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Optimal Control

A general formula for decay rates of nonlinear dissipative systems

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Abstract

This work is concerned with stabilization of hyperbolic systems by a nonlinear feedback which can be localized on part of the boundary or locally distributed. We present here a general formula which gives the energy decay rates in terms of the behavior of the nonlinear feedback close to the origin. This formula allows us to unify for instance the cases where the feedback has a polynomial growth at the origin, with the cases where it goes exponentially fast to zero at the origin. We give also two other significant examples of nonpolynomial growth at the origin. We also show that we either obtain or improve significantly the decay rates of Lasiecka and Tataru (Differential Integral Equations 8 (1993) 507–533) and Martinez (Rev. Mat. Comput. 12 (1999) 251–283). ***To cite this article: F. Alabau-Boussouira, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Une formule générale pour le taux de décroissance des systèmes dissipatifs non linéaires. On étudie le problème de la stabilisation des équations de type hyperbolique par un feedback qui peut être frontière ou bien localement distribué. L'objet de cette Note est de montrer qu'il existe une formule générale qui permet d'obtenir un taux de décroissance de l'énergie en fonction du comportement au voisinage de zéro du terme de dissipation non linéaire. Cette formule permet d'unifier tous les cas et notamment ceux pour lesquels le feedback croît polynomialement et ceux pour lesquels il s'écrase exponentiellement en zéro. On donne aussi deux autres exemples significatifs de croissance non polynomiale. On montre pour tous ces exemples que l'on retrouve ou obtient de meilleurs taux de décroissance que ceux de Lasiecka et Tataru (Differential Integral Equations 8 (1993) 507–533) et Martinez (Rev. Mat. Comput. 12 (1999) 251–283). ***Pour citer cet article : F. Alabau-Boussouira, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Version française abrégée

Introduction

Il existe de nombreux résultats de stabilisation pour les équations hyperboliques linéaires ou semi-linéaires et pour des dissipations linéaires et non linéaires. Nous renvoyons à, e.g., [1,11,6,8,9,7] pour des résultats de

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stabilisation concernant le problème de la localisation de la région active du feedback. On renvoie à, e.g., [11,6,10] pour la stabilisation des équations semi-linéaires. Cette Note concerne la stabilisation par des feedbacks non linéaires. On renvoie à [12,10,5] pour des résultats de stabilisation avec taux de décroissance polynomial dans le cas de feedbacks avec croissance polynomiale à l'origine. On renvoie à [6] et [9] pour des taux de décroissance dans les cas où aucune hypothèse de croissance à l'origine n'est faite sur le feedback. Lasiecka et Tataru [6], ont traité le cas d'une équation des ondes semi-linéaire sous des hypothèses géométriques plus générales avec un feedback frontière non linéaire sans hypothèse de croissance à l'origine. Le taux de décroissance obtenu pour l'énergie dépend de celui d'une équation différentielle dont les coefficients font intervenir implicitement la nonlinéarité du feedback. Martinez [9] a obtenu une formule explicite qui donne de très bonnes estimations du taux de décroissance. Nous donnons dans cette Note, une formule générale qui unifie tous les cas, allant par exemple de la croissance polynomiale à la croissance exponentielle du feedback. De plus, cette formule conduit à des taux de décroissance aisément calculables quand on l'applique à des exemples concrets. Elle nous permet aussi de retrouver ou d'améliorer les taux de décroissance connus dans la littérature.

On donne notre résultat sur un problème spécifique : l'équation des ondes avec dissipation non linéaire localement distribuée. Mais l'approche est générale et s'adapte aussi bien au cas de la stabilisation frontière et à d'autres équations (e.g., Petrovsky).

Soit Ω un ouvert borné non vide dans R^N ayant une frontière Γ de classe C^2. On considère le système (1). On définit l'énergie associée à une solution u de (1)

$$E(t) = \frac{1}{2}\left(\int_\Omega |u_t|^2 + |\nabla u|^2\right) \mathrm{d}x.$$

Résultat principal

Le résultat principal de cette Note est le

Théorème 0.1. *On fait les hypothèses* (HG) *et les hypothèses* (HF). *On suppose que $a \in \mathcal{C}^0(\Omega)$ et qu'il existe une constante $a_- > 0$, telle que*

$$a(x) \geqslant a_- \quad \text{sur } \omega.$$

On suppose de plus que g vérifie les propriétés suivantes :

$$g'(0) = 0,$$

il existe $r_0 > 0$ tel que g est de classe $\mathcal{C}^2$ sur $(0, r_0]$ et la fonction h définie par

$$h(x) = g'(x) + \frac{g(x)}{x} \text{ est strictement croissante sur } [0, r_0].$$

Alors pour t suffisamment grand et pour tout $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ l'énergie de la solution de (1) *vérifie l'estimation* (2) (*cf. version anglaise*).

On donne dans la version anglaise quatre exemples significatifs d'applications qui montrent que le résultat ci-dessus donne des taux de décroissance précis.

1. Introduction

When a dissipative feedback acts on a system which was initially conservative, a natural question which arises is whether this dissipation is sufficient to lead to precise decay rates of the energy. Several results are available in this direction. We refer here, due to the constraint of the length to some of them only. We refer, e.g., to [2] for strong stabilization results. We refer to, e.g., [1,11,6,8,9,7] for stabilization results connected to the problem of the localization of the active region of the feedback. We refer to, e.g., [11,6,4,10] for stabilization results for semilinear equations. We are concerned here with stabilization results for nonlinear feedbacks. We refer to [12,10,5] for the obtaining of precise decay rates of the energy for feedbacks with a polynomial growth at the origin. We refer to [6] and [9] for decay rates in the case where no growth condition on the feedback at the origin is assigned a priori. In [6], Lasiecka and Tataru consider the case of a wave equation including distributed and boundary semilinear terms with a nonlinear boundary feedback. The authors show that the energy decays as fast as the solution of an associated differential equation whose coefficients depend on g. Martinez [9] gives an explicit formula which leads to very good estimates of decay rates of the energy. Our work situates in this context. Our purpose is to produce a formula which unifies all the cases. More precisely it should be the same for very general feedbacks, starting from polynomial case up to exponential case or faster than any polynomial but less than exponential for instance. Moreover, this formula when applied to any specific example under interest should lead to precise easily computable decay rates. It should also allow us to recover the decay rates which are known in the existing literature (e.g., polynomial or exponential cases) or to improve the existing decay rates.

For the sake of clearness, we give our result on a specific model, namely the wave equation with nonlinear locally distributed damping. However, this approach is general, and work for nonlinear boundary stabilization as well as for other equations such as Petrovsky equations. Let Ω be a non-empty bounded open set in $\mathbb{R}^N$ having a boundary Γ of class C^2.

We consider the following system:

$$\begin{cases} \partial_{tt}u - \Delta u + \rho(x, u_t) = 0 \quad \text{in } \Omega \times \mathbb{R}, \\ u = 0 \quad \text{on } \Sigma = \Gamma \times \mathbb{R}, \\ (u, \partial_t u)(0) = \big(u^0, u^1\big) \quad \text{on } \Omega. \end{cases} \tag{1}$$

We define the energy of a solution by

$$E(t) = \frac{1}{2}\left(\int_\Omega |u_t|^2 + |\nabla u|^2\right) \mathrm{d}x.$$

2. Geometric assumptions for piecewise multipliers and assumptions on the feedback

We make the following assumptions on the feedback:

$$\text{(HF)} \quad \begin{cases} \rho \in \mathcal{C}(\overline{\Omega} \times \mathbb{R}) \text{ and is monotone increasing with respect to the second variable} \\ \exists a \in L^\infty(\Omega),\ a \geqslant 0 \text{ a.e. and a strictly increasing odd function } g \in \mathcal{C}^1(\mathbb{R}) \text{ such that} \\ a(x)|v| \leqslant \big|\rho(x, v)\big| \leqslant C\, a(x)|v| \quad \text{if } |v| \geqslant 1, \\ a(x)g\big(|v|\big) \leqslant \big|\rho(x, v)\big| \leqslant C\, a(x)g^{-1}\big(|v|\big) \quad \text{if } |v| \leqslant 1, \end{cases}$$

where g^{-1} denotes the inverse function of g and where C is a positive constant.

We use the following notations. If $\Omega_j \subset \Omega$ is a Lipschitz domain, we denote by Γ_j its boundary and by ν_j the outward unit normal to Γ_j. Moreover, if U is a subset of $\mathbb{R}^N$ and $x \in R^N$, we set $d(x, U) = \inf_{y \in U} |x - y|$, and $\mathcal{N}_\varepsilon(U) = \{x \in \mathbb{R}^N,\ d(x, U) \leqslant \varepsilon\}$.

Remark 1. We refer to, e.g., [3] for the existence, uniqueness and regularity of solutions of (1), assuming that (HF) holds. It shows that the above problem is well-posed for initial data in the energy space $H_0^1(\Omega) \times L^2(\Omega)$.

We make the following geometric assumptions on Ω and ω as in [8] and [9]:

$$\text{(HG)}\qquad \begin{cases} \exists \varepsilon > 0,\ \text{domains } \Omega_j \subset \Omega \text{ with Lipschitz boundary } \Gamma_j \text{ for } 1 \leqslant j \leqslant J \text{ and points } x_j \text{ in } \mathbb{R}^N \\ \text{such that } \Omega_i \cap \Omega_j = \emptyset \quad \text{if } i \neq j, \\ \Omega \cap \mathcal{N}_\varepsilon\big[\bigcup_j \gamma_j(x_j) \cup (\Omega - \bigcap_j \Omega_j)\big] \subset \omega, \end{cases}$$

where $\gamma_j(x_j) = \{x \in \Gamma_j,\ (x - x_j) \cdot \nu_j(x) > 0\}$. One can remark that these assumptions are a generalization of Zuazua's assumptions in [11], where he proved stabilization of a semilinear wave equation by a damping locally distributed on a set ω provided that this set contains a neighbourhood of $\{x \in \partial\Omega, (x - x_0) \cdot \nu > 0\}$, x_0 being a fixed point in $\mathbb{R}^N$.

Theorem 2.1. *Assume the above hypotheses* (HF) *and* (HG). *Assume also that $a \in \mathcal{C}^0(\Omega)$ and is such that there exists a constant $a_- > 0$, with*

$$a(x) \geqslant a_- \quad \text{on } \omega.$$

Moreover assume that the function g satisfies in addition the following properties:

$$g'(0) = 0,$$

there exists $r_0 > 0$ such that g is of class $\mathcal{C}^2$ on $(0, r_0]$ and the function h defined by

$$h(x) = g'(x) + \frac{g(x)}{x} \text{ is strictly increasing on } [0, r_0].$$

Then the energy of the solutions of (1) *satisfies, for all $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, the following estimate*

$$E(t) \leqslant C\big(E(0)\big)\big(z(t)\big)^2 \frac{z(t)g'(z(t)) - g(z(t))}{z(t)g'(z(t)) + g(z(t))}, \tag{2}$$

where $C(E(0))$ is a constant which depends continuously on $E(0)$, and where $z(t)$ can be computed by the following formula:

$$\frac{2z(t)}{z(t)g'(z(t)) + g(z(t))} + 4\alpha\big(z(t)\big) = \frac{t}{T_0}, \tag{3}$$

with the function α defined on $(0, r_0)$ by the following integral expression

$$\alpha(\tau) = \int_\tau^{r_0} \frac{g(t)(t^2 g''(t) + t g'(t) - g(t))}{(t g'(t) + g(t))^2 (t g'(t) - g(t))} \, \mathrm{d}t, \tag{4}$$

and where T_0 does not depend on $E(0)$.

Remark 2. If $g'(0) \neq 0$, then g has a linear growth at the origin. In this case, it is well known that the energy of the system decays exponentially.

3. Examples of application

We show by giving below four significant examples how this general formula leads to precise decay rates. We also compare the resulting decay rates with the ones obtained in [9], namely

$$E(t) \leqslant C\left(g^{-1}\left(\frac{1}{t}\right)\right)^2, \tag{5}$$

where C is a constant which depends on $E(0)$ in a continuous way. This formula is completely explicit.

In all what follows, we just give the expression of the function g on an interval of the form $(0, r_0]$ where $r_0 > 0$ is chosen sufficiently small so that the hypotheses of Theorem 2.1 hold. The function is then suitably extended to $\mathbb{R}$. Moreover, $C_1(E(0))$ will stand for a generic constant which depends on $E(0)$, whereas C_2 will be a generic positive constant which does not depend on $E(0)$.

Corollary 3.1. *We assume the above geometric hypotheses and* (HF)*, then the energy of solutions of* (1) *for the four examples given below, satisfies*

$$E(t) \leqslant C_1\big(E(0)\big)\left(\Theta^{-1}\left(\frac{C_2}{t}\right)\right)^2, \tag{6}$$

for t sufficiently large and for all $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$.

Example 1. Let g be given by $g(x) = x^p$ where $p > 1$. Then

$$E(t) \leqslant C_1\big(E(0)\big)t^{-2/(p-1)}, \tag{7}$$

for t sufficiently large and for all $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$.

Example 2. Let g be given by $g(x) = \mathrm{e}^{-1/x}$.

Then the energy of solution of (1) satisfies the estimate (6), where the function Θ is defined by

$$\Theta(x) = \frac{\mathrm{e}^{-1/x}}{x^2}, \quad x > 0.$$

Moreover the following property holds

$$\lim_{t \longrightarrow \infty}\left(\frac{\Theta^{-1}(1/t)}{g^{-1}(1/t)}\right) = 1$$

so that the estimate given here is

$$E(t) \leqslant C_1\big(E(0)\big)\big(\ln(t)\big)^{-2}.$$

Example 3. Let g be given by $g(x) = x^p(\ln(\frac{1}{x}))^q$ where $p > 1$ and $q > 0$.

Then the energy of solution of (1) satisfies the estimate (6), where the function Θ is defined by

$$\Theta(x) = x^{p-1}\left(\ln\left(\frac{1}{x}\right)\right)^q, \quad x \in (0, 1).$$

Moreover the following property holds

$$\lim_{t \longrightarrow \infty}\left(\frac{\Theta^{-1}(1/t)}{g^{-1}(1/t)}\right) = 0. \tag{8}$$

Example 4. Let g be given by $g(x) = \mathrm{e}^{-(\ln(1/x))^p}$ where $1 < p < 2$.

Then the energy of solution of (1) satisfies the estimate (6), where the function Θ is defined by

$$\Theta(x) = x^{-1}\left(\ln\left(\frac{1}{x}\right)\right)^{p-1}\mathrm{e}^{-(\ln(1/x))^p}, \quad x \in (0, 1).$$

Moreover the property (8) holds.

Remark 3. The estimate (5) for Exemple 1 is

$$E(t) \leqslant C_1\big(E(0)\big)t^{-2/p}, \tag{9}$$

which is weaker than (7). In [9], Martinez has shown that it is possible to improve (9), so as to get as close as possible of the decay rate (7) without obtaining it. Example 2 shows that if g goes very fast to zero at the origin (exponentially in that case) then (5) and (2) are asymptotically equivalent as t goes to infinity. Examples 2 and 4 show that it is possible to have other types of growth at the origin (between two polynomials for Example 3 and faster than any polynomial but less than an exponential for Example 4) for which the decay given by (2)–(4) is better than the one given by (5).

Idea of the proof. The proof is based on the multiplier method combined with the choice of appropriate weight functions to handle the growth of the nonlinearity g at the origin.

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