Available online at www.sciencedirect.com
SCIENCE DIRECT®
C. R. Acad. Sci. Paris, Ser. I 338 (2004) 23–26
COMPTES RENDUS MATHEMATIQUE

Partial Differential Equations

A simple proof of an inequality of Bourgain, Brezis and Mironescu

Jean Van Schaftingen

Département de mathématique, Université catholique de Louvain, 2, chemin du Cyclotron, B-1348 Louvain-la-Neuve, Belgium

Received and accepted 24 October 2003

Presented by Haïm Brezis

Abstract

A simpler proof of a recent inequality of Bourgain, Brezis and Mironescu is given. ***To cite this article: J. Van Schaftingen, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Une preuve simple d'une inégalité de Bourgain, Brezis et Mironescu. Nous donnons une preuve plus simple d'une inégalité récente de Bourgain, Brezis et Mironescu. ***Pour citer cet article : J. Van Schaftingen, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Version française abrégée

Bourgain, Brezis et Mironescu ont établi dans [1] l'inégalité suivante.

Proposition 0.1. *Soit Γ une courbe fermée, orientée et rectifiable de $\mathbb{R}^3$, et soit $\vec{t}$ le vecteur unité tangent à Γ. Si $\vec{k} \in W^{1,3}(\mathbb{R}^3;\mathbb{R}^3)$, alors*

$$\left|\int_\Gamma \vec{k}\cdot\vec{t}\,\right| \leqslant C\|k\|_{W^{1,3}}|\Gamma|.$$

La preuve de la Proposition 0.1 dans [1] est assez complexe. Nous en donnons une preuve élémentaire qui se généralise à des surfaces de dimension quelconque dans un espace de dimension quelconque.

Proposition 0.2. *Soit Γ une courbe fermée, orientée et lipschitzienne dans $\mathbb{R}^N$, $N \geqslant 2$; soit $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N)$. Si $\nabla u \in L^N(\mathbb{R}^N)$, alors*

E-mail address: vanschaftingen@math.ucl.ac.be (J. Van Schaftingen).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2003.10.036

$$\left|\int_\Gamma u\,\mathrm{d}\gamma\right| \leqslant C_N \|\nabla u\|_{L^N(\mathbb{R}^N)}|\Gamma|, \tag{1}$$

où $|\Gamma|$ désigne la longueur de la courbe Γ.

La preuve de la Proposition 0.2 commence par établir, avec la stratégie de [1], la formule (3). Ensuite, alors que Bourgain, Brezis et Mironescu utilisent une décomposition de Littlewood–Paley, nous utilisons les inégalités de Morrey et de Hölder pour conclure.

La preuve se généralise sans difficulté à des surfaces k-dimensionnelles (Proposition 3.2).

1. Introduction

Bourgain, Brezis and Mironescu proved in [1] the following inequality.

Proposition 1.1. *Let Γ be a closed, oriented, rectifiable curve of $\mathbb{R}^3$, and denote by $\vec{t}$ the unit tangent vector along Γ; let $\vec{k} \in W^{1,3}(\mathbb{R}^3; \mathbb{R}^3)$. Then*

$$\left|\int_\Gamma \vec{k}\cdot\vec{t}\right| \leqslant C\|k\|_{W^{1,3}}|\Gamma|.$$

The proof of Proposition 1.1 in [1] is technically involved. We provide an elementary proof and a generalization to k-dimensional surfaces and N-dimensional space. For simplicity, we begin with the case of a curve in $\mathbb{R}^N$.

Proposition 1.2. *Let Γ be an oriented, compact and closed Lipschitz curve of $\mathbb{R}^N$, $N \geqslant 2$; let $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N)$. If $\nabla u \in L^N(\mathbb{R}^N)$, then*

$$\left|\int_\Gamma u\,\mathrm{d}\gamma\right| \leqslant C_N \|\nabla u\|_{L^N(\mathbb{R}^N)}|\Gamma|, \tag{2}$$

where $|\Gamma|$ denotes the length of curve Γ.

Remark 1. When $N = 1$, the left-hand side of (2) is 0; when $N = 2$ and γ is a Jordan curve, Proposition 1.2 is a simple consequence of Green theorem and the isoperimetric inequality; and when $N = 3$ it is equivalent to Proposition 1.1.

2. Proof of Proposition 1.2

Proof. Without loss of generality, the curve Γ is connected and is the image of S^1 by a Lipschitz map γ. We assume first that u and $\gamma : S^1 \to \mathbb{R}^N$ are of class C^1. We start with the same strategy as [1], bounding

$$e\cdot\int_{S^1} u\bigl(\gamma(x)\bigr)\dot\gamma(x)\,\mathrm{d}x,$$

for an arbitrary unit-norm vector $e \in \mathbb{R}^N$.

Let

$$\Gamma_t = \bigl\{x \in S^1 \mid e\cdot\gamma(x) = t\bigr\}.$$

Since $e \cdot \gamma(s)$ is of class C^1, Sard's lemma implies that for almost every $t \in \mathbb{R}$, Γ_t is finite and $e \cdot \dot{\gamma}(s) \neq 0$ if $\gamma(s) \in \Gamma_t$. We have then

$$e \cdot \int_{S^1} u\big(\gamma(x)\big)\dot{\gamma}(x)\,\mathrm{d}x = \int_{S^1} u\big(\gamma(x)\big)\, e \cdot \dot{\gamma}(x)\,\mathrm{d}x = \int_{\mathbb{R}} \sum_{x \in \Gamma_t} \sigma(x) u(x)\,\mathrm{d}t, \tag{3}$$

where $\sigma(x) = \mathrm{sign}(e \cdot \dot{\gamma}(x))$. Since Γ is closed, for almost every $t \in \mathbb{R}$ we can write $\Gamma_t = \{P_1, \ldots, P_{r(t)}\} \cup \{N_1, \ldots, N_{r(t)}\}$ so that $\sigma(P_i) = 1$, $\sigma(N_i) = -1$ and $\sum_{i=1}^{r(t)} |\gamma(P_i) - \gamma(N_i)|$ is minimal (in particular, it is bounded by $|\Gamma|$).

In order to estimate $\sum_{x \in \Gamma_t} \sigma(x) u(x)$, we proceed differently from [1]. They used a Littlewood–Paley decomposition. Instead, we apply Morrey's inequality (see, e.g., [2, Theorem IX.12]) in $\mathbb{R}^{N-1}$ for $u_t = u|_{\{y \in \mathbb{R}^N \mid e \cdot y = t\}}$ before applying the discrete Hölder inequality to the sum

$$\sum_{x \in \Gamma_t} \sigma(x) u(x) \leqslant C_N \|\nabla u_t\|_N \sum_{i=1}^{r(t)} \big|\gamma(P_i) - \gamma(N_i)\big|^{1/N} \leqslant C_N \|\nabla u_t\|_p |\Gamma|^{1/N} r(t)^{(N-1)/N}.$$

We are now ready to estimate the integral of (3):

$$\begin{aligned}\int_{\mathbb{R}} \sum_{x \in \Gamma_t} \sigma(x) u(x)\,\mathrm{d}t &\leqslant C_N |\Gamma|^{1/N} \int_{\mathbb{R}} \|\nabla u_t\|_N r(t)^{(N-1)/N}\,\mathrm{d}t \\ &\leqslant C_N |\Gamma|^{1/N} \Big(\int_{\mathbb{R}} \|\nabla u_t\|_N^N\,\mathrm{d}t\Big)^{1/N} \Big(\int_{\mathbb{R}} r(t)\,\mathrm{d}t\Big)^{(N-1)/N} \leqslant C'_{p,N} |\Gamma| \|\nabla u\|_N\end{aligned}$$

since $2\int_{\mathbb{R}} r(t)\,\mathrm{d}t = \int_{S^1} |e \cdot \gamma'(x)|\,\mathrm{d}x \leqslant \int_{S^1} |\gamma'(x)|\,\mathrm{d}x = |\Gamma|$.

The result is extended to general Γ and u by standard smoothing arguments.

3. Generalization to surfaces

Proposition 1.2 generalizes straightforwardly to k-dimensional surfaces defined as follows.

Definition 3.1. A pair $\Gamma = (M, \gamma)$ is a C^r k-dimensional Lipschitz surface of $\mathbb{R}^N$ if

(i) M is a compact oriented k-dimensional C^r manifold without boundary,
(ii) $\gamma : M \to \mathbb{R}^N$ is a Lipschitz function.

When $\Gamma = (M, \gamma)$ is a k-dimensional Lipschitz surface of $\mathbb{R}^N$, it is possible, since M is oriented, to define the integral of Borel function $u : \mathbb{R}^N \to \mathbb{R}$ as the k-vector $\int_\Gamma u\,\mathrm{d}\gamma(x)$, where $\mathrm{d}\gamma(x)[a_1, \ldots, a_k] = \gamma'(x)a_1 \wedge \cdots \wedge \gamma'(x)a_k$, and the mass of Γ as $|\Gamma| = \int_M |\mathrm{d}\gamma|$, where $|\cdot|$ denotes the Euclidean norm of a k-vector.

Proposition 3.2. *Let Γ be a C^k k-dimensional surface. Then*

$$\Big|\int_M u\,\mathrm{d}\gamma\Big| \leqslant C_N \|\nabla u\|_N |\Gamma|, \tag{4}$$

where the norm on the left is the comass-norm (see [3]).

Proof. Since the proof is similar to the proof of Proposition 1.2, we only give a sketch. For an arbitrary simple unit covector $e = e_1 \wedge \cdots \wedge e_k$, we write

$$e \cdot \int_{\Gamma} u \,\mathrm{d}\gamma = \int_{\mathbb{R}^k} \sum_{x \in \Gamma_y} \sigma(x) u(x) \,\mathrm{d}y,$$

where $\Gamma_y = \{x \in M \mid e_i \cdot \gamma(x) = y_i,\ 1 \leqslant i \leqslant k\}$ and $\sigma(x) = \operatorname{sign}(e_1 \gamma'(x) \wedge \cdots \wedge e_k \gamma'(x))$. This formula is valid by Sard's lemma because M is a C^k manifold. Then, using Morrey's and Hölder's inequalities, with the notations of the proof of Proposition 1.2,

$$\int_{\mathbb{R}^k} \sum_{x \in \Gamma_y} \sigma(x) u(x) \,\mathrm{d}y \leqslant \int_{\mathbb{R}} \|\nabla u_{y_1}\|_N \left(\int_{\mathbb{R}^{k-1}} \sum_{i=1}^{r(y)} \bigl|\gamma(P_i) - \gamma(N_i)\bigr| \,\mathrm{d}y'' \right)^{1/(N-1)} \left(\int_{\mathbb{R}^{k-1}} r(y) \,\mathrm{d}y'' \right)^{N/(N-1)} \mathrm{d}y_1,$$

where $y'' = (y_2, \ldots, y_k)$ and one concludes using $\int_{\mathbb{R}^{k-1}} \sum_{i=1}^{r(y)} |\gamma(P_i) - \gamma(N_i)| \,\mathrm{d}y'' \leqslant |\Gamma|$, Hölder's inequality and $2 \int_{\mathbb{R}^k} r(y) \,\mathrm{d}y \leqslant |\Gamma|$.

Remark 2. Proposition 3.2 can also be proved by induction on k. The case $k = 1$ is Proposition 1.2 and for $k > 1$, Γ is cut into slices of dimension $k - 1$, for which the estimate of Proposition 3.2 holds. The integration of this estimate, with Hölder's inequality, gives the conclusion.

Remark 3. The inequality (4) is the limit case of

$$\left| \int_{\Gamma} u \,\mathrm{d}\gamma \right| \leqslant C_{p,N} \delta(\Gamma)^{1-N/p} |\Gamma| \|\nabla u\|_p,$$

where $p > N$, $\nabla u \in L^p(\mathbb{R}^N)$ and $\delta(\Gamma)$ denotes the diameter of Γ. It is a simple consequence of Morrey's inequality.

Acknowledgements

The author wishes to thank Haïm Brezis for proposing the problem and for comments. He also acknowledges a Research Fellow grant of the Belgian Fonds National de la Recherche Scientifique.

References

[1] J. Bourgain, H. Brezis, P. Mironescu, $H^{1/2}$ maps with value into the circle; minimal connections, lifting, and the Ginzburg–Landau equation, Inst. Hautes Études Sci. Publ. Math., in press.
[2] H. Brezis, Analyse fonctionnelle, in: Collect. Math. Appl. Maîtrise, Masson, Paris, 1983.
[3] H. Federer, Geometric Measure Theory, Springer-Verlag, New York, 1969.