n+1}$ for $n \geqslant e$.

Let $G_0 \in \mathcal{O}_{X_\infty} \setminus \mathcal{P}$ and $\mathcal{I}$ as in (ii) of Lemma 3.1, and let $n_0 \in \mathbb{N}$ be such that G_0 and a system of generators of $\mathcal{I}$ are in $\mathcal{O}_{n_0}$. Let $n_1 = \sup\{n_0, e\}$ and $G = G_0 \cdot Q_{1,e}$. For $n \geqslant n_1$, let us consider the morphisms

$$(\mathcal{O}_n)_G/\mathcal{P}_n[\mathbf{X}_{2,n+1}, \ldots, \mathbf{X}_{d+1,n+1}] \xrightarrow{\delta_n} (\mathcal{O}_{n+1})_G/\big(\mathcal{P}_n + (F'_{n+e+1})\big)(\mathcal{O}_{n+1})_G \xrightarrow{\gamma_n} (\mathcal{O}_{n+1})_G/\mathcal{P}_{n+1},$$

where $\delta_n(\mathbf{X}_{j,n+1}) = X_{j,n+1}$. By the elimination argument, $\overline{j_n(N_\alpha)} \cap D(G) = j_n(N_\alpha) \cap D(G)$ for $n \geqslant n_1$. By Lemma 4.1 in [1], $\gamma_n \circ \delta_n$ is an isomorphism. This follows from the following two arguments: the first one is Hensel's Lemma, which implies that $j_{n+1}(N_\alpha) \cap D(G)$ is defined as subset of $(j_n(N_\alpha) \cap D(G)) \times \mathbb{A}_k^{d+1}$ by $F'_{n+e+1} = 0$, i.e., $(\mathcal{O}_{n+1})_G/\mathcal{P}_{n+1} \cong ((\mathcal{O}_n)_G/\mathcal{P}_n)[\underline{\mathbf{X}}_{n+1}]/\sqrt{(F'_{n+e+1})}$ ([1], p. 219, ls. 16–18). The second one is the elimination of $\mathbf{X}_{1,n+1}$ in the equation $F'_{n+e+1} = 0$, since $Q_{1,e}$ is invertible, hence $(\mathcal{O}_{n+1})_G/\mathcal{P}_{n+1} \cong ((\mathcal{O}_n)_G/\mathcal{P}_n)[\underline{\mathbf{X}}_{2,n+1}, \ldots, \underline{\mathbf{X}}_{d+1,n+1}]$. This elimination argument also implies that $X_{1,n+1}$ belongs to the image of δ_n, i.e., δ_n is surjective. Therefore, both δ_n and γ_n are isomorphisms, thus $\mathcal{P}_{n+1}(\mathcal{O}_{n+1})_G = (\mathcal{P}_n + (F'_{n+e+1}))(\mathcal{O}_{n+1})_G$ and it follows that $\mathcal{P}(\mathcal{O}_{X_\infty})_G$ is finitely generated by the generators of $\mathcal{P}_{n_1}$ and the generators of $\mathcal{Q}$. □

Remark 2. We may have that, for $n \gg 0$, $\mathcal{P}_n(\mathcal{O}_m)_G \neq \mathcal{P}_m(\mathcal{O}_m)_G$ for all $m \geqslant n+1$. See Example 4.5 in [4].

Lemma 3.3. *Let E_α, E_β, $E_\alpha \neq E_\beta$, be essential divisors over X and let us keep the notation as before. We have*:

(i) $\widehat{\mathcal{O}_{X_\infty,z_\alpha}}$ *is Noetherian.*
(ii) *If $N_\alpha \subset N_\beta$, then* $\dim \widehat{\mathcal{O}_{N_\beta,z_\alpha}} \geqslant 1$.

Here the completions are with respect to the topology defined by the maximal ideal.

Proof. The ring $\widehat{\mathcal{O}_{X_\infty,z_\alpha}}$ is a complete local ring with maximal ideal $\mathcal{P}\widehat{\mathcal{O}_{X_\infty,z_\alpha}}$, which is finitely generated by Lemma 3.2. Therefore (i) holds. For (ii), let $R = \mathcal{O}_{N_\beta,z_\alpha}$ and $M = \mathcal{P}\mathcal{O}_{N_\beta,z_\alpha}$ its maximal ideal, which is finitely generated. Since $E_\alpha \neq E_\beta$, $N_\alpha \neq N_\beta$, and R is not a field. Since R is a domain, we have $M^n \neq (0)$ for all $n \geqslant 1$. By (i), $\widehat{R}$ is a Noetherian ring. If it were artinian, then $M^n\widehat{R} = 0$ for some n, thus $M^n = M^{n+1}$, hence $M^n = (0)$ by Nakayama, which is a contradiction. □

End of proof of Theorem 2.1. To construct the diagram in Fig. 1, by Lemma 3.3, we may apply the curve selection lemma to $Spec\,\widehat{\mathcal{O}_{N_\beta,z_\alpha}}$ and obtain a morphism $\psi : Spec\,\kappa[[\xi]] \to Spec\,\widehat{\mathcal{O}_{N_\beta,z_\alpha}}$ such that the image of the closed point is the closed point in $Spec\,\widehat{\mathcal{O}_{N_\beta,z_\alpha}}$, the image of the generic point is not the closed point in $Spec\,\widehat{\mathcal{O}_{N_\beta,z_\alpha}}$, and κ is a finite extension of k_α. The composition of ψ and the morphism $Spec\,\widehat{\mathcal{O}_{N_\beta,z_\alpha}} \to Spec\,\mathcal{O}_{N_\beta,z_\alpha}$ induces the morphism $Spec\,\kappa[[\xi,t]] \to Spec\,\mathcal{O}_{N_\beta,z_\alpha}[[t]]$ in Fig. 1. □

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