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Asymptotically balanced schemes for non-homogeneous hyperbolic systems – application to the Shallow Water equations

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Abstract

In this work we introduce a class of balanced numerical schemes, up to second order, for the solution of general non-homogeneous hyperbolic systems of conservation laws. We give a general technique to build such schemes. We also prove that they balance up to second order a large class of steady solutions in the whole domain but some subset whose measure tends to zero as the grid size decreases to zero. We finally present an application to Shallow Water equations that exhibit the good performances of some of the schemes introduced. ***To cite this article: T. Chacón Rebollo et al., C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Schémas asymptotiquement equilibrés pour systèmes hyperboliques non-homogénes – application aux équations de Saint-Venant. Dans ce travail nous introduisons une classe de schémas numériques equilibrés au second ordre pour la solution de systèmes hyperboliques de lois de conservation. Nous donnons une technique générale pour construire ce type de schémas. Nous prouvons que ces schémas équilibrent au second ordre une grande classe de solutions stationaires, dans tout le domaine, excepté un sous-ensemble de mesure qui tend vers zéro lorsque la taille de la maille tend vers zéro. Nous présentons finalement une application aux équations de Saint-Venant qui montre les bonnes performances de quelques-uns des schémas présentés. ***Pour citer cet article : T. Chacón Rebollo et al., C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Version française abrégée

Dans ce travail nous introduisons quelques schémas numériques equilibrés pour la résolution de systèmes hyperboliques de lois de conservation non homogènes.

C'est un fait bien connu, et largement traité dans la littérature à cet égard, que pour qu'un schéma numérique fournisse des solutions avec un niveau de précision acceptable, il doit approcher les équilibres (solutions

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stationnaires) du système, au moins au deuxième ordre. Autrement, les solutions obtenues présentent de fortes erreurs, étant souvent physiquement inacceptables.

Le point-clé pour construire des schémas equilibrés est d'introduire un décentrage du terme source compatible avec le terme de diffusion numérique du schéma (cf. [1,5,6]).

Dans ce travail nous présentons d'abord une technique générale pour construire des termes sources décentrés, de façon à garantir que les équilibres sont calculés au moins au second ordre (schéma (2)–(7)). L'idée consiste à rajouter un terme correcteur au système équivalent. Ceci s'applique a des systèmes hyperboliques de lois de conservations non-homogènes quelconques, à des schémas de type Flux-difference et Flux-splitting et à une grande classe de solutions stationnaires (Théorème 3.3). Essentiellement, nous démontrons que notre schéma équilibre les états stationnaires à l'ordre 2 dans les zones du domaine où les vitesses caractéristiques gardent un signe constant (même zéro). Asymptotiquement (lorsque la taille de la maille tends vers zéro), on équilibre ces états partout sauf sur un ensemble de mesure nulle. L'intérêt de ce résultat est qu'on n'a pas besoin de connaître les solutions stationnaires auparavant.

Nous donnons également des conditions suffisantes pour que notre schéma équilibre de façon exacte une solution stationnaire connue.

Finalement, la technique proposée est appliquée aux équations de Saint-Venant : nous proposons une discrétisation concrète du terme source, et montrons quelques résultats numériques assez satisfaisants pour des tests significatifs (Section 4).

1. Introduction

In this paper we study the numerical discretization of non-homogeneous hyperbolic systems of conservation laws. It is well known that in order that the scheme is accurate enough, it should solve some equilibria of the system up to second order. There is a wide literature which treats this problem. Bermúdez–Vázquez in [1] define the C-property; which asks that the scheme calculates a given stationary solution for a given system at grid nodes, exactly or up to second order. In [3] an extension of a flux-splitting scheme to non-homogeneous 1D SWE is derived. In [2], based on the ideas of [3], a systematic technique to build numerical schemes which verify the C-property is introduced. In [5] Greenberg–Leroux introduce the concept of a well-balanced scheme. In this paper, for non-homogeneous scalar equations, the authors introduce a numerical scheme based on the resolution of regularized Riemann problems, which preserve the balance between internal forces and the source term of the hyperbolic conservation law. The numerical scheme balances the equilibria of non-homogeneous equations. In the case of such a system, in [6], Jin introduces the interface-method. He proves that the proposed scheme balances up to second order all stationary solutions for the system of Shallow Water equations, at cell interfaces. In [7] Perthame–Simeoni prove an extension of the theorem of Lax–Wendroff to non-homogeneous scalar conservation laws. The remarkable fact, in our context, is that the theorem holds, provided the scheme balances certain stationary solutions.

In the present work we introduce, for general non-homogeneous hyperbolic systems, numerical schemes which balance virtually all stationary solutions up to second order.

We consider a general formulation of numerical schemes, under which can be written a wide class of known and new schemes. We present an application to Shallow Water equations that exhibits the good performances of the schemes introduced.

2. Motivation

In this section, we give the general structure of the schemes that we propose.

We consider a non-homogeneous hyperbolic system,

$$\frac{\partial W}{\partial t} + \frac{\partial F}{\partial x} = G. \tag{1}$$

The structure of schemes in conservative form for the hyperbolic system is obtained by integrating the differential system in the control volume, in the 1D case, $(x_{i-1/2}, x_{i+1/2})$, where $x_i = i\,\Delta x$. Then, for non-homogeneous hyperbolic systems with an explicit discretization in time, if Δt is the time step, we have

$$\frac{W_i^{n+1} - W_i^n}{\Delta t} + \frac{\phi_{i+1/2}^n - \phi_{i-1/2}^n}{\Delta x} = G_{C,i}^n, \tag{2}$$

where $G_{C,i}^n$ is a centered (second order) approximation of $\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} G(x, W_h^n)\,\mathrm{d}x$, W_h^n being a function built from $\{W_i^n\}_i$. The numerical flux function, $\phi_{i+1/2}$ is an approximation of F at $x = x_{i+1/2}$.

We consider upwind schemes, that treat the convection dominance by introducing some numerical diffusion. For these schemes, the numerical flux function has the structure

$$\phi_{i+1/2} = F_C(W_i, W_{i+1}) - \frac{1}{2} D(W_i, W_{i+1})(W_{i+1} - W_i), \tag{3}$$

where by $F_C(W_i, W_{i+1})$, we denote a centered (second order) approximation of $F(W)$ in $x = x_{i+1/2}$, and $D(W_i, W_{i+1})$ is a first-order approximation of the diffusion matrix $\mathcal{D}(\widetilde{W}_{i+1/2})$, where $\widetilde{W}_{i+1/2}$ is an intermediate state between W_i and W_{i+1}.

These schemes can be interpreted as second order approximations in the space of a parabolic equivalent system. If we also take a centered approximation of the source term, then this equivalent system is

$$\frac{\partial W}{\partial t} + \frac{\partial}{\partial x} F(W) - \nu \frac{\partial}{\partial x}\left(\mathcal{D}(W)\frac{\partial}{\partial x} W\right) = G(x, W), \tag{4}$$

where ν is equal to half of the space step ($\nu = \Delta x/2$).

Notice that if W is a stationary solution of (1) then, it verifies $\frac{\partial}{\partial x} F(W) = G(x, W)$. Then W is not a stationary solution of (4), as there is a first order error due to the upwinding of the flux. It is well known that this is a source of large errors in the numerical solution. These large errors disappear when the scheme has a high precision in space (cf. [1,5]). We shall develop here schemes with second order accuracy in space. For this purpose, we propose to add a correcting term (that we denote by *VST*, Viscosity Source Term) to the equivalent system:

$$\frac{\partial W}{\partial t} + \frac{\partial}{\partial x} F(x, W) - \nu \frac{\partial}{\partial x}\left(\mathcal{D}(x, W)\frac{\partial W}{\partial x}\right) = G(x, W) + \mathit{VST}. \tag{5}$$

The term *VST*, for stationary solutions of (1) must verify $-\nu\frac{\partial}{\partial x}(\mathcal{D}(W)\frac{\partial W}{\partial x}) = \mathit{VST}$.

To give the definition of *VST* we consider a smooth stationary solution W of (1). By $A(W)$, we denote the Jacobian matrix of F. If $A(W)$ is non singular, then

$$A(W)\frac{\partial W}{\partial x} = G(x, W) \quad \Rightarrow \quad -\nu\frac{\partial}{\partial x}\left(\mathcal{D}(W)\frac{\partial W}{\partial x}\right) = -\nu\frac{\partial}{\partial x}\big(\mathcal{D}(W)A^{-1}(W)G(x, W)\big).$$

With this definition of *VST*, we can write the system (5) (with a conservative structure) as

$$\frac{\partial W}{\partial t} + \frac{\partial}{\partial x}\left[F(W) - \nu\mathcal{D}(W)\left(\frac{\partial W}{\partial x} - A^{-1}G(x, W)\right)\right] = G(x, W). \tag{6}$$

We can conclude then that to build balanced numerical schemes, up to second order, for the hyperbolic system (1), it is enough to use approximations of order two in space of (6). This motivates the definition of $\phi_{i+1/2}$, as a second order approximation of $F(W) - \nu\mathcal{D}(W)(\frac{\partial W}{\partial x} - A^{-1}G(x, W))$ in $x = x_{i+1/2}$. Concretely, we define

$$\phi_{i+1/2} = F_C(W_i, W_{i+1}) - \nu D(W_i, W_{i+1})\left(\frac{W_{i+1} - W_i}{\Delta x} - \widetilde{A^{-1}}(W_i, W_{i+1}) G_D(x_i, x_{i+1}, W_i, W_{i+1})\right), \tag{7}$$

where G_D and $\widetilde{A^{-1}}$ are first-order approximations of G and $A^{-1}(\widetilde{W}_{i+1/2})$, respectively.

There is still the difficulty of defining *VST* when A is singular. If by λ_j, $j = 1, \ldots, N$, we denote the eigenvalues of A, we shall assume that there exists N continuous surfaces $\gamma_j \subset \mathbb{R}^N$, $j = 1, \ldots, N$, such that λ_j is invertible and C^2 in $\mathbb{R}^N \setminus \gamma_j$.

We propose to define the matrix $\widetilde{A^{-1}}(U, V)$ as $\widetilde{A^{-1}}(U, V) = X(\widetilde{W}(U, V))\widetilde{\Lambda^{-1}}(U, V)X^{-1}(\widetilde{W}(U, V))$ where $\widetilde{W}(U, V)$ is an intermediate state between U and V, by X we denote the matrix defined by the eigenvectors of A and

$$\widetilde{\Lambda^{-1}}(U, V) = \mathrm{Diag}\big(\widetilde{\lambda_1^{-1}}(U, V), \ldots, \widetilde{\lambda_N^{-1}}(U, V)\big) \quad \text{with } \widetilde{\lambda_j^{-1}} = \begin{cases} \lambda_j^{-1}\big(\widetilde{W}\big) & \text{if } L[U, V] \subset \mathbb{R}^N \setminus \gamma_j, \\ 0 & \text{otherwise,} \end{cases}$$

where $L[U, V]$ is the segment in $\mathbb{R}^N$ that connects U and V.

This definition of $\widetilde{A^{-1}}$ states that when some eigenvalue of A vanishes, then no upwinding of the source term in characteristics variables must be performed. For scalar equations, we may prove that this definition of $\widetilde{A^{-1}}$ ensures the stability of the scheme.

This choice allows us to balance the scheme, even in some cases when A is singular.

3. Balance properties

In this section we prove that the scheme (2)–(7) balances up to second order the system (1) for a wide class of stationary solutions. We also prove that under some natural hypotheses upon matrix D, the defined scheme balances system (1) for all stationary solutions in all $[0, L]$ but in a set whose measure tends to zero as $\Delta x \to 0$. We will call them 'asymptotically balanced schemes'.

Firstly, we observe that matrix $\mathcal{D}(W)$ is only a Lipschitz function, with non-continuous derivative at the points where its eigenvalues vanish. A possible choice of the matrix is $D(W) = |A(W)|$; for example this happens for the scheme of Roe. Thus, we cannot hope that the introduced scheme balances all stationary solutions, with order two, in all the domain $[0, L]$. We introduce the following definitions,

Definition 3.1 (*Balanced schemes*). We say that the numerical scheme (2)–(7) is balanced on a smooth stationary solution $W(x)$ of the hyperbolic system (1) if the associated consistency error has second order accuracy.

Definition 3.2 (*Asimptotically balanced schemes*). We say that the scheme (2)–(7) is asymptotically balanced on a smooth stationary solution $W(x)$ of the hyperbolic system (1) if there exists an increasing sequence $\{K_n\}_n$ of compact subsets of $[0, L]$ such that

(1) $\mu([0, L] \setminus \bigcup_n K_n) = 0$, where by μ we denote Lebesgue measure in $\mathbb{R}$.
(2) For all n there exists a $\delta_n > 0$ such that if $0 < \Delta x < \delta_n$, then the scheme balances the system in K_n.

Assuming that A^{-1} and $\mathcal{D}$ present singularities only on curves in $\mathbb{R}^N$, we obtain the following result:

Theorem 3.3. *We consider* $W : [0, L] \to \mathbb{R}^N$ *a stationary solution of class* C^2 *of the system* (1). *Then,*

(a) *If* $A(W)$ *does not have singular points, the scheme* (2)–(7) *balances system* (1) *on* W *up to second order.*
(b) *If the set of points* x *where* $A(W(x))$ *is singular has measure zero, then the scheme* (2)–(7) *asymptotically balances system* (1) *on* W.
(c) *If* $\mathcal{D}$ *has the same eigenvectors as* A *and their eigenvalues vanish at the same points, then the scheme* (2)–(7) *asymptotically balances system* (1) *on* W.

The proof of this theorem is based on the fact that in regions of $[0, L]$ where A is inversible, a smooth solution of the original system (1) is also a solution of the equivalent system (6). Point (c) applies to steady solutions W for which the set of point where $A(W)$ is singular has non-zero measure. The complete proof of the theorem can be found in [4].

The interest of point (c) is that it provides a wide class of schemes that balance all (smooth) steady solutions of system (1).

We also have the following result that yields sufficient conditions for the exact calculation of stationary solutions.

Theorem 3.4. *Let W be a stationary solution of* (1). *Assume that F_C, G_C, and G_D evaluated on W verify*

$$\frac{F_C(W_i, W_{i+1}) - F_C(W_{i-1}, W_i)}{\Delta x} = G_{C,i}, \qquad \frac{W_{i+1} - W_i}{\Delta x} = \widetilde{A^{-1}}(W_i, W_{i+1}) G_D(x_i, x_{i+1}, W_i, W_{i+1}).$$

Then, the scheme defined by (2)–(7) *balances in an exact way the system* (1) *for the stationary solution at grid points. This happens independently of the choice of the upwinding matrix $D(U, V)$.*

4. Application to Shallow Water equations

In this section we apply the proposed scheme to Shallow Water equations, which model the behaviour of water flows in channels. The unknowns are h, the height of the water, and q the discharge. The physical flux function and the source term for variable depth, are defined by

$$F(W) = \begin{pmatrix} q \\ \dfrac{q^2}{h} + \dfrac{1}{2} g h^2 \end{pmatrix}, \quad G(x, W) = \begin{pmatrix} 0 \\ g h H'(x) \end{pmatrix} \quad \text{where } W = \begin{pmatrix} h \\ q \end{pmatrix},$$

g is the gravity constant and $H(x) = \bar{h} - z_b(x)$ is the depth of the channel, about a fixed reference level.

The different elements that determine scheme (2)–(7) that we use are defined as follows:

$$F_C(W_i, W_{i+1}) = \big(F(W_{i+\alpha}) + F(W_{i+1-\alpha})\big)/2, \quad \text{where } W_{i+\alpha} = (1-\alpha) W_i + \alpha W_{i+1} \text{ with } \alpha \in [0, 1].$$

The first component of G_D and G_C is equal to zero, in both cases, and the second components are

$$\big[G_D(x_i, x_{i+1}, W_i, W_{i+1})\big]_2 = g h_{i+1/2} \frac{H_{i+1} - H_i}{\Delta x},$$

$$[G_{C,i}]_2 = \frac{g}{2} \left\{ \big[\alpha h_{i+\alpha/2} + (1-\alpha) h_{i+(1-\alpha)/2}\big] \frac{H_{i+1} - H_i}{\Delta x} + \big[\alpha h_{i-\alpha/2} + (1-\alpha) h_{i+(\alpha-1)/2}\big] \frac{H_i - H_{i-1}}{\Delta x} \right\}.$$

Matrix D has the same eigenvectors as matrix A and its eigenvalues are given by $d_j(W_i, W_{i+1}) = |\lambda_j(W_{i+1/2})|$. With this choice, at least for the linear homogeneous case, the scheme is L^2 stable.

Using Theorem 3.3 we prove that the scheme is asymptotically balanced for all smooth stationary solutions. Moreover, the scheme exactly calculates water at rest, $(h, q) = (H, 0)$. It is enough to observe that the conditions of the Theorem 3.4 are verified.

Numerical test. We test our scheme for two stationary solutions, in a domain of length $L = 25$ m and bottom function defined by the maximum between zero and the 'bump' $z_b(x) = 0.2 - 0.05\,(x - 10)^2$. For this test the discharge must be constant, so this test is interesting in order to observe the performance of the scheme near the bottom bump.

We use a CFL condition equal to 0.8, a step discretization equal to $\Delta x = 0.25$ and as initial condition water at rest with constant free surface at 0.5. As boundary conditions, we consider two different cases: for Test 1.a we impose $h = 0.66$ m downstream when the flow is subcritical and $q = 1.53$ m^2/s upstream. Test 1.b is obtained by imposing $h = 0.33$ downstream and $q = 0.18$ m^2/s upstream. In both cases we consider that the scheme converges to a stationary solution when the relative error between two consecutive approximations in time is less than 10^{-5}.

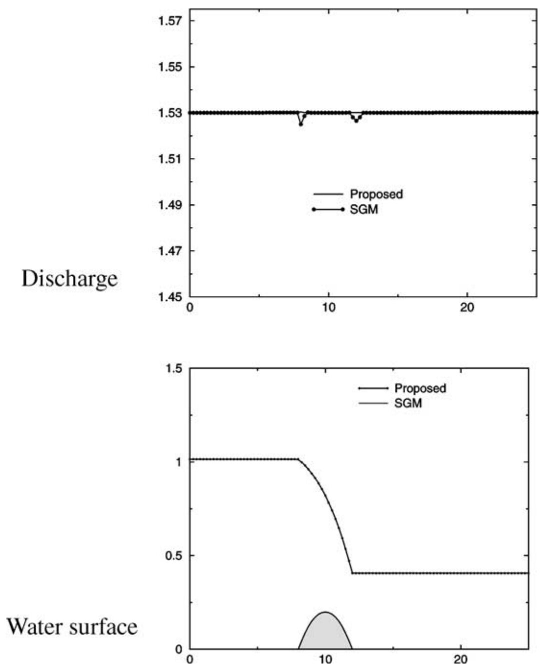

Fig. 1. Test 1.a.

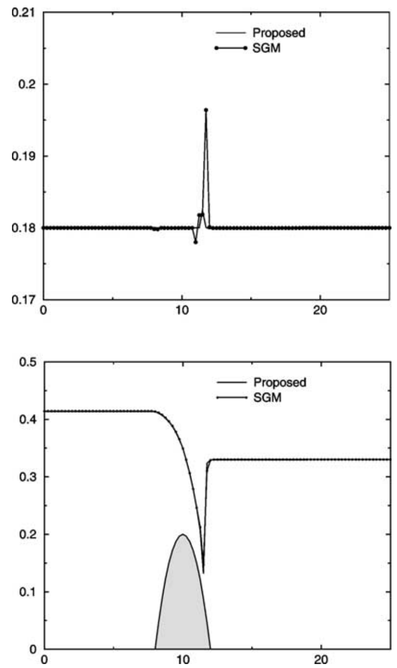

Fig. 2. Test 1.b.

In Figs. 1 and 2 we compare the calculated discharge for the Surface Gradient Method (SGM) (cf. [8]) and the proposed scheme with $\alpha = 1/8$. In both cases the proposed scheme provides some improvement. This is particularly relevant for Test 1.b, as in this case there is a hydraulic jump (discontinuity of the free surface) near $x = 12$.

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