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Numerical Analysis

Approximation of multi-scale elliptic problems using patches of finite elements

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Abstract

In this paper we present a method to solve numerically elliptic problems with multi-scale data using multiple levels of not necessarily nested grids. The method consists in calculating successive corrections to the solution in patches whose discretizations are not necessarily conforming. It resembles the FAC method (see Math. Comp. 46 (174) (1986) 439–456) and its convergence is obtained by a domain decomposition technique (see Math. Comp. 57 (195) (1991) 1–21). However it is of much more flexible use in comparison to the latter. ***To cite this article: R. Glowinski et al., C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Résumé

Approximation de problèmes elliptiques multi-échelles utilisant des patches d'éléments finis. Dans cette Note nous présentons une méthode faisant apparaître plusieurs niveaux de grilles non nécessairement emboîtées pour résoudre numériquement des problèmes elliptiques à données multi-échelles. La méthode consiste à calculer des corrections successives de la solution par sous-domaines discrétisés de façon non nécessairement conforme. Elle s'apparente à la méthode FAC (voir Math. Comp. 46 (174) (1986) 439–456) et sa convergence s'obtient par une technique de décomposition de domaines (voir Math. Comp. 57 (195) (1991) 1–21). Toutefois elle permet une plus grande souplesse d'utilisation que ces dernières citées. ***Pour citer cet article : R. Glowinski et al., C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

This Note presents several results whose proofs are to be found in [4].

Let V be a Hilbert space with scalar product $(\cdot,\cdot)$ and denote by $\|\cdot\|$ the induced norm. If $\mathcal{L}(V)$ is the space of linear and continuous operators from V into V, we denote by $\|B\| = \sup_{v\in V,\,\|v\|=1}\|Bv\|$ the norm of $B \in \mathcal{L}(V)$.

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Consider V_1, V_2 two closed subspaces of V. We call $P_j : V \to V_j \subset V$ the orthogonal projectors from V into V_j, $j = 1, 2$. If I denotes the identity operator in V and ω is a real parameter, we define the operator $B \in \mathcal{L}(V)$ by

$$B = (I - \omega P_2)(I - \omega P_1). \tag{1}$$

Introduce the number

$$\gamma = \sup_{\substack{v_1 \in V_1,\, v_1 \neq 0 \\ v_2 \in V_2,\, v_2 \neq 0}} \frac{(v_1, v_2)}{\|v_1\| \|v_2\|}. \tag{2}$$

Remark 1. Constant γ is necessarily included in interval [0; 1]. If $V_1 \cap V_2 \neq \{0\}$, then we have $\gamma = 1$. Moreover $\gamma = 0$ if and only if V_1 is orthogonal to V_2.

In the sequel we assume that the following hypothesis is satisfied:

Hypothesis (H). There exists a constant C_0 such that for all $v \in V$ there exist $v_1 \in V_1$, $v_2 \in V_2$ satisfying $v = v_1 + v_2$ and

$$\|v_1\|^2 + \|v_2\|^2 \leqslant C_0^2 \|v\|^2. \tag{3}$$

Remark 2. If (H) is satisfied, we have necessarily $V = V_1 + V_2$. Moreover if $V = V_1 \oplus V_2$, we have necessarily $C_0 \geqslant 1$. If V_1 is orthogonal to V_2, we can take $C_0 = 1$.

Adapting the proof of [3] to the present setting we obtain:

Lemma 1. *If hypothesis* (H) *is satisfied and if* $0 < \omega < 2$, *then the norm of the operator* B *given by* (1) *verifies*

$$\|B\| \leqslant \left(1 - \frac{(2-\omega)\omega}{C_0^2(1+\omega\gamma)^2}\right)^{1/2} < 1. \tag{4}$$

Remark 3. Note that this estimate is not optimal. In particular if $V_1 = V_2 = V$ and $\omega = 1$, the optimum is $\|B\| = 0$ which is not reached with (4).

Remark 4. By applying the Cauchy–Schwarz inequality together with (2) it is easy to show the following: if $V = V_1 \oplus V_2$ and if $\gamma < 1$, then there exists a constant $C_0 \leqslant \sqrt{1/(1-\gamma)}$ which satisfies hypothesis (H).

Now we apply the result of Lemma 1 to the following "multi-scale" situation.

Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain and consider a bilinear, symmetric, continuous and coercive form

$$a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}. \tag{5}$$

If $f \in H^{-1}(\Omega)$, due to Riesz' representation theorem ($a(\cdot,\cdot)$ being a scalar product on $H_0^1(\Omega)$) there exists a unique $u \in H_0^1(\Omega)$ such that

$$a(u, v) = \langle f | v \rangle, \quad \forall v \in H_0^1(\Omega). \tag{6}$$

Let us point out that (6) is the weak formulation of a problem of type $\mathcal{L}(u) = f$ in Ω, $u = 0$ on the boundary $\partial\Omega$ of Ω, where $\mathcal{L}(\cdot)$ is a second order, linear, symmetric, strongly elliptic operator. An approximation of u by "finite elements" consists in introducing a triangulation $\mathcal{T}_H$ of $\overline{\Omega}$, defining $V_H = \{g : \overline{\Omega} \to \mathbb{R}$ continuous such that

$g|_K \in \mathbb{P}_1(K),\ \forall K \in \mathcal{T}_H$ and $g = 0$ on $\partial\Omega\}$, where $\mathbb{P}_1(K)$ is the space of polynomials of degree $\leqslant 1$ on triangle $K \in \mathcal{T}_H$, and calculating $u_H \in V_H$ satisfying

$$a(u_H, v) = \langle f|v\rangle, \quad \forall v \in V_H. \tag{7}$$

Consider now $\Lambda \subset \Omega$ another polygonal domain wherein we would like to obtain a "better" precision on the solution u than the one given by u_H. Take note that $\bar{\Lambda}$ is not necessarily the union of several triangles K of $\mathcal{T}_H$. Besides Λ can be determined in practice by an a posteriori error estimator for example. Let $\mathcal{T}_h$ be a triangulation of $\bar{\Lambda}$ and consider $V_h = \{g : \overline{\Omega} \to \mathbb{R}$ continuous such that $g|_K \in \mathbb{P}_1(K),\ \forall K \in \mathcal{T}_h$ and $g = 0$ on $\overline{\Omega} \setminus \Lambda\}$. Setting $V_{Hh} = V_H + V_h$ we search as approximation for u the function u_{Hh} satisfying

$$a(u_{Hh}, v) = \langle f|v\rangle, \quad \forall v \in V_{Hh}. \tag{8}$$

A priori $V_H \cap V_h$ does not necessarily reduce to the element zero and it is impossible, practically speaking, to exhibit a "finite element"-type basis of the space V_{Hh}. That is why we suggest the following algorithm to compute u_{Hh}:

(1) Set $u^0 = u_H$ and choose $\omega \in (0, 2)$.
(2) For $n = 1, 2, 3, \ldots$ find
 (i) $w_h \in V_h$ such that $a(w_h, v) = \langle f|v\rangle - a(u^{n-1}, v),\ \forall v \in V_h$;
 $u^{n-1/2} = u^{n-1} + \omega w_h$;
 (ii) $w_H \in V_H$ such that $a(w_H, v) = \langle f|v\rangle - a(u^{n-1/2}, v),\ \forall v \in V_H$;
 $u^n = u^{n-1/2} + \omega w_H$.

Remark 5. When implementing the algorithm, the coarse and the fine parts of u^n and $u^{n-1/2}$ are stored separately. In practice this is efficient for calculating the scalar product $a(\cdot,\cdot)$ in the right-hand side of (i) and (ii).

It is readily seen that this algorithm is a domain decomposition method (Schwarz method) with complete overlapping but without any conformity between the meshes $\mathcal{T}_H$ and $\mathcal{T}_h$! It resembles the FAC method (see [7]) or possibly a hierarchical method (see [2]) with a mortar method (see [1]).

When applying Lemma 1 with $V = H_0^1(\Omega)$, we shall use $a(\cdot,\cdot)$ as scalar product.

If $P_h : V_{Hh} \to V_h$ and $P_H : V_{Hh} \to V_H$ are orthogonal projectors from V_{Hh} to V_h and V_H respectively with regard to the scalar product $a(\cdot,\cdot)$, it is easy to verify that

$$u_{Hh} - u^n = (I - \omega P_H)(I - \omega P_h)\big(u_{Hh} - u^{n-1}\big),$$

where I denotes the identity operator in V_{Hh}. Setting $B = (I - \omega P_H)(I - \omega P_h)$ we obtain that $u_{Hh} - u^n = B^n(u_{Hh} - u_H)$. By applying Lemma 1 to $V = V_{Hh}$, $V_1 = V_h$, $V_2 = V_H$, we have proved:

Theorem 2. *If $\omega \in (0, 2)$, then the algorithm* (i), (ii) *converges, i.e.,* $\lim_{n\to\infty} \|u^n - u_{Hh}\|_{H^1(\Omega)} = 0$. *The convergence factor is bounded by* $(1 - \frac{(2-\omega)\omega}{C_0^2(1+\omega\gamma)^2})^{1/2}$.

Remark 6. A priori $\omega = \frac{1}{1+\gamma}$ gives the best factor which minimizes $(1 - \frac{(2-\omega)\omega}{C_0^2(1+\omega\gamma)^2})^{1/2}$. With this ω we obtain $\|B\| \leqslant (1 - \frac{1}{C_0^2(1+2\gamma)})^{1/2}$. If we have a priori $V_H \cap V_h \neq \{0\}$, then $\gamma = 1$, $\omega = \frac{1}{2}$ and $\|B\| \leqslant (1 - \frac{1}{3C_0^2})^{1/2}$.

Remark 7. In the case where the boundary $\partial\Lambda$ of Λ is given by the union of edges of triangles $K \in \mathcal{T}_H$, we can estimate C_0 (see [4]) by splitting $v \in V_{Hh}$ into $v = v_h + v_H$, where $v_H = r_H v$ is the interpolant of v in V_H and $v_h = v - r_H v \in V_h$. It is often possible as in [2,6] to estimate $\frac{a(v_h, v_H)}{\|v_h\|\|v_H\|}$ better than 1.

Remark 8. If we can obtain an estimate for γ with $\gamma < 1$, then by taking $\omega = \frac{1}{1+\gamma}$, $C_0 = \sqrt{\frac{1}{1-\gamma}}$ (see Remark 4), we obtain $\|B\| \leqslant (\frac{3\gamma}{1+2\gamma})^{1/2}$.

Let $a_{ij} \in W^{1,\infty}(\Omega)$, $1 \leqslant i, j \leqslant 2$, verifying $a_{ij} = a_{ji}$ and the hypothesis of strong ellipticity,

$$\sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j \geqslant \alpha \sum_{i=1}^{2} \xi_i^2, \quad \forall(\xi_1, \xi_2) \in \mathbb{R}^2, \text{ a.e. in } \Omega, \tag{9}$$

where α is a positive constant. If $\mathcal{L}$ is the elliptic operator given by $\mathcal{L}(u) = -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial u}{\partial x_j})$, the associated bilinear form is given by $a(u, v) = \sum_{i,j=1}^{2} \int_\Omega a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}\,\mathrm{d}\mathbf{x}$. In this case we obtain the following result:

Theorem 3. *Assume that* (9) *is satisfied and that there exists $K \in \mathcal{T}_H$ such that $\bar{\Lambda} \subset K$. Then*: *if $\beta = [\sum_{j=1}^{2}(\sum_{i=1}^{2}\|\frac{\partial a_{ij}}{\partial x_i}\|_{L^\infty(\Lambda)})^2]^{1/2}$, then $\gamma \leqslant \frac{\beta d}{\alpha}$ where d is the diameter of $\bar{\Lambda}$. By assuming that $\beta < \frac{\alpha}{d}$, we can choose $C_0 = \sqrt{\frac{1}{1-\gamma}}$ (see Remark* 4) *with $\gamma = \frac{\beta d}{\alpha}$ and consequently we obtain $\|B\| \leqslant (1 - \frac{\alpha-\beta d}{\alpha+2\beta d})^{1/2}$ when $\omega = \frac{\alpha}{\alpha+\beta d}$ (see Remark* 8).

Remark 9. If the a_{ij}'s are constant over Λ, $1 \leqslant i, j \leqslant 2$, we clearly have $\beta = 0$, $\gamma = 0$, $C_0 = 1$. In this case V_H and V_h are orthogonal and, since $B = 0$, algorithm (i), (ii) converges in only one iteration.

Numerical example. We illustrate the above presented algorithm with the following example: Consider the Poisson–Dirichlet problem $-\Delta u = f$ in the domain $\Omega = (-1, 1)^2$, $u = 0$ on its boundary $\partial\Omega$. Take $f = f_1 + f_2$ with $f_1 = 2k^2\pi^2\cos(k\pi x_1)\cos(k\pi x_2)$ and $f_2 = -4\eta\chi(r)\exp(\frac{1}{\varepsilon^2})\exp(\frac{-1}{|\varepsilon^2-r^2|})\frac{r^2+r^4-\varepsilon^4}{|\varepsilon^2-r^2|^4}$ where $r = \sqrt{x_1^2 + x_2^2}$ and $\chi(r) = 1$ if $r \leqslant \varepsilon$, $\chi(r) = 0$ if $r > \varepsilon$; k, η and ε are parameters. The exact solution to the problem is given by $u = \cos(k\pi x_1)\cos(k\pi x_2) + \eta\chi(r)\exp(\frac{1}{\varepsilon^2})\exp(\frac{-1}{|\varepsilon^2-r^2|})$.

We choose $k = 0.5$, $\eta = 10$ and $\varepsilon = 0.25$. Away from the origin $(0, 0)$ the solution is smooth. In a region close to $(0, 0)$ where the solution is peaking, we need to apply a patch with a finer mesh. For the triangulation of Ω, we use a coarse uniform grid with mesh size H. We consider a patch $\Lambda = (-\varepsilon, \varepsilon)^2$ ($\supset \operatorname{supp}(f_2)$) with a fine uniform triangulation of size h. We consider first a case where the fine triangulation is nested in the coarse one, then a non-nested case where we slightly stretch the coarse triangulation so that the mesh size is $\widetilde{H} = H/(1 + H)$. The mesh sizes H and h are chosen in a way that the origin $(0, 0)$ is always a grid point. Furthermore, for numerical quadratures and calculating the reference solution, we introduce a global fine uniform triangulation wherein the fine grid is nested. It is an extension of the fine triangulation to the domain Ω in order to minimize the projection errors introduced when comparing the results against the reference solution.

We use the software `Freefem++` [5] to generate the grids and implement the algorithm. We outline here the errors of the solution calculated on the reference grid with $h = 1/12$, $1/24$, $1/48$, $1/96$ and $1/192$. The relative L^2-norm error gives the values 6.41E−2, 2.22E−2, 6.99E−3, 1.87E−3, 4.75E−4; the relative H^1-norm error yields 2.20E−1, 8.13E−2, 2.16E−2 and 5.49E−3, 1.37E−3.

Tables 1 and 2 indicate the convergence of the algorithm to the reference solution with successive smaller H and fixed ratio $H/h = 3$. The stopping criteria for the algorithm is $|e_n - e_{n-1}|/e_0 < 10^{-3}$ where e_n is the relative error at iteration n, $n = 1, 2, \ldots$. In the nested case (Table 1), we observe optimal convergence in the mesh size: we have h^2-accuracy for the L^2-norm and h-accuracy for the H^1-norm. In the non-nested case (Table 2) the rate of convergence deteriorates: both L^2- and H^1-norm only have rate of convergence one.

In Tables 3 and 4 we change the ratio H/h to 6 to investigate its effect on the sensitivity of convergence. This only slightly affects the results.

Table 1
Relative L^2-, H^1- and L^∞-error and convergence of the algorithm for successive smaller H with ratio $H/h = 3$ and nested meshes

H	1/4			1/8			1/16			1/32		
iter.	L^2	H^1	L^∞	L^2	H^1	L^∞	L^2	H^1	L^∞	L^2	H^1	L^∞
0	3.58E−1	7.33E−1	4.46E−1	1.58E−1	5.50E−1	2.24E−1	1.31E−1	4.98E−1	2.26E−1	4.55E−2	2.99E−1	9.83E−2
0.5	7.10E−2	5.29E−2	2.39E−2	2.87E−2	3.21E−2	9.82E−3	2.71E−3	6.56E−3	1.74E−3	6.72E−4	2.81E−3	3.58E−4
1	2.71E−2	2.86E−2	5.21E−3	7.64E−3	1.67E−2	2.25E−3	1.72E−3	5.59E−3	3.46E−4	4.26E−4	2.70E−3	6.59E−5
2	2.61E−2	2.27E−2	3.64E−3	6.83E−3	1.17E−2	1.10E−3	1.70E−3	5.45E−3	2.67E−4	4.26E−4	2.70E−3	6.31E−5
conv.	2.61E−2	2.25E−2	3.66E−3	6.79E−3	1.14E−2	1.07E−3	1.70E−3	5.45E−3	2.67E−4	4.26E−4	2.70E−3	6.31E−5
iter.	3	3	3	3	3	3	2	2	2	2	2	2

Table 2
Relative L^2-, H^1- and L^∞-error and convergence of the algorithm for successive smaller H with ratio $H/h = 3$ and non-nested meshes

H	1/4			1/8			1/16			1/32		
iter.	L^2	H^1	L^∞	L^2	H^1	L^∞	L^2	H^1	L^∞	L^2	H^1	L^∞
0	2.98E−1	6.38E−1	3.60E−1	1.33E−1	4.46E−1	1.74E−1	1.22E−1	4.54E−1	2.26E−1	4.35E−2	2.55E−1	9.43E−2
0.5	8.98E−2	6.64E−2	2.98E−2	2.57E−2	5.66E−2	2.34E−2	1.03E−2	3.39E−2	1.43E−2	2.13E−3	1.34E−2	3.83E−3
1	7.97E−2	1.04E−1	5.48E−2	4.26E−2	9.09E−2	3.55E−2	2.08E−2	5.90E−2	2.32E−2	1.06E−2	3.43E−2	1.22E−2
2	6.31E−2	7.65E−2	4.06E−2	3.48E−2	7.41E−2	2.89E−2	1.77E−2	4.87E−2	2.02E−2	9.76E−3	3.09E−2	1.12E−2
conv.	1.59E−2	1.42E−2	2.51E−3	6.69E−3	2.49E−2	6.84E−3	4.64E−3	1.78E−2	4.00E−3	2.83E−3	1.15E−2	3.91E−3
iter.	13	14	17	30	22	13	22	17	17	36	22	27

Table 3
Relative L^2-, H^1- and L^∞-error and convergence of the algorithm for successive smaller H with ratio $H/h = 6$ and nested meshes

H	1/4			1/8			1/16			1/32		
iter.	L^2	H^1	L^∞	L^2	H^1	L^∞	L^2	H^1	L^∞	L^2	H^1	L^∞
0	4.06E−1	8.56E−1	5.33E−1	2.07E−1	6.34E−1	2.68E−1	1.46E−1	5.27E−1	2.32E−1	4.96E−2	3.13E−1	1.07E−1
0.5	7.19E−2	5.79E−2	2.46E−2	2.89E−2	3.12E−2	9.96E−3	2.94E−3	6.81E−3	1.87E−3	7.32E−4	2.93E−3	3.90E−4
1	2.95E−2	3.18E−2	6.36E−3	8.36E−3	1.71E−2	2.32E−3	1.86E−3	5.79E−3	3.60E−4	4.61E−4	2.81E−3	7.10E−5
2	2.86E−2	2.39E−2	4.05E−3	7.42E−3	1.18E−2	1.22E−3	1.84E−3	5.64E−3	2.77E−4	4.61E−4	2.81E−3	6.82E−5
conv.	2.87E−2	2.35E−2	4.06E−3	7.35E−3	1.14E−2	1.09E−3	1.84E−3	5.64E−3	2.77E−4	4.61E−4	2.81E−3	6.82E−5
iter.	3	3	3	3	3	3	2	2	2	2	2	2

Table 4
Relative L^2-, H^1- and L^∞-error and convergence of the algorithm for successive smaller H with ratio $H/h = 6$ and non-nested meshes

H	1/4			1/8			1/16			1/32		
iter.	L^2	H^1	L^∞	L^2	H^1	L^∞	L^2	H^1	L^∞	L^2	H^1	L^∞
0	3.38E−1	7.87E−1	4.47E−1	1.76E−1	5.48E−1	2.54E−1	1.37E−1	4.93E−1	2.25E−1	4.70E−1	2.75E−1	1.01E−1
0.5	6.79E−2	5.89E−2	2.56E−2	2.29E−2	4.45E−2	1.32E−2	5.24E−3	2.05E−2	7.36E−3	3.85E−3	1.75E−2	4.34E−3
1	3.36E−2	5.16E−2	2.27E−2	2.99E−2	7.05E−2	3.25E−2	1.35E−2	4.34E−2	2.07E−2	9.29E−3	3.44E−2	1.24E−2
2	3.00E−2	3.96E−2	1.97E−2	2.60E−2	6.21E−2	2.92E−2	1.23E−2	3.90E−2	1.87E−2	8.68E−3	3.19E−2	1.14E−2
conv.	1.93E−2	2.21E−2	8.01E−3	8.17E−3	1.81E−2	4.71E−3	3.34E−3	1.43E−2	2.05E−3	1.73E−3	1.02E−2	3.48E−3
iter.	16	15	17	21	20	26	24	18	27	40	27	26

Our algorithm can easily be generalized to multiple levels. For the 3-level case, we introduce a second patch $\Lambda_2 = (-2\varepsilon/3, 2\varepsilon/3)^2 \subset \Lambda$ with a uniform triangulation of size h_2. For creating the non-nested grids, we slightly stretch the triangulations in the same way as for the 2-level case (i.e., $\tilde{h} = h/(1+h/2)$, $\tilde{h}_2 = h_2/(1+h_2)$). The reference grid is also the extension of the finest mesh on Ω. Table 5 illustrates the results with ratios $H/h = h/h_2 = 3$ with nested resp. non-nested grids. In the nested case (first two columns of Table 5) we observe the same rates of convergence as with the 2-level algorithm.

Table 5
Relative L^2-, H^1- and L^∞-error and convergence of the algorithm for successive smaller H with ratio $H/h = 3$, $h/h_2 = 3$ and nested meshes

	nested case						non-nested case					
H	1/4			1/8			1/4			1/8		
iter.	L^2	H^1	L^∞	L^2	H^1	L^∞	L^2	H^1	L^∞	L^2	H^1	L^∞
0	4.33E−1	8.83E−1	5.51E−1	2.18E−1	6.49E−1	3.02E−1	4.33E−1	8.83E−1	5.51E−1	2.18E−1	6.49E−1	3.02E−1
1/3	2.75E−1	2.75E−1	1.21E−1	5.83E−2	9.13E−2	3.22E−2	2.39E−1	2.39E−1	9.19E−2	5.62E−2	1.37E−1	2.84E−2
2/3	7.27E−2	8.38E−2	2.47E−2	2.94E−2	3.75E−2	9.99E−3	9.07E−2	1.76E−1	3.40E−2	3.13E−2	1.26E−1	2.29E−2
1	3.42E−2	6.91E−2	1.89E−2	8.96E−3	2.67E−2	3.86E−3	6.41E−2	1.55E−1	3.70E−1	2.83E−1	1.15E−1	1.82E−2
2	2.94E−2	2.57E−1	4.83E−3	7.56E−3	1.22E−2	1.19E−3	4.60E−2	1.10E−1	2.85E−2	2.51E−2	1.03E−1	1.62E−2
conv.	2.93E−2	2.31E−2	4.04E−3	7.49E−3	1.14E−2	1.07E−3	2.76E−2	4.29E−2	1.14E−2	1.13E−2	5.21E−2	8.94E−3
iter.	4	4	4	4	4	3	8	13	14	26	27	17

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