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Optimal Control

On the null controllability of a one-dimensional fluid–solid interaction model

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Abstract

We analyze the null controllability of a one-dimensional nonlinear system which models the interaction of a fluid and a particle. The fluid is governed by the Burgers equation and the control is exerted on the boundary points. We present two main results: the global null controllability of a linearized system and the local null controllability of the nonlinear original model. The proofs rely on appropriate global Carleman inequalities and fixed point arguments. ***To cite this article: A. Doubova, E. Fernández-Cara, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Résumé

Sur la contrôlabilité nulle d'un modèle unidimensionnelle d'interaction fluide–solide. On analyse la contrôlabilité nulle d'un système unidimensionnel non linéaire qui modélise l'intéraction d'un fluide et d'une particule. Le fluide est gouverné par l'équation de Burgers et le contrôle est exercé sur la frontière. On présente deux résultats principaux : la contrôlabilité nulle globale d'un système linéarisé et la contrôlabilité nulle locale du système non linéaire originaire. Les démonstrations de ces résultats reposent sur des inégalités de Carleman et des arguments de point fixe appropriés. ***Pour citer cet article : A. Doubova, E. Fernández-Cara, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Version française abrégée

Soient $T > 0$ et $Q = (-1, 1) \times (0, T)$. On considère le modèle suivant pour l'intéraction d'un fluide et une particule :

$$\begin{cases} u_t - u_{yy} + uu_y = 0, & (y,t) \in Q,\ y \neq h(t), \\ u(-1,t) = \alpha(t), \quad u(1,t) = \beta(t), & t \in (0,T), \\ u(h(t),t) = h'(t), \quad [u_y](h(t),t) = h''(t), & t \in (0,T), \\ u(y,0) = u^0(y), & y \in (-1,1), \\ h(0) = 0, \quad h'(0) = h^1. \end{cases} \tag{1}$$

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Ici, $u(y,t)$ est la vitesse de la particule du fluide qui se trouve au point y au temps t, $h(t)$ est la position de la particule solide à l'instant t, α et β sont les contrôles (deux fonctions au moins dans $L^2(0,T)$), $u^0 \in L^\infty(-1,1)$, $h^1 \in \mathbf{R}$ et, de façon générale, $[g](y)$ dénote le *saut* de g au point y.

On s'intéresse à l'analyse de la contrôlabilité exacte à zéro de (1), c'est-à-dire à l'existence de contrôles α, β tels que (1) possède une solution $(u, h') \in C^0([0,T]; L^2(-1,1)) \times C^0([0,T])$ avec

$$u(y,T) = 0 \quad \text{et} \quad h'(T) = 0.$$

Le système considéré peut être regardé comme une version préliminaire simplifiée d'autres modèles bidimensionnels ou tridimensionnels en espace. Par exemple, pour un fluide gouverné par les équations de Navier–Stokes avec une sphère rigide plongée à l'intérieure, on peut se demander comment faut-il agir sur le fluide pour arrêter la sphère. Ceci justifie l'intérêt de l'analyse de la contrôlabilité nulle de (1).

Dans (1), le domaine spatial est variable. Sous l'hypothèse $|h(t)| \leqslant 1 - b$, où b est une constante positive et petite, on peut introduire le changement de variable $x = (y - h)/(1 - \kappa h)$, où κ est le signe de x. Ceci permet de reécrire (1) dans un domaine indépendant de t :

$$\begin{cases} (1-\kappa h)u_t - \dfrac{1}{1-\kappa h}u_{xx} - (1-\kappa x)h'u_x + uu_x = 0, & (x,t) \in Q,\ x \neq 0, \\ u(-1,t) = \alpha(t), \quad u(1,t) = \beta(t), & t \in (0,T), \\ u(0,t) = h'(t), \quad \left[\dfrac{1}{1-\kappa h}u_x\right](0,t) = h''(t), & t \in (0,T), \\ u(x,0) = u^0(x), & x \in (-1,1), \\ h(0) = 0, \quad h'(0) = h^1. \end{cases} \tag{2}$$

Pour $u \in L^\infty(Q)$ et $h \in W^{1,\infty}(0,T)$ données avec $|h(t)| \leqslant 1 - b$, on considère aussi le système *linéarisé*

$$\begin{cases} (1-\kappa h)v_t - \dfrac{1}{1-\kappa h}v_{xx} - (1-\kappa x)h'v_x + \dfrac{1}{2}(uv)_x = 0, & (x,t) \in Q,\ x \neq 0, \\ v(-1,t) = \alpha(t), \quad v(1,t) = \beta(t), & t \in (0,T), \\ v(0,t) = k'(t), \quad \left[\dfrac{1}{1-\kappa h}v_x\right](0,t) = k''(t), & t \in (0,T), \\ v(x,0) = v^0(x), & x \in (-1,1), \\ k(0) = 0, \quad k'(0) = k^1. \end{cases} \tag{3}$$

Le premier résultat principal de cette Note es le suivant :

Théorème 0.1. *Pour tout $v^0 \in L^2(-1,1)$, $k^1 \in \mathbf{R}$ et pour tout $u \in L^\infty(Q)$ et $h \in W^{1,\infty}(0,T)$ avec $|h(t)| \leqslant 1 - b$, le système linéaire* (3) *est exactement contrôlable à zéro.*

Pour la démonstration de ce résultat, on établit une inégalité globale de type Carleman pour un problème adjoint associé et on adapte les arguments habituels.

Notre deuxième résultat principal est le suivant :

Théorème 0.2. *Il existe $\varepsilon > 0$ tel que, si les données initiales $u^0 \in L^\infty(-1,1)$ et $h^1 \in \mathbf{R}$ satisfont $\|u^0\|_{L^\infty} + |h^1| \leqslant \varepsilon$, alors le système non linéaire* (2) *est exactement contrôlable à zéro.*

Pour la démonstration, on utilise le Théorème 0.1, un argument de point fixe approprié et un résultat de régularité de la solution de (2). Les démostrations détaillées des théorèmes précédents seront données dans [2].

1. Introduction and main results

Let us set $T>0$ and $Q=(-1,1)\times(0,T)$. We will consider a nonlinear system that models the interaction of a one-dimensional fluid evolving in $(-1,1)$ and a solid particle. It will be assumed that the velocity of the fluid is governed by the viscous Burgers equation at both sides of the point mass location $y=h(t)$. For simplicity, it will be also assumed that the fluid density is constant and the solid particle has unit mass. The system is thus the following:

$$\begin{cases} u_t-u_{yy}+uu_y=0, & (y,t)\in Q,\ y\neq h(t),\\ u(-1,t)=\alpha(t), \quad u(1,t)=\beta(t), & t\in(0,T),\\ u(h(t),t)=h'(t), \quad [u_y](h(t),t)=h''(t), & t\in(0,T),\\ u(y,0)=u^0(y), & y\in(-1,1),\\ h(0)=0, \quad h'(0)=h^1. \end{cases} \tag{4}$$

Here, $u(y,t)$ is the velocity of the fluid particle we find at y at time t, $h(t)$ is the position occupied by the solid particle at time t, α and β are the controls (two functions at least in $L^2(0,T)$), $u^0\in L^\infty(-1,1)$, $h^1\in\mathbf{R}$ and, in general, $[g](y)$ denotes the *jump* of g at y.

In this Note, we are concerned with the null controllability properties of (4). We say that (4) is null controllable (at time T) if, for each $u^0\in L^\infty(-1,1)$ and $h^1\in\mathbf{R}$, there exist controls α,β and an associated solution $(u,h')\in C^0([0,T];L^2(-1,1))\times C^0([0,T])$ such that

$$u(y,T)=0 \quad\text{and}\quad h'(T)=0.$$

The previous system can be viewed as a preliminary simplified version of other more complicate models in higher dimensions. For example, for a system governed by the Navier–Stokes equations with a solid sphere inside the fluid, a related question is how to act on the fluid particles to stop the sphere. This justifies the relevance of the controllability analysis of (4).

The existence, uniqueness and regularity properties of the solutions to (4) have been studied in [5]. With $(-1,1)$ replaced by $\mathbf{R}$, this problem has been analyzed in [6].

In (4), the spatial domain depends on t. Assuming that $|h(t)|\leqslant 1-b$ with b a positive small constant, we can introduce the following change of variable: for any $y\in(-1,h(t))\cup(h(t),1)$, we put $x=(y-h)/(1-\kappa h)$, where κ is the sign of x. This leads to a reformulation of (4) in a domain which is independent of t:

$$\begin{cases} (1-\kappa h)u_t-\dfrac{1}{1-\kappa h}u_{xx}-(1-\kappa x)h'u_x+uu_x=0, & (x,t)\in Q,\ x\neq 0,\\ u(-1,t)=\alpha(t), \quad u(1,t)=\beta(t), & t\in(0,T),\\ u(0,t)=h'((t), \quad \left[\dfrac{1}{1-\kappa h}u_x\right](0,t)=h''(t), & t\in(0,T),\\ u(x,0)=u^0(x), & x\in(-1,1),\\ h(0)=0, \quad h'(0)=h^1. \end{cases} \tag{5}$$

For any $u\in L^\infty(Q)$ and any $h\in W^{1,\infty}(0,T)$ with $|h(t)|\leqslant 1-b$, we will also consider the following *linearized system*:

$$\begin{cases} (1-\kappa h)v_t-\dfrac{1}{1-\kappa h}v_{xx}-(1-\kappa x)h'v_x+\dfrac{1}{2}(uv)_x=0, & (x,t)\in Q,\ x\neq 0,\\ v(-1,t)=\alpha(t), \quad v(1,t)=\beta(t), & t\in(0,T),\\ v(0,t)=k'(t), \quad \left[\dfrac{1}{1-\kappa h}v_x\right](0,t)=k''(t), & t\in(0,T),\\ v(x,0)=v^0(x), & x\in(-1,1),\\ k(0)=0, \quad k'(0)=k^1. \end{cases} \tag{6}$$

The main results of this Note are the following:

Theorem 1.1. *For any $v^0 \in L^2(-1,1)$ and $k^1 \in \mathbf{R}$ and for any $u \in L^\infty(Q)$ and $h \in W^{1,\infty}(0,T)$ with $|h(t)| \leqslant 1-b$, the linear system* (6) *is null controllable.*

Theorem 1.2. *There exists $\varepsilon > 0$ such that, whenever the initial data $u^0 \in L^\infty(-1,1)$ and $h^1 \in \mathbf{R}$ satisfy $\|u^0\|_{L^\infty} + |h^1| \leqslant \varepsilon$, the nonlinear system* (5) *is null controllable.*

The proofs of Theorems 1.1 and 1.2 will be respectively sketched in Sections 2 and 3. More details will be given in [2], where other related aspects will be also considered.

2. The null controllability of the linearized system

Let $\delta > 0$ be given and let us set

$$\omega = (-1-\delta, -1-\delta/2) \cup (1+\delta/2, 1+\delta), \qquad Q' = (-1-\delta, 1+\delta) \times (0,T)$$

and $Q'_* = \{(x,t) \in Q'\colon\ x \neq 0\}$. We will consider the following auxiliary distributed controlled system:

$$\begin{cases} (1-\kappa h)v_t - \dfrac{1}{1-\kappa h} v_{xx} - (1-\kappa x)h' v_x + \dfrac{1}{2}(\tilde u v)_x = f 1_\omega, & (x,t) \in Q'_*, \\ v(-1-\delta,t) = 0, \quad v(1+\delta,t) = 0, & t \in (0,T), \\ v(0,t) = k'(t), \quad \left[\dfrac{1}{1-\kappa h} v_x\right](0,t) = k''(t), & t \in (0,T), \\ v(x,0) = \tilde v^0(x), & x \in (-1-\delta, 1+\delta), \\ k(0) = 0, \quad k'(0) = k^1. \end{cases} \tag{7}$$

Here, $\tilde u$ and $\tilde v^0$ are the extensions by zero of u and v^0, respectively. Obviously, the null controllability of (7) with controls $f \in L^2(\omega \times (0,T))$ implies the boundary null controllability of (6).

In order to prove that (7) is null-controllable, let us introduce the associated adjoint system

$$\begin{cases} -\big((1-\kappa h)z\big)_t - \dfrac{1}{1-\kappa h} z_{xx} + \big((1-\kappa x)h' z\big)_x - \dfrac{1}{2}\tilde u z_x = 0\,, & (x,t) \in Q'_*, \\ z(-1-\delta,t) = 0, \quad z(1+\delta,t) = 0, & t \in (0,T), \\ z(0,t) = \lambda'(t), \quad \left[\dfrac{1}{1-\kappa h} z_x\right](0,t) = -\lambda''(t), & t \in (0,T), \\ z(x,T) = z^0(x), & x \in (-1-\delta, 1+\delta), \\ \lambda(T) = 0, \quad \lambda'(T) = \lambda^1, \end{cases} \tag{8}$$

where (z^0, λ^1) is given in $L^2(-1-\delta, 1+\delta) \times \mathbf{R}$. We then have the following result:

Proposition 2.1. *There exists a positive constant C, only depending on T, b, $\|h'\|_{L^\infty(0,T)}$ and $\|u\|_{L^\infty(Q)}$ (and nondecreasing with respect to the last two arguments) such that the following holds for any solution to* (8):

$$\big\|\big(z(\cdot,0), \lambda'(0)\big)\big\|^2_{L^2\times\mathbf{R}} \leqslant C \iint\limits_{\omega\times(0,T)} |z|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{9}$$

This is an observability estimate for the solutions to (8). Its proof relies on a global Carleman estimate of the form

$$\iint_{Q'} \rho^{-2s}\big(s\phi|z_x|^2+(s\phi)^3|z|^2\big)\,\mathrm{d}x\,\mathrm{d}t+\int_0^T \rho(0,t)^{-2s}\big(s\phi(0,t)\big)^3\big|\lambda'(t)\big|^2\,\mathrm{d}t$$
$$\leqslant C\bigg(\iint_{Q'} \rho^{-2s}\bigg|(1-\kappa h)z_t+\frac{1}{1-\kappa h}z_{xx}\bigg|^2\,\mathrm{d}x\,\mathrm{d}t+\iint_{\omega\times(0,T)} \rho^{-2s}(s\phi)^3|z|^2\,\mathrm{d}x\,\mathrm{d}t\bigg),$$

that can be established for large s by choosing appropriately the weight functions $\rho=\rho(x,t)$ and $\phi=\phi(t)$.

Now, a standard argument similar to those in [3] and [4] shows that (7) is null-controllable, with controls in $L^2(\omega\times(0,T))$ satisfying

$$\iint_{\omega\times(0,T)} |f|^2\,\mathrm{d}x\,\mathrm{d}t\leqslant C\big\|\big(v^0,k^1\big)\big\|^2_{L^2\times\mathbf{R}}, \tag{10}$$

where C is essentially the same constant in (9) (recall that C only depends on T, b, $\|h'\|_{L^\infty(0,T)}$ and $\|u\|_{L^\infty(Q)}$).

This ends the proof of Theorem 1.1.

3. The local null controllability of the nonlinear system

Let us now present the main steps of the proof of Theorem 1.2. As in Section 2, we will introduce an auxiliary system:

$$\begin{cases}(1-\kappa h)u_t-\dfrac{1}{1-\kappa h}u_{xx}-(1-\kappa x)h'u_x+uu_x=f1_\omega, & (x,t)\in Q'_*,\\ u(-1-\delta,t)=0,\quad u(1+\delta,t)=0, & t\in(0,T),\\ u(0,t)=h'(t),\quad \bigg[\dfrac{1}{1-\kappa h}u_x\bigg](0,t)=h''(t), & t\in(0,T),\\ u(x,0)=\tilde u^0(x), & x\in(-1-\delta,1+\delta),\\ h(0)=0,\quad h'(0)=h^1,\end{cases} \tag{11}$$

where $\tilde u^0$ is the extension by zero of u^0. Again, the null controllability of (11) with controls $f\in L^2(\omega\times(0,T))$ implies the boundary null controllability of (4).

To prove that (11) is null-controllable, we proceed as follows. Let us set

$$Z=\big\{(w,\ell):\ w\in L^2(Q'),\ \ell\in H^1(0,T),\ \ell(0)=0\big\} \tag{12}$$

and let us fix $b\in(0,1)$ and $R>0$. Let (w,ℓ) be given in Z and consider the linear system (7), with $(\tilde v^0,k^1)$, $\tilde u$, h and h' respectively replaced by $(\tilde u^0,h^1)$, $T_R(w)$, $T_{1-b}(\ell)$ and $T_R(\ell')$.

Here, we are using the following notation: for any $K>0$, T_K is the real function

$$T_K(s)=\begin{cases}K & \text{if } s>K,\\ s & \text{if } -K\leqslant s\leqslant K,\\ -K & \text{if } s<-K.\end{cases}$$

In view of Theorem 1.1, there exist controls $f\in L^2(\omega\times(0,T))$ and associated states (v,k) solving (7) for these $(\tilde v^0,k^1)$, $\tilde u$, h and h' and satisfying

$$v(x,T)=0\quad \text{for } x\in(-1-\delta,1+\delta),\quad k'(T)=0. \tag{13}$$

Furthermore, in view of the proof we have sketched in Section 2, we know that the controls f can be chosen satisfying

$$\iint_{\omega\times(0,T)} |f|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant C(T,b,R)\big\|\big(u^0,h^1\big)\big\|^2_{L^2\times\mathbf{R}}. \tag{14}$$

We will denote by $A(w,\ell)$ (resp. $\Lambda(w,\ell)$) the family of these controls (resp. the family of the associated states).

Using (14), it is not difficult to deduce that the states in $\Lambda(w,\ell)$ satisfy an estimate

$$\big\|(v,k)\big\|_E \leqslant C(T,b,R)\big\|\big(u^0,h^1\big)\big\|_{L^2\times\mathbf{R}}, \tag{15}$$

where E is a subspace of Z such that the embedding $E \hookrightarrow Z$ is compact. Consequently, we can apply Kakutani's fixed point theorem to the multivalued mapping $\Lambda : Z \mapsto Z$, see for instance [1]. This leads to the existence of a control f and a solution to the nonlinear system

$$\begin{cases} \big(1-\kappa T_{1-b}(h)\big)u_t - \dfrac{1}{1-\kappa T_{1-b}(h)}u_{xx} & \\ \quad -(1-\kappa x)T_R(h')u_x + \dfrac{1}{2}\big(T_R(u)u\big)_x = f 1_\omega, & (x,t)\in Q'_*,\\ u(-1-\delta,t)=0, \quad u(1+\delta,t)=0, & t\in(0,T),\\ u(0,t)=h'(t), \quad \left[\dfrac{1}{1-\kappa T_{1-b}(h)}u_x\right](0,t)=h''(t), & t\in(0,T),\\ u(x,0)=\tilde{u}^0(x), & x\in(-1-\delta,1+\delta),\\ h(0)=0, \quad h'(0)=h^1, & \end{cases} \tag{16}$$

such that (14) holds, $u(x,T)=0$ for $x\in(-1-\delta,1+\delta)$ and $h'(T)=0$.

Finally, notice that the solution to (16) also satisfies

$$\big\|(u,h)\big\|_{L^\infty(Q')\times W^{1,\infty}(0,T)} \leqslant C(T,b,R)\big\|\big(u^0,h^1\big)\big\|_{L^\infty\times\mathbf{R}}. \tag{17}$$

This is a consequence of (14) and the particular form of the nonlinearities in (16). Therefore, (u,h) is in fact a solution to the nonlinear system (11) whenever (u^0,h^1) is sufficiently small in the sense of Theorem 1.2. We have thus proved that this system is locally null-controllable. This ends the proof of Theorem 1.1.

Remark 1. It would be interesting to know whether Theorem 1.2 is still true when the control is exerted only at $x=1$ (for example, fixing $\alpha\equiv 0$ in (4)). To our knowledge, this is unknown.

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