Available online at www.sciencedirect.com
SCIENCE DIRECT

C. R. Acad. Sci. Paris, Ser. I 337 (2003) 667–673

COMPTES RENDUS MATHEMATIQUE

Probability Theory/Statistics

Limiting laws associated with Brownian motion perturbated by normalized exponential weights

Bernard Roynette [a], Pierre Vallois [a], Marc Yor [b]

[a] *Département de mathématiques, institut Elie Cartan, BP 239, 54506 Vandœuvre-lès-Nancy cedex, France*
[b] *Laboratoire de probabilités et modèles aléatoires, Université Paris VI et VII, case 188, 75252 Paris cedex 05, France*

Received 11 June 2003; accepted 17 September 2003

Presented by Marc Yor

Abstract

We perturb Brownian motion on the time interval $[0,t]$ by an exponential weight; we show that for a large class of these weights the corresponding probability laws converge weakly as $t\to\infty$. ***To cite this article: B. Roynette et al., C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Résumé

Lois limites associées au mouvemnt brownien perturbé par des poids exponentiels normalisés. Nous perturbons le mouvement brownien sur l'intervalle de temps $[0,t]$ par un poids exponentiel ; nous montrons que pour une large classe de tels poids, les lois de probabilités correspondantes convergent étroitement lorsque $t\to\infty$. ***Pour citer cet article : B. Roynette et al., C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Version française abrégée

On perturbe la mesure de Wiener W_x sur $\mathcal{C}(\mathbb{R}_+,\mathbb{R})$ à l'aide du poids exponentiel : $\exp(-\frac{1}{2}\int_0^t u(X_h)\,\mathrm{d}h)$, normalisé par son espérance, $u:\mathbb{R}\to\mathbb{R}$ désignant une fonction borélienne localement bornée. On note $P_{x,t}^{(u)}$ la probabilité ainsi obtenue, et on s'intéresse à l'existence de la limite étroite $Q_x^{(u)}$ de cette famille de probabilités, lorsque $t\to\infty$.

Dans cette Note, on présente l'existence de cette loi limite lorsque u satisfait à l'une des conditions suivantes :

$$u(x)\geqslant 0\,; \qquad \int_{-\infty}^{+\infty}\big(1+|x|\big)u(x)\,\mathrm{d}x<\infty, \tag{C$_1$}$$

E-mail addresses: roynette@iecn.u-nancy.fr (B. Roynette), vallois@iecn.u-nancy.fr (P. Vallois).

doi:10.1016/j.crma.2003.09.025

$$u(x) \geqslant 0\,; \qquad \int_{-\infty}^{0} |x| u(x)\,\mathrm{d}x < \infty; \qquad \varliminf_{x \to +\infty} x^{2\alpha} u(x) \geqslant \gamma > 0, \quad \text{pour un } \alpha < 1, \tag{C$_2$}$$

$$u(x) \geqslant 0\,; \qquad \lim_{|x| \to +\infty} u(x) = l > 0, \tag{C$_3$}$$

$$u(x) = \frac{C''(x)}{C(x)}, \quad \text{avec } C \text{ une fonction positive de classe } C^2, \tag{C$_4$}$$

telle que

(a) C est paire,
(b) C décroit sur $\mathbb{R}_+$,
(c) $\int_0^\infty (C(x))^p \,\mathrm{d}x < \infty$, pour un $p \in (0,1)$,

$$u(x) = \frac{\lambda}{1 + |x|^\alpha}, \quad \lambda > 0,\ 0 < \alpha < 2, \tag{C$_5$}$$

$$u(x) = \frac{\lambda}{\varepsilon + x^2}, \quad \text{with } \varepsilon = 0 \text{ or } 1. \tag{C$_6$}$$

Supposant l'une quelconque de ces conditions satisfaite, la probabilité $Q_x^{(u)}$ est localement équivalente à la mesure de Wiener, et peut être définie par :

$$Q_x^{(u)}|_{\mathcal{F}_s} = \frac{C(X_s)}{C(x)} \exp\left(-\frac{1}{2} \int_0^s u(X_h)\,\mathrm{d}h\right) \cdot W_x|_{\mathcal{F}_s},$$

où $\mathcal{F}_s = \sigma\{X_h;\ h \leqslant s\}$, pour une fonction $C : \mathbb{R} \to \mathbb{R}_+$ solution de l'équation de Sturm–Liouville : $C''(x) = C(x)u(x)$, et satisfaisant certaines conditions frontières en $\pm\infty$.

Bien que ces questions aient été étudiées plus ou moins explicitement par plusieurs auteurs, la littérature est très éparpillée, et il nous a semblé utile de faire le point sur ce sujet. Nous indiquons les réfrénces que nous connaissons ; en l'absence de référence, les résulats que nous présentons semblent nouveaux.

1. A general framework, and a general statement

On the canonical space $\Omega = C(\mathbb{R}_+, \mathbb{R})$ of continuous functions $\omega : t \to \omega(t),\ t \geqslant 0$, we define the filtration $\mathcal{F}_t = \sigma(X_s;\ s \leqslant t),\ t \geqslant 0$, where $X_t(\omega) = \omega(t)$. Let $x \in \mathbb{R}$, and we denote by W_x the distribution on $(\Omega, \mathcal{F}_\infty)$ which makes $(X_t,\ t \geqslant 0)$ a one-dimensional Brownian motion, starting at x.

1.1. In this Note, for a Borel function $u : \mathbb{R} \to \mathbb{R}$, such that, for every t:

$$\mathcal{E}_x^{(u)}(t) = E_{W_x}\left[\exp\left(-\frac{1}{2} \int_0^t u(X_h)\,\mathrm{d}h\right)\right] < \infty,$$

we introduce the family of probabilities, indexed by u, x, t:

$$P_{x,t}^{(u)} = \frac{\exp(-\frac{1}{2}\int_0^t u(X_h)\,\mathrm{d}h)}{\mathcal{E}_x^{(u)}(t)} \cdot\ W_x|_{\mathcal{F}_t},$$

and we are interested in the existence of the weak limit, as $t \to \infty$:

$$P_{x,t}^{(u)} \underset{(w)}{\longrightarrow} Q_x^{(u)}, \tag{1}$$

in the following sense:

$$\text{for } s \geqslant 0,\ \text{and } \Lambda_s \in \mathcal{F}_s, \quad P_{x,t}^{(u)}(\Lambda_s) \underset{t\to\infty}{\longrightarrow} Q_x^{(u)}(\Lambda_s). \tag{2}$$

1.2. Under various conditions on u, we indicate below that the convergence result (2) holds, and that:

$$Q_x^{(u)}|_{\mathcal{F}_t} = \frac{C(X_t)}{C(x)} \exp\left(-\frac{1}{2}\int_0^t u(X_h)\,\mathrm{d}h\right) \cdot W_x|_{\mathcal{F}_t}, \tag{3}$$

where $C:\mathbb{R} \to \mathbb{R}$ solves the Sturm–Liouville equation

$$C''(x) = C(x)u(x), \tag{4}$$

together with some boundary conditions at $\pm\infty$.

Then, from Girsanov's theorem, $Q_x^{(u)}$ is the law of the diffusion Y which solves:

$$Y_t = x + \beta_t + \int_0^t \left(\frac{C'}{C}\right)(Y_s)\,\mathrm{d}s,$$

for $(\beta_t;\ t \geqslant 0)$ a Brownian motion.

1.3. In the next paragraphs, various hypotheses on u are presented, under which the above holds; in fact, we even consider more generally a Radon measure $\mu(\mathrm{d}x)$, instead of $u(x)\,\mathrm{d}x$, and the quantity $\int_0^t u(X_s)\,\mathrm{d}s$ is then replaced by $\int_{-\infty}^{+\infty} L_t^x \mu(\mathrm{d}x)$, where $\{L_t^x\}$ is the family of local times of X. We then write $P^{(\mu)},\ Q^{(\mu)},\ \mathcal{E}^{(\mu)}$ instead of $P^{(u)},\ Q^{(u)},\ \mathcal{E}^{(u)}$.

The key to our results is that, in each case, there exists a function $\varphi:\mathbb{R}_+ \to \mathbb{R}_+$ such that:

$$\varphi(t)\mathcal{E}_x^{(\mu)}(t) \underset{t\to\infty}{\longrightarrow} C(x). \tag{5}$$

Some of these results may be found in the literature, but they are quite scattered; we have indicated the references known to us. Otherwise, the results we present here are, to our knowledge, new.

2. The integrable case

2.1. Under the hypothesis: $\int_{-\infty}^{+\infty}(1+|x|)\,\mathrm{d}\mu(x) < \infty$, for μ a positive Radon measure, the convergence (5) holds with $\varphi(t) = \sqrt{t}$; C is strictly positive, convex, and is the unique solution of the Sturm–Liouville equation (4), such that:

$$\lim_{x\to+\infty} C'(x) = \sqrt{\frac{2}{\pi}}, \qquad \lim_{x\to-\infty} C'(x) = -\sqrt{\frac{2}{\pi}}. \tag{6}$$

Consequently, C satisfies:

$$C(x) \underset{|x|\to+\infty}{\sim} \sqrt{\frac{2}{\pi}}\,|x|. \tag{7}$$

2.2. In order to prove (5) in this case, we rely upon the simplest form of the Tauberian theorem, and we show the following:

$$E_{W_0}\left[\exp\left(-\frac{1}{2}\int_{-\infty}^{+\infty} L^x_{S_\theta}\mu(\mathrm{d}x)\right)\right] \underset{\theta\to 0}{\sim} \theta H_{(\mu)}, \tag{8}$$

where S_θ denotes an exponential variable with parameter $\theta^2/2$, independent of X under W_0, and

$$-H_{(\mu)} = \frac{\phi_{\mu_+}(\infty)+\phi_{\mu_-}(\infty)}{\phi'_{\mu_+}(0)+\phi'_{\mu_-}(0)}, \tag{9}$$

where: $\mu_+ = \mu_{|\mathbb{R}_+}$, μ_- is the image of $\mu_{|\mathbb{R}_-}$ by $x \to -x$, and for a Radon measure ν on $\mathbb{R}_+$, $\phi_\nu : \mathbb{R}_+ \to \mathbb{R}_+$ is the unique decreasing function which solves:

$$\phi''(\mathrm{d}x) = \phi(x)\nu(\mathrm{d}x); \quad \phi(0) = 1. \tag{10}$$

Proofs of (8) may be achieved by using the explicit form of the law of the local times of X up to S_θ (see, e.g., Biane and Yor [1]), or by expressing the left-hand side of (8) as a ratio of integrals with respect to Itô's characteristic measure of excursions (see, e.g., Jeanblanc, Pitman and Yor [4], and/or Revuz and Yor [6]).

3. The unilateral case

Here we assume that $\mu(\mathrm{d}x) = u(x)\,\mathrm{d}x$ and u is 'small' near $-\infty$, and 'large' near $+\infty$; precisely:

$$\int_{-\infty}^{0} |x|u(x)\,\mathrm{d}x < \infty; \qquad \varliminf_{x\to+\infty} x^{2\alpha}u(x) \geqslant \gamma > 0, \quad \text{for some } \alpha < 1. \tag{11}$$

Then, the same statement holds as in 2.1, except that (6) and (7) should be changed into:

$$\lim_{x\to-\infty} C'(x) = -\sqrt{\frac{2}{\pi}}; \qquad \lim_{x\to+\infty} C(x) = 0; \tag{12}$$

$$C(x) \underset{x\to-\infty}{\sim} \sqrt{\frac{2}{\pi}}\,|x|; \qquad C(x) \leqslant k'\,\mathrm{e}^{-k''x^{1-\alpha}}, \quad x \geqslant 0, \tag{13}$$

for some $k', k'' > 0$.

4. The bilateral case

The assumption here is:

$$u(x) \geqslant 0; \qquad \lim_{|x|\to+\infty} u(x) = \infty. \tag{14}$$

This case is the most well known, and has been studied by Kac [5] and Carmona [3] among others.

In particular, the operator:

$$\Delta^{(u)} : f \to -\frac{1}{2}f'' + uf,$$

has a discrete spectrum: $0 < \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \cdots$ of eigenvalues, and

$$\exp(\lambda_1 t)\mathcal{E}^{(u)}_x(t) \quad \text{converges as } t \to \infty. \tag{15}$$

We then refer the reader to 5.3 for the rest of the discussion.

5. The 'signed' case

5.1. Here, we start with a function $C:\mathbb{R}\to\mathbb{R}_+$ such that:

(a) C is even: $C(x)=C(-x)$;
(b) C decreases on $\mathbb{R}_+$,
(c) $\int_0^\infty (C(x))^p\,\mathrm{d}x<\infty$, for some $p\in(0,1)$.

We then define: $u(x)=C''(x)/C(x)$ which is, in general, 'signed'.
Then, (5) holds with $\varphi(t)$ which does not depend on t: $\varphi(t)=1/\overline{C}$, where $\overline{C}=\int_{-\infty}^{+\infty}C(x)\,\mathrm{d}x$.

5.2. As an example, we consider:

$$C(x)=\exp\bigl(-\gamma|x|^\alpha\bigr),\quad \text{for some } \alpha\geqslant 2;$$

then,

$$\frac{C'(x)}{C(x)}=-\gamma\alpha|x|^{\alpha-1}\operatorname{sgn}(x);$$

$$u(x)=\frac{C''(x)}{C(x)}=\left(\frac{C'}{C}\right)'(x)+\left(\frac{C'}{C}\right)^2(x)=-\gamma\alpha(\alpha-1)|x|^{\alpha-2}+\gamma^2\alpha^2|x|^{2(\alpha-1)}.$$

Note in particular that for $\alpha=2$, $Q_x^{(u)}$ is the law of the Ornstein–Uhlenbeck process with parameter (2γ), starting at x.

5.3. For a function $u:\mathbb{R}\to\mathbb{R}_+$ such that

$$u \text{ is continuous, even, increasing when restricted to } \mathbb{R}_+ \text{ and admits a finite limit as } |x|\to\infty. \tag{16}$$

There exists a constant $\gamma_{(u)}>0$ such that: $\exp(\frac{\gamma_{(u)}}{2}t)\mathcal{E}_x^{(u)}(t)$ converges to a strictly positive limit $D(x)$.
The function D is related to u via:

$$u(x)-\gamma_{(u)}=\frac{D''(x)}{D(x)},$$

and the study is reduced to that of 5.1 with D instead of C.
The same discussion applies in the bilateral case (Section 4) with $\gamma_{(u)}=2\lambda_1$.

6. The function u converges 'slowly' to 0 as $|x|\to\infty$

We consider the particular function:

$$u(x)=\frac{\lambda}{1+|x|^\alpha},\quad \text{for } \lambda,\alpha>0. \tag{17}$$

(a) The case $\alpha>2$ belongs to the integrable case (see Section 2);
(b) For $0<\alpha<2$, we could obtain:

$$\lim_{t\to+\infty}\left\{t^{(\alpha-2)/(\alpha+2)}\ln\left(E_x\left[\exp\left(-\frac{\lambda}{2}\int_0^t\frac{\mathrm{d}s}{1+|B_s|^\alpha}\right)\right]\right)\right\}=-\frac{1}{2}\rho(\lambda), \tag{18}$$

where $\rho(\lambda)\in\,]0,+\infty[$ does not depend on x and is equal to:

$$\inf_{\psi\in\Lambda_0}\left\{\int_0^1 \psi'(s)^2\,\mathrm{d}s + \lambda\int_0^1 \frac{\mathrm{d}s}{|\psi(s)|^\alpha}\right\},$$

and $\Lambda_0 = \{f:[0,1]\mapsto\mathbb{R},\ f$ continuous, and $f(0)=0\}$.

Our proof of relation (18) is based on large deviation technique. Unfortunately, we only obtain logarithmic equivalents.

(c) For $\alpha = 2$, we also consider:

$$u(x) = \frac{\lambda}{\varepsilon + x^2};\quad \lambda > 0,\ \varepsilon = 0 \text{ or } 1. \tag{19}$$

Let $n = (1+\sqrt{1+4\lambda})/4$ and $F(a,b,c;x)$ be the hypergeometric function with parameters $a,\ b,\ c$. Then the following result holds:

$$\lim_{t\to+\infty}\left\{t^n\ E_x\left[\exp\left(-\frac{\lambda}{2}\int_0^t \frac{\mathrm{d}s}{\varepsilon + B_s^2}\right)\right]\right\} = A_n C_n(x),\quad x>0, \tag{20}$$

with $A_n = \frac{1}{2^n}\frac{\Gamma(n+1/2)}{\Gamma(2n+1/2)}$, $C_n(x) = x^{2n}$ if $\varepsilon = 0$ and

$$C_n(x) = \begin{cases} \gamma_n F(n-\frac{1}{2}, -n, \frac{1}{2}; -x^2) & \text{if } n \text{ is an integer,}\\ \gamma_n (1+x^2)^{-n+1/2} F(n-\frac{1}{2}, \frac{1}{2}+n, \frac{1}{2}; \frac{x^2}{1+x^2}) & \text{otherwise} \end{cases}$$

with $\gamma_n = \frac{\Gamma(n-1/2)\Gamma(n+1/2)}{\Gamma(2n-1/2)\Gamma(1/2)}$ when $\varepsilon = 1$.

If $\varepsilon = 0$, identity (20) may be obtained as a consequence of the local absolute continuity relationship between the laws of Bessel processes (see [6], Exercise 1.22, Chapter XI, p. 451).

More completely, the same argument allows us to obtain the weak convergence of each of the following family of probabilities indexed by (ν,λ), as $t\to\infty$:

$$P_{r,t}^{(\nu,\lambda)} = \frac{\exp\{-(\lambda/2)\int_0^t (1/R_s^2)\,\mathrm{d}s\}}{\mathcal{E}_r^{(\nu,\lambda)}(t)}\cdot P_{r,t}^{(\nu)}, \tag{21}$$

where $\lambda > 0,\ \nu > -1,\ \mathcal{E}_r^{(\nu,\lambda)}(t)$ is the normalisation factor and $P_{r,t}^{(\nu)}$ denotes the law of the Bessel process $(R_s,\ s\leqslant t)$ starting from $r>0$, with index ν.

The absolute continuity argument implies:

$$P_{r,t}^{(\nu,\lambda)} = \frac{(1/R_t)^{\theta-\nu}}{E_{r,t}^{(\theta)}[(1/R_t)^{\theta-\nu}]}\cdot P_{r,t}^{(\theta)}, \tag{22}$$

where $\theta = \sqrt{\lambda+\nu^2}$.

Finally, a scaling argument yields:

$$P_{r,t}^{(\nu,\lambda)}|_{\mathcal{R}_s} \underset{t\to\infty}{\longrightarrow} P_{r,s}^{(\theta)}, \tag{23}$$

and the Brownian case considered in (19), (20) (with $\varepsilon = 0$) corresponds to: $\nu = -1/2$, hence $\theta = \sqrt{\lambda + 1/4}$.

When $\varepsilon = 1$, the process $(B_s^2;\ s\geqslant 0)$ in (20) may be replaced by the square of a Bessel process with dimension $\delta\geqslant 1$. Relation (20) is still valid, $n,\ A_n$ and $C_n(x)$ being changed by $n_\delta = (2-\delta+\sqrt{(2-\delta)^2+4\lambda})/4$, respectively $(1/2^{n_\delta})\Gamma(n_\delta+\delta/2)/\Gamma(2n_\delta+\delta/2)$, and

$$\begin{cases} \gamma F(n_\delta + \frac{\delta}{2} - 1, -n_\delta, \frac{\delta}{2}; -x^2) & \text{if } n_\delta \text{ is an integer,}\\ \gamma (1+x^2)^{-n_\delta+1-\delta/2} F(n_\delta + \frac{\delta}{2} - 1, \frac{\delta}{2} + n_\delta, \frac{\delta}{2}; \frac{x^2}{1+x^2}) & \text{otherwise,} \end{cases}$$

where $\gamma = \Gamma(n_\delta - 1 + \delta/2)\Gamma(n_\delta+\delta/2)/(\Gamma(2n_\delta - 1 + \delta/2)\Gamma(\delta/2))$.

References

[1] P. Biane, M. Yor, Sur la loi des temps locaux du mouvement brownien réel, pris en un temps exponentiel, in: Sem. Probab. XXII, in: Lecture Notes in Math., Vol. 1321, Springer, 1988, pp. 454–466.
[2] P. Biane, M. Yor, Valeurs principales des temps locaux browniens, Bull. Sci. Math. 111 (1987) 23–101.
[3] R. Carmona, Processus de diffusions gouvernés par la forme de Dirichlet de l'opérateur de Schrödinger, in: Sem. Probab. XIII, in: Lecture Notes in Math., Vol. 721, Springer, 1988, pp. 557–569;
R. Carmona, Opérateur de Schrödinger à résolvante compacte, in: Sem. Probab. XIII, in: Lecture Notes in Math., Vol. 721, Springer, 1988, pp. 570–573.
[4] M. Jeanblanc, J. Pitman, M. Yor, The Feynman–Kac formula and decomposition of Brownian paths, Comput. Appl. Math. 16 (1997) 27–52.
[5] M. Kac, Integration in Function Spaces and some of its Applications, in: Lezioni Fermiane, Acc. Nat. dei Lin., Scuola Normale Superiore di Pisa, 1980.
[6] D. Revuz, M. Yor, Continuous Martingales and Brownian Motion, 3rd edition, Springer, 1999.