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A class of stochastic differential equations with non-Lipschitzian coefficients: pathwise uniqueness and no explosion

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Abstract

A new result for the pathwise uniqueness of solutions of stochastic differential equations with non-Lipschitzian coefficients is established. Furthermore, we prove that the solution has no explosion under the growth $\xi \log \xi$. ***To cite this article: S. Fang, T. Zhang, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Résumé

Une classe d'équations différentielles stochastiques à coefficients non lipschitziens : unicité forte et non explosion. La condition lipschitzienne locale sera affaiblie dans l'établissemnt de l'unicité trajectorielle d'une e.d.s ; de plus, nous montrerons que la solution a un temps de vie infini sous la croissance $\xi \log \xi$. ***Pour citer cet article : S. Fang, T. Zhang, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

1. Introduction

Let $\sigma : \mathbf{R}^d \to \mathbf{R}^d \otimes \mathbf{R}^m$ and $b : \mathbf{R}^d \to \mathbf{R}^d$ be continuous functions. Consider the following Itô s.d.e.:

$$\mathrm{d}x_t(w) = \sigma\big(x_t(w)\big)\,\mathrm{d}w_t + b\big(x_t(w)\big)\,\mathrm{d}t, \quad x_0(w) = x_0, \tag{1}$$

where $t \to w(t)$ is a $\mathbf{R}^m$-valued standard Brownian motion. It is well known (see [5], [2, p. 159]) that the s.d.e. (1) has a weak solution up to a lifetime ζ, and $\zeta \equiv +\infty$ if σ and b are of linear growth. Morever if the s.d.e. (1) has the pathwise uniqueness, then it admits a strong solution (see [2, p. 149], [4, p. 341]). So the study of pathwise uniqueness is of great interest. It is well known that the pathwise uniqueness holds for (1) in the case that the coefficients are locally Lipschitzian. However, there few results on the pathwise uniqueness belyond the Lipschitz (or locally) conditions except in the one dimension case, some results have been obtained for certain Hölder coefficients (see [4, Chapter IX-3], [2, p. 168]). The purpose of this work is to establish the pathwise uniqueness of the solutions under the following assumptions,

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$$\begin{cases} \|\sigma(x)-\sigma(y)\|^2 \leqslant C|x-y|^2 \log \dfrac{1}{|x-y|}, & \text{for } |x-y|<1,\\ |b(x)-b(y)| \leqslant C|x-y| \log \dfrac{1}{|x-y|}, & \text{for } |x-y|<1, \end{cases} \tag{H1}$$

where $|\cdot|$ denotes the Euclidean distance in $\mathbf{R}^d$ and $\|\sigma\|^2 = \sum_{i=1}^d \sum_{j=1}^m \sigma_{ij}^2$. Our results are valid for any dimension. The main obstacle is that the traditional way of using Gronwall lemma wouldn't work. To overcome the difficulty, a special function has been constructed. This work is motivated by a study of the canonical Brownian motion on the diffeomorphism group $\mathrm{Diff}(S^1)$ of the circle (see [3,1]).

2. Pathwise uniqueness

Theorem 2.1. *Let* $(x_t(w))_{t\geqslant 0}$ *and* $(y_t(w))_{t\geqslant 0}$ *be two solutions* (*of continuous samples, without explosion*) *of the s.d.e.* (1) *such that* $x_0(w)=y_0(w)$. *Then under* (H1), *we have almost surely* $x_t(w)=y_t(w)$, $t\geqslant 0$.

Proof. Let $\eta_t(w)=x_t(w)-y_t(w)$ and $\xi_t(w)=|\eta_t(w)|^2$. We have $\mathrm{d}\eta_t(w)=(\sigma(x_t)-\sigma(y_t))\,\mathrm{d}w_t+(b(x_t)-b(y_t))\,\mathrm{d}t$, $\eta_0(w)=0$. Then

$$\mathrm{d}\xi_t = 2\langle \eta_t, (\sigma(x_t)-\sigma(y_t))\,\mathrm{d}w_t\rangle + 2\langle \eta_t, b(x_t)-b(y_t)\rangle \mathrm{d}t + \|\sigma(x_t)-\sigma(y_t)\|^2\,\mathrm{d}t, \tag{2}$$

and the stochastic contraction $\mathrm{d}\xi_t\cdot\mathrm{d}\xi_t$ is given by

$$\mathrm{d}\xi_t\cdot\mathrm{d}\xi_t = 4\left|(\sigma^*(x_t)-\sigma^*(y_t))\eta_t\right|^2\mathrm{d}t, \tag{3}$$

where σ^* denotes the transpose matrix of σ. Let $\rho>0$. Define the function $\psi_\rho:[0,1]\to\mathbf{R}$ by

$$\psi_\rho(\xi) = \int_0^\xi \frac{\mathrm{d}s}{s\log(1/s)+\rho}. \tag{4}$$

It is clear that for any $0<\xi<1$, $\psi_\rho(\xi)\uparrow\psi_0(\xi)=\int_0^\xi \frac{\mathrm{d}s}{s\log(1/s)}=+\infty$, as $\rho\downarrow 0$. Define

$$\Phi_\rho(\xi)=\mathrm{e}^{\psi_\rho(\xi)}. \tag{5}$$

Then we have

$$\Phi_\rho'(\xi)\left(\xi\log\frac{1}{\xi}+\rho\right)=\Phi_\rho(\xi), \tag{6}$$

and

$$\Phi_\rho''(\xi)=\frac{\Phi_\rho(\xi)(2-\log(1/\xi))}{(\xi\log(1/\xi)+\rho)^2}\leqslant 0 \quad \text{if } \xi<\mathrm{e}^{-2}. \tag{7}$$

Let $\tau=\inf\{t>0,\ \xi_t\geqslant \mathrm{e}^{-2}\}$.

By Itô formula, (2) and (3), we get

$$\begin{aligned}\Phi_\rho(\xi_{t\wedge\tau}) &= 1+2\int_0^{t\wedge\tau}\Phi_\rho'(\xi_s)\langle\eta_s,(\sigma(x_s)-\sigma(y_s))\,\mathrm{d}w_s\rangle + 2\int_0^{t\wedge\tau}\Phi_\rho'(\xi_s)\langle\eta_s, b(x_s)-b(y_s)\rangle\,\mathrm{d}s\\ &\quad+\int_0^{t\wedge\tau}\Phi_\rho'(\xi_s)\|\sigma(x_s)-\sigma(y_s)\|^2\,\mathrm{d}s+2\int_0^{t\wedge\tau}\Phi_\rho''(\xi_s)\left|(\sigma^*(x_s)-\sigma^*(y_s))\eta_s\right|^2\mathrm{d}s.\end{aligned}$$

Using (7) and hypothesis (H1), it follows that $\mathbb{E}(\Phi_\rho(\xi_{t\wedge\tau}))\leqslant 1+C\mathbb{E}(\int_0^{t\wedge\tau}\Phi_\rho'(\xi_s)\xi_s\log\frac{1}{\xi_s}\,\mathrm{d}s)$ which is smaller by (6) than $1+C\int_0^t\mathbb{E}(\Phi_\rho(\xi_{s\wedge\tau}))\,\mathrm{d}s$.

Now by Gronwall lemma, we get $\mathbb{E}(\Phi_\rho(\xi_{t\wedge\tau})) \leqslant e^{Ct}$ or

$$\mathbb{E}\big(e^{\Psi_\rho(\xi_{t\wedge\tau})}\big) \leqslant e^{Ct}. \tag{8}$$

Letting $\rho \downarrow 0$ in (8), $\mathbb{E}(e^{\Psi_0(\xi_{t\wedge\tau})}) \leqslant e^{Ct}$ which implies that for any t given,

$$\xi_{t\wedge\tau} = 0 \quad \text{almost surely.} \tag{9}$$

If $P(\tau < +\infty) > 0$, then for some $T > 0$ big enough $P(\tau \leqslant T) > 0$. It follows from (9) that on $\{\tau \leqslant T\}$, $\xi_\tau = 0$ which is absurd by the definition of τ. Therefore $\tau = +\infty$ almost surely and for any t given, $\xi_t = 0$ almost surely. Now by continuity of the sample paths, the two solutions are indistinguishable. □

3. Criterion for non-explosion

Theorem 3.1. *Let σ and b be continuous functions satisfying*

$$\begin{cases} \|\sigma(x)\|^2 \leqslant C\big(|x|^2\log|x| + 1\big), \\ |b(x)| \leqslant C\big(|x|\log|x| + 1\big). \end{cases} \tag{H2}$$

Then the s.d.e. (1) *has no explosion*: $P(\zeta = +\infty) = 1$.

Proof. Consider $\psi(\xi) = \int_0^\xi \frac{ds}{s|\log s|+1}$ and $\Phi(\xi) = e^{\psi(\xi)}$, $\xi \geqslant 0$. We have

$$\Phi'(\xi)\big(\xi|\log\xi| + 1\big) = \Phi(\xi), \tag{10}$$

$$\Phi''(\xi) = -\frac{\Phi(\xi)\log\xi}{(\xi\log\xi + 1)^2}, \quad \text{for } \xi > 1. \tag{11}$$

Note that the function Φ is not in $\mathcal{C}^2(\mathbf{R}_+)$. We need to modify Φ slightly in the neighbourhood of 1. Fix a small $\delta > 0$, take $\widetilde{\Phi} \in \mathcal{C}^2(\mathbf{R}_+)$ such that

$$\widetilde{\Phi} \geqslant \Phi, \quad \widetilde{\Phi}(\xi) = \Phi(\xi) \quad \text{for } \xi \notin [1-\delta, 1+\delta]. \tag{12}$$

Denote $K_1 = \sup_{\xi\in[1-\delta,1+\delta]}(|\widetilde{\Phi}'(\xi)| + |\widetilde{\Phi}''(\xi)|)$, $K_2 = \sup_{\xi\in[1-\delta,1+\delta]}(\xi|\log\xi|)$. Then

$$\big|\widetilde{\Phi}'(\xi)\big| \leqslant \frac{K_1(K_2+1)}{\Phi(1-\delta)} \cdot \frac{\Phi(\xi)}{\xi|\log\xi| + 1}, \quad \xi \in [1-\delta, 1+\delta], \tag{13}$$

and

$$\big|\widetilde{\Phi}''(\xi)\big| \leqslant \frac{K_1(K_2+1)^2}{\Phi(1-\delta)} \cdot \frac{\Phi(\xi)}{(\xi|\log\xi| + 1)^2}, \quad \xi \in [1-\delta, 1+\delta]. \tag{14}$$

Let $\eta_t(w) = x_t(w) - x_0$ and $\xi_t(w) = |\eta_t(w)|^2$. Define $\tau_R = \inf\{t > 0,\ \xi_t \geqslant R\}$, $R > 0$. Then $\tau_R \uparrow \zeta$ as $R \uparrow +\infty$. Let $I_w = \{t > 0,\ \xi_t(w) \in [e^{-2}, 1+\delta]\}$.

By (12) and (11),

$$\widetilde{\Phi}''(\xi_t) = \Phi''(\xi_t) \leqslant 0 \quad \text{for } t \notin I_w. \tag{15}$$

Combining (12) and (14), there exists a constant C_1 such that

$$\big|\widetilde{\Phi}''(\xi_t)\big| \leqslant \frac{C_1\Phi(\xi_t)}{(\xi_t|\log\xi_t| + 1)^2}, \quad t \in I_w. \tag{16}$$

By (10) and (13), for some constant C_2, we have

$$\big|\widetilde{\Phi}'(\xi_t)\big| \leqslant \frac{C_2\Phi(\xi_t)}{\xi_t|\log\xi_t| + 1}, \quad t > 0. \tag{17}$$

Now by Itô formula, we have

$$\widetilde{\Phi}(\xi_{t\wedge\tau_R}) = 1 + 2\int_0^{t\wedge\tau_R} \widetilde{\Phi}'(\xi_s)\big\langle\eta_s, \sigma(x_s)\,\mathrm{d}w_s\big\rangle + 2\int_0^{t\wedge\tau_R} \widetilde{\Phi}'(\xi_s)\big\langle\eta_s, b(x_s)\big\rangle\,\mathrm{d}s$$
$$+ \int_0^{t\wedge\tau_R} \widetilde{\Phi}'(\xi_s)\big\|\sigma(x_s)\big\|^2\,\mathrm{d}s + 2\int_0^{t\wedge\tau_R} \widetilde{\Phi}''(\xi_s)\big|\sigma^*(x_s)\eta_s\big|^2\,\mathrm{d}s. \tag{18}$$

By (15) and (16),

$$\int_0^{t\wedge\tau_R} \widetilde{\Phi}''(\xi_s)\big|\sigma^*(x_s)\eta_s\big|^2\,\mathrm{d}s \leqslant \int_0^{t\wedge\tau_R} \mathbf{1}_{I_w}(s)\frac{C_1\Phi(\xi_s)}{(\xi_s|\log\xi_s|+1)^2}\big|\sigma^*(x_s)\eta_s\big|^2\,\mathrm{d}s. \tag{19}$$

By (H2), there exists $C_1 > 0$ such that

$$\begin{cases} \big\|\sigma(x)\big\|^2 \leqslant C_1\big(|x-x_0|^2\log|x-x_0|+1\big), \\ \big|b(x)\big| \leqslant C_1\big(|x-x_0|\log|x-x_0|+1\big). \end{cases}$$

It follows that

$$\frac{|\sigma^*(x_s)\eta_s|^2}{(\xi_s|\log\xi_s|+1)^2} \leqslant C_1\frac{\xi_s(\xi_s|\log\xi_s|+1)}{(\xi_s|\log\xi_s|+1)^2}$$

which is dominated by a constant C_3. According to (19), we get

$$\int_0^{t\wedge\tau_R} \widetilde{\Phi}''(\xi_s)\big|\sigma^*(x_s)\eta_s\big|^2\,\mathrm{d}s \leqslant C_3\int_0^{t\wedge\tau_R} \Phi(\xi_s)\,\mathrm{d}s. \tag{20}$$

In the same way, for some constant $C_4 > 0$, we have

$$\frac{|\langle\eta_s, b(x_s)\rangle| + \|\sigma(x_s)\|^2}{\xi_s|\log\xi_s|+1} \leqslant C_4, \quad s > 0. \tag{21}$$

Now using (18), (17), (21) and (20), we get

$$\mathbb{E}\big(\Phi(\xi_{t\wedge\tau_R})\big) \leqslant \mathbb{E}\big(\widetilde{\Phi}(\xi_{t\wedge\tau_R})\big) \leqslant 1 + C_5\int_0^t \mathbb{E}\big(\Phi(\xi_{s\wedge\tau_R})\big)\,\mathrm{d}s,$$

which implies that $\mathbb{E}(\Phi(\xi_{t\wedge\tau_R})) \leqslant \mathrm{e}^{C_5 t}$. Letting $R \to +\infty$, by Fatou lemma, we get

$$\mathbb{E}\big(\Phi(\xi_{t\wedge\zeta})\big) \leqslant \mathrm{e}^{C_5 t}. \tag{22}$$

Now if $P(\zeta < +\infty) > 0$, then for some $T > 0$, $P(\zeta \leqslant T) > 0$. Taking $t = T$ in (22), we get $\mathbb{E}(\mathbf{1}_{(\zeta\leqslant T)}\Phi(\xi_\zeta)) \leqslant \mathrm{e}^{C_5 t}$ which is impossible, because of $\Phi(\xi_\zeta) = +\infty$. □

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