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Partial Differential Equations

About a Liouville phenomenon

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Abstract

This work is devoted to the study of a new Liouville-type phenomenon for entire subsolutions of elliptic partial differential equations of the form

$$A(u)=0.$$

Typical examples of the operator $A(u)$ are the p-Laplacian for $p>1$ and its well-known modifications. ***To cite this article: V.V. Kurta, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Autour d'un phénomène de type Liouville. Ce travail est consacré à l'étude d'un nouveau phénomène de type Liouville pour des sous-solutions entières d'équations aux dérivées partielles elliptiques de la forme

$$A(u)=0.$$

Des exemples typiques de l'opérateur $A(u)$ sont le p-laplacien pour $p>1$ et ses modifications bien connues. ***Pour citer cet article : V.V. Kurta, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

1. Introduction

Due to the famous Liouville theorem it is well known that any subharmonic function on $\mathbb{R}^2$ bounded below by a constant is itself a constant. On the other hand it is also well known that for $n \geqslant 3$ there exist non-constant subharmonic functions on $\mathbb{R}^n$ bounded below by a constant. The purpose of this work is to determine for $n \geqslant 3$ "the sharp distance at infinity" between the non-constant subharmonic functions on $\mathbb{R}^n$ bounded below by a constant and this constant itself in the form of a Liouville-type theorem and to characterize basic properties of quasilinear elliptic partial differential operators, which make it possible to obtain such a Liouville-type theorem for subsolutions of quasilinear elliptic partial differential equations of the form

$$A(u)=0 \tag{1}$$

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on $\mathbb{R}^n$.

Typical examples of the operator $A(u)$ are the p-Laplacian

$$\Delta_p(u) := \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) \tag{2}$$

for $p > 1$ and its well-known modification (see, e.g., [1, p. 155])

$$\tilde{\Delta}_p(u) := \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right) \tag{3}$$

for $n \geqslant 2$ and $p > 1$.

2. Definitions

Let $A(u)$ be a differential operator defined formally by

$$A(u) = \sum_{i=1}^{n} \frac{\mathrm{d}}{\mathrm{d}x_i} A_i(x, u, \nabla u). \tag{4}$$

Here and in what follows $n \geqslant 2$. We assume that the functions $A_i(x, \eta, \xi)$, $i = 1, \ldots, n$, satisfy the usual Carathéodory conditions on $\mathbb{R}^n \times \mathbb{R}^1 \times \mathbb{R}^n$. Namely, they are continuous in η, ξ for a.e. $x \in \mathbb{R}^n$ and measurable in x for any $\eta \in \mathbb{R}^1$ and $\xi \in \mathbb{R}^n$.

Definition 2.1. Let $\alpha > 1$ be a given number. The operator $A(u)$, defined by (4), belongs to the class $\mathcal{A}(\alpha)$ if for all $\eta \in \mathbb{R}^1$, all $\xi, \psi \in \mathbb{R}^n$, and almost all $x \in \mathbb{R}^n$ the inequality

$$0 \leqslant \sum_{i=1}^{n} \xi_i A_i(x, \eta, \xi), \tag{5}$$

with equality only in the case when $\xi = 0$, and the inequality

$$\left|\sum_{i=1}^{n} \psi_i A_i(x, \eta, \xi)\right|^{\alpha} \leqslant \mathcal{K}|\psi|^{\alpha}\left(\sum_{i=1}^{n} \xi_i A_i(x, \eta, \xi)\right)^{\alpha-1}, \tag{6}$$

with a certain positive constant $\mathcal{K}$, hold.

It is easy to see that condition (6) is fulfilled whenever the inequality

$$\left(\sum_{i=1}^{n} A_i^2(x, \eta, \xi)\right)^{\alpha/2} \leqslant \mathcal{K}\left(\sum_{i=1}^{n} \xi_i A_i(x, \eta, \xi)\right)^{\alpha-1} \tag{7}$$

holds for all $\eta \in \mathbb{R}^1$, all $\xi, \psi \in \mathbb{R}^n$, and almost all $x \in \mathbb{R}^n$. Hence, the operator $A(u)$ defined by (4) and satisfying conditions (5) and (7) belongs to the class $\mathcal{A}(\alpha)$.

Remark 1. Conditions (6) and (7) on the behavior of the coefficients of partial differential operators were introduced in [2].

It is not difficult to verify that for any given $p > 1$ the differential operators (2) and (3) as well as the differential operator defined by (4) and satisfying the well-known growth conditions

$$\left(\sum_{i=1}^{n} A_i^2(x, \eta, \xi)\right)^{1/2} \leqslant \mathcal{K}_1|\xi|^{p-1} \tag{8}$$

and

$$|\xi|^p \leqslant \mathcal{K}_2 \sum_{i=1}^{n} \xi_i A_i(x,\eta,\xi), \tag{9}$$

with some positive constants $\mathcal{K}_1, \mathcal{K}_2$, belong to the class $\mathcal{A}(\alpha)$ with $\alpha = p$.

It is also easy to see that a linear divergent elliptic partial differential operator

$$L := \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial}{\partial x_i}\right) \tag{10}$$

with $a_{ij}(x)$ measurable bounded coefficients and with the (possibly non-uniformly) positive-definite quadratic form

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \tag{11}$$

belongs to the class $\mathcal{A}(\alpha)$ with $\alpha = 2$ but does not satisfy condition (9) for any fixed $p > 1$.

In connection with this we give another example of an operator that belongs to the class $\mathcal{A}(\alpha)$ with a certain $\alpha > 1$ but does not satisfy condition (9). Let $a(x,\eta,\xi)$ be a positive bounded function which satisfies the Carathéodory conditions on $\mathbb{R}^n \times \mathbb{R}^1 \times \mathbb{R}^n$. It is evident that for a given $p > 1$ the weighted p-Laplacian

$$\bar{\Delta}_p(u) := \operatorname{div}\big(a(x,u,\nabla u)|\nabla u|^{p-2}\nabla u\big) \tag{12}$$

belongs to the class $\mathcal{A}(\alpha)$ with $\alpha = p$ but does not satisfy condition (9) for any fixed $p > 1$ if the function $a(x,\eta,\xi)$ is assumed to be only positive, but not bounded below away from zero.

It can happen that an operator $A(u)$ given by (4) belongs simultaneously to several different classes $\mathcal{A}(\alpha)$. For example, the well-known mean curvature operator

$$\Xi(u) := \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) \tag{13}$$

belongs to the classes $\mathcal{A}(\alpha)$ for all $1 < \alpha \leqslant 2$; similarly its modification for $p \geqslant 2$,

$$\Xi_p(u) := \operatorname{div}\left(\frac{|\nabla u|^{p-2}\nabla u}{\sqrt{1+|\nabla u|^2}}\right), \tag{14}$$

belongs to the classes $\mathcal{A}(\alpha)$ for all $\alpha \in (p-1, p]$ and $p \geqslant 2$.

Obviously, operators (13) and (14) do not satisfy conditions (9), (10) for any fixed $p \geqslant 1$.

Definition 2.2. Let $\alpha > 1$ be a given number, and let the operator $A(u)$, given by (4), belong to the class $\mathcal{A}(\alpha)$. A function $u : \mathbb{R}^n \to (-\infty, +\infty)$ is called an entire subsolution of Eq. (1) if it belongs to the space $W^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$ and satisfies the integral inequality

$$\int_{\mathbb{R}^n} \sum_{i=1}^{n} \varphi_{x_i} A_i(x,u,\nabla u)\,\mathrm{d}x \geqslant 0 \tag{15}$$

for every non-negative function $\varphi \in W^{1,\alpha}(\mathbb{R}^n)$ with compact support.

3. Results

Theorem 3.1. *Let $\alpha \geqslant n$ be a given number, and let the operator $A(u)$, given by* (4), *belong to the class $\mathcal{A}(\alpha)$. Let $u(x)$ be an entire subsolution of* (1) *bounded below by a constant. Then $u(x) = \mathrm{const.}$, a.e. on $\mathbb{R}^n$.*

Theorem 3.2. *Let $\alpha \in (1, n)$ be a given number, and let the operator $A(u)$, given by* (4), *belong to the class $\mathcal{A}(\alpha)$. Let $u(x)$ be an entire subsolution of* (1) *bounded below by a constant c and such that $u \in L^{\infty}_{\mathrm{loc}}(\mathbb{R}^n)$. Then either $u(x) = c$, a.e. on $\mathbb{R}^n$, or the equality*

$$\liminf_{r\to+\infty}\Big[\sup_{r\leqslant|x|\leqslant 2r}\big(u(x)-c\big)\Big]r^{(n-\alpha)/(\alpha-1-\nu)} = +\infty \tag{16}$$

holds with any fixed $\nu \in (0, \alpha - 1)$.

Theorem 3.3. *Let $\alpha \in (1, n)$ be a given number, and let the operator $A(u)$, given by* (4), *belong to the class $\mathcal{A}(\alpha)$. Let $u(x)$ be an entire subsolution of* (1), *bounded below by a constant c. Then either $u(x) = c$, a.e. on $\mathbb{R}^n$, or the equality*

$$\liminf_{r\to+\infty} r^{-\alpha}\int\limits_{r\leqslant|x|\leqslant 2r}\big(u(x)-c\big)^{\alpha-1-\nu}\,\mathrm{d}x = +\infty \tag{17}$$

holds with any fixed $\nu \in (0, \alpha - 1)$.

Remark 2. It is important to note that for any given $\alpha \in (1, n)$ the function

$$u(x) = \big(1 + |x|^{\alpha/(\alpha-1)}\big)^{(\alpha-n)/\alpha} \tag{18}$$

is a non-negative entire subsolution of the equation

$$\Delta_p(u) = 0 \tag{19}$$

with $p = \alpha$ such that the equality

$$\liminf_{r\to+\infty}\Big[\sup_{r\leqslant|x|\leqslant 2r}\big(u(x)-0\big)\Big]r^{(n-\alpha)/(\alpha-1)} = C \tag{20}$$

holds with a certain positive constant C.

Remark 3. The statements of Theorems 3.2 and 3.3 with $\alpha = 2$ are new results even for entire classical subsolutions of the equation

$$\Delta u = 0. \tag{21}$$

Remark 4. Similar results to those of Theorem 3.1 for entire continuous subsolutions of (1) were obtained in [3].

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