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Number Theory

Higher analogues of Stickelberger's theorem

Grzegorz Banaszak

Department of Mathematics, Adam Mickiewicz University, Poznań, Poland

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Abstract

Let l be an odd prime number, F denote any totally real number field and E/F be an Abelian CM extension of F of conductor $\mathbf{f}$. In this paper we prove that for every n odd and almost all prime numbers l we have $S_n(E/F, l) \subset Ann_{\mathbb{Z}_l[G(E/F)]} H^2(\mathcal{O}_E[1/l];\ \mathbb{Z}_l(n+1))$ where $S_n(E/F, l)$ is the Stickelberger ideal (Ann. of Math. 135 (1992) 325–360; J. Coates, p-adic L-functions and Iwasawa's theory, in: Algebraic Number Fields by A. Fröhlich, Academic Press, London, 1977). In addition if we assume the Quillen–Lichtenbaum conjecture then $S_n(E/F, l) \subset Ann_{\mathbb{Z}_l[G(E/F)]} K_{2n}(\mathcal{O}_E)_l$. ***To cite this article: G. Banaszak, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Résumé

Analogues plus hauts du théorème de Stickelberger. Soit l un nombre premier impair, soit F un corps de nombres totalement réel et soit E/F une extension abélienne de conducteur $\mathbf{f}$, où E est un corps de nombres de type CM. Dans cette Note nous prouvons que $S_n(E/F, l) \subset Ann_{\mathbb{Z}_l[G(E/F)]} H^2(\mathcal{O}_E[1/l]; \mathbb{Z}_l(n+1))$ pour tout entier impair $n > 0$ et pour presque tout nombre premier l, où $S_n(E/F, l)$ est l'idéal de Stickelberger (Ann. of Math. 135 (1992) 325–360 ; J. Coates, p-adic L-functions and Iwasawa's theory, in : Algebraic Number Fields by A. Fröhlich, Academic Press, London, 1977). Si nous supposons la conjecture de Quillen–Lichtenbaum alors $S_n(E/F, l) \subset Ann_{\mathbb{Z}_l[G(E/F)]} K_{2n}(\mathcal{O}_E)_l$. ***Pour citer cet article : G. Banaszak, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Version française abrégée

Soit l un nombre premier impair, soit F un corps de nombres totalement réel et soit E/F une extension abélienne de conducteur $\mathbf{f}$, où E est un corps de nombres totalement réel ou un corps de nombres de type CM. Lors du congrès *Algebraic K-theory and Arithmetic*, Newton Institute, Cambridge University, octobre 2002, Victor Snaith a indroduit un idéal fractionnaire $\mathcal{I}_{-n}(l) \in \mathbb{Q}_l[G(E/F)]$ (cf. [12]) pour chaque entier positif n et a proposé les conjectures suivantes

E-mail address: Banaszak@math.amu.edu.pl (G. Banaszak).

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Conjecture A.

$$\left(Ann_{\mathbb{Z}_l[G(E/F)]}H^1\left(\mathcal{O}_E\left[\frac{1}{l}\right];\mathbb{Z}_l(n+1)\right)_l\right)\mathcal{I}_{-n}(l)\subset Ann_{\mathbb{Z}_l[G(E/F)]}H^2\left(\mathcal{O}_E\left[\frac{1}{l}\right];\mathbb{Z}_l(n+1)\right).$$

Conjecture B.

$$\left(Ann_{\mathbb{Z}_l[G(E/F)]}K_{2n+1}(\mathcal{O}_E)_l\right)\mathcal{I}_{-n}(l)\subset Ann_{\mathbb{Z}_l[G(E/F)]}K_{2n}(\mathcal{O}_E)_l.$$

La Conjecture B est une généralisation de la conjecture de Coates et Sinnott [5]. Dans le cas où n est impair et E un corps de nombres totalement réel, l'idéal $Ann_{\mathbb{Z}_l[G(E/F)]}H^1(\mathcal{O}_E[\frac{1}{l}];\mathbb{Z}_l(n+1))_l\mathcal{I}_{-n}(l)$ est égal à l'idéal de Stickelberger $S_n(E/F,\, l)$ d'après V. Snaith [12], Exemple 5.8. Dans le cas où E est un corps de nombres de type CM le rapport entre $Ann_{\mathbb{Z}_l[G(E/F)]}H^1(\mathcal{O}_E[\frac{1}{l}];\mathbb{Z}_l(n+1))_l\mathcal{I}_{-n}(l)$ et $S_n(E/F,l)$ n'est pas connu encore. Dans cette Note nous prouvons le théorème suivant.

Théorème 0.1. *Soit E/F un extension abélienne de corps de nombres totalement réel F par un corps de nombres E de type CM. Soit n un entier impair >0 et soit l un nombre premier l tel que*

$$\left|n\sharp G(E/F)\right|_l^{-1}\frac{|\prod_{v|l}w_n(E_v)|_l^{-1}}{|w_n(E)|_l^{-1}}=1. \tag{1}$$

Alors

$$S_n(E/F,l)\subset Ann_{\mathbb{Z}_l[G(E/F)]}H^2\left(\mathcal{O}_E\left[\frac{1}{l}\right];\mathbb{Z}_l(n+1)\right).$$

Si nous supposons la conjecture de Quillen–Lichtenbaum alors

$$S_n(E/F,l)\subset Ann_{\mathbb{Z}_l[G(E/F)]}K_{2n}(\mathcal{O}_E)_l.$$

Un calcul facile permet de déduire que l'égalité (1) ci-dessus est satisfaite pour presque tout nombre premier l, plus précisément pour tout l premier au nombre $n\,\sharp\, G(E/F)$ qui n'est pas ramifié en $E/\mathbb{Q}$, et tel que $(l-1)\nmid n$.

1. Introduction

Let l be an odd prime number, F denote any totally real number field and E/F be an Abelian CM or totally real extension of F of conductor $\mathbf{f}$. At the conference *Algebraic K-theory and Arithmetic*, held in Newton Institute of Cambridge University in October 2002, Victor Snaith introduced a fractional ideal $\mathcal{I}_{-n}(l)\in\mathbb{Q}_l[G(E/F)]$ (see also [12]) for any nonnegative integer n and stated the following two conjectures

Conjecture A.

$$\left(Ann_{\mathbb{Z}_l[G(E/F)]}H^1\left(\mathcal{O}_E\left[\frac{1}{l}\right];\mathbb{Z}_l(n+1)\right)_l\right)\mathcal{I}_{-n}(l)\subset Ann_{\mathbb{Z}_l[G(E/F)]}H^2\left(\mathcal{O}_E\left[\frac{1}{l}\right];\mathbb{Z}_l(n+1)\right).$$

Conjecture B.

$$\left(Ann_{\mathbb{Z}_l[G(E/F)]}K_{2n+1}(\mathcal{O}_E)_l\right)\mathcal{I}_{-n}(l)\subset Ann_{\mathbb{Z}_l[G(E/F)]}K_{2n}(\mathcal{O}_E)_l.$$

Conjecture B is a generalization from base field $\mathbb{Q}$ to any totally real base field F of the conjecture of Coates and Sinnott [5]. For any ideal $\mathbf{b}$ of $\mathcal{O}_F$ relatively prime to $\mathbf{f}$, define Stickelberger's element (see [4], p. 297, or [1], p. 341 for details)

$$\Theta_n(\mathbf{b},\mathbf{f})=\left(N_{F/\mathbb{Q}}\mathbf{b}^{n+1}-(\mathbf{b};\,E)\right)\sum_{\mathbf{a}}\zeta_{\mathbf{f}}(\mathbf{a};\,-n)(\mathbf{a};\,E)^{-1}$$

which is contained in the group ring $\mathbb{Z}_l[G(E/F)]$ for every l odd by results of Deligne and Ribet [6]. Let $S_n(E/F,l)$ be the ideal in $\mathbb{Z}_l[G(E/F)]$ generated by the elements $\Theta_n(\mathbf{b},\mathbf{f})$ for all ideals $\mathbf{b}$ of $\mathcal{O}_F$ relatively prime to $\mathbf{f}$. $S_n(E/F,l)$ is called the Stickelberger's ideal for the extension E/F and the integer n. In the case of n odd and E totally real, Snaith proved in [12], Example 5.8, that the ideal $Ann_{\mathbb{Z}_l[G(E/F)]}H^1(\mathcal{O}_E[1/l];\mathbb{Z}_l(n+1))_l\mathcal{I}_{-n}(l)$ is equal to $S_n(E/F,l)$. In the case of E a CM field the relation between $Ann_{\mathbb{Z}_l[G(E/F)]}H^1(\mathcal{O}_E[1/l];\mathbb{Z}_l(n+1))_l\mathcal{I}_{-n}(l)$ and $S_n(E/F,l)$ is yet unknown. In this paper I show that substituting the ideal $(Ann_{\mathbb{Z}_l[G(E/F)]}H^1(\mathcal{O}_E[1/l];\mathbb{Z}_l(n+1))_l)\mathcal{I}_{-n}(l)$ in Conjecture A (resp. $(Ann_{\mathbb{Z}_l[G(E/F)]}K_{2n+1}(\mathcal{O}_E)_l)\mathcal{I}_{-n}(l)$ in Conjecture B) with $S_n(E/F,l)$ the following theorem holds.

Theorem 1.1. *Let E/F be arbitrary Abelian extension of a totally real field F by a CM field E. Let n be odd and l be such an odd prime number that*

$$\big|n\sharp G(E/F)\big|_l^{-1}\frac{|\prod_{v|l}w_n(E_v)|_l^{-1}}{|w_n(E)|_l^{-1}}=1. \tag{1}$$

Then

$$S_n(E/F,l)\subset Ann_{\mathbb{Z}_l[G(E/F)]}H^2\left(\mathcal{O}_E\left[\frac{1}{l}\right];\mathbb{Z}_l(n+1)\right)$$

and if in addition Quillen–Lichtenbaum conjecture holds then

$$S_n(E/F,l)\subset Ann_{\mathbb{Z}_l[G(E/F)]}K_{2n}(\mathcal{O}_E)_l.$$

Remark 1. The equality (1) holds for almost all prime numbers l. In particular (1) holds for every l which does not ramify in E, does not divide the number $n\sharp G(E/F)$ and such that $(l-1)\nmid n$. (cf. proof of Proposition 2.1).

The key step in the proof of Theorem 1.1 is the result from [1], Section 4 (see Proposition 3.2 below) which states that for every l not dividing $\sharp G(E/F)$ the Stickelberger's ideal $S_n(E/F,l)$, for $l\nmid n$ (resp. the ideal $nS_n(E/F,l)$, for $l|n$) annihilates the group $D_{n+1}(E)_l$ of divisible elements in $K_{2n}(E)_l$. This annihilation result relies on important results of Deligne and Ribet [6] and Wiles [15]. In the proof of the second part of the theorem, I need to assume Quillen–Lichtenbaum conjecture. One expects that the results and the work in progress of M. Levine, F. Morel, M. Rost, A. Suslin, V. Voevodsky might verify Quillen–Lichtenbaum conjecture in a near future.

In the special case $F=\mathbb{Q}$ and E a cyclotomic extension of $\mathbb{Q}$, Snaith [11] obtained results towards Coates–Sinnott and his conjectures for all $n>0$ (see Theorem 1.6, Corollary 1.7 and Theorem 4.6, loc. cit.). He also needs to assume Quillen–Lichtenbaum conjecture to get results towards Conjecture B in this special case. For n odd, $F=\mathbb{Q}$ and $E/\mathbb{Q}$ abelian the annihilation results in direction of Coates–Sinnott conjecture have already been obtained in [1,2,5,8]. Nevertheless the interesting part of Snaith's work [11] is that the case of n even is also dealt with.

2. Wild kernel and divisible elements

In this section I will state two propositions which describe the size of the group of divisible elements in K-theory with respect to the wild kernel and give an idea of where the number on the left side of (1) comes from.

Let L be any number field. Let $D_{n+1}(L)$ denote the group of divisible elements in $K_{2n}(L)$ (see [1,2]). Then by Theorem 3(i), p. 289, [2], $D_{n+1}(L)_l$ is also the group of divisible elements in $H^2_{\mathrm{cont}}(G_L;\mathbb{Z}_l(n+1))_l$ (cf. [10]).

Let $K_{2m}^{w}(\mathcal{O}_L)_l$ be the wild kernel introduced in [2], p. 281 for any $m > 0$ and let $WK_m(L)$ be the wild kernel introduced in [3], p. 229 for any $m \geqslant 0$.

Proposition 2.1. *For any number field L, any $m > 0$ and any l odd such that*

$$\frac{|\prod_{v|l} w_n(L_v)|_l^{-1}}{|w_n(L)|_l^{-1}} = 1 \tag{2}$$

(*cf.* [2], *p.* 299) *we have equality*

$$K_{2m}^{w}(\mathcal{O}_L)_l = K_{2m}(\mathcal{O}_L)_l. \tag{3}$$

In particular if l does not ramify in L and $(l-1)\nmid n$, then the equality (3) *holds.*

Proof. This follows from Theorem 4, p. 299, [2]. Note that $\mathbb{Q}(\mu_l^{\otimes n}) \subset \mathbb{Q}(\mu_l)$ so if l does not ramify in L then for each $v|l$ we get

$$\mathbb{Q}_l\big(\mu_l^{\otimes n}\big) \cap L_v \subset \mathbb{Q}_l(\mu_l) \cap L_v = \mathbb{Q}_l,$$

which shows that if $(l-1)\nmid n$, then

$$\big|w_n(L_v)\big|_l = \sharp H^0\big(G_{L_v}; \mathbb{Q}_l/\mathbb{Z}_l(n)\big) = 1. \qquad \square$$

Proposition 2.2. *Let L be any number field, and let l be any odd prime number. Assume that the Quillen–Lichtenbaum conjecture holds. Then for any $m > 0$ there is the following equality*

$$K_{2m}^{w}(\mathcal{O}_L)_l = WK_{2m}(L)_l. \tag{4}$$

For any $m \geqslant 0$ there is the following equality

$$WK_{2m}(L)_l = D_{m+1}(L)_l. \tag{5}$$

Proof. Consider the following commutative diagram (cf. [2] and [3])

$$\begin{array}{ccccccc}
0 & \longrightarrow & WK_{2m}(L)_l & \longrightarrow & K_{2n}(L)_l & \longrightarrow & \bigoplus_v K_{2n}(L_v^h)_l \\
 & & \downarrow & & = \downarrow & & \cong \downarrow \\
0 & \longrightarrow & K_{2m}^{w}(\mathcal{O}_L)_l & \longrightarrow & K_{2n}(L)_l & \longrightarrow & \bigoplus_v H^1(G_{L_v^h}; \mathbb{Q}_l/\mathbb{Z}_l(n+1))/Div
\end{array}$$

Direct sums in the diagram are over all finite primes v in L. To prove the Proposition 2.2 we need only to explain why the right vertical arrow in the diagram is an isomorphism. Namely, for each v the field L_v^h (= quotient field of the henselization of $\mathcal{O}_L$ at the prime v) is a direct limit of finite extensions L'/L. Since each L' is a number field then under the Quillen–Lichtenbaum conjecture the natural map

$$K_{2n}(L')_l \longrightarrow H^1\big(G_{L'}; \mathbb{Q}_l/\mathbb{Z}_l(n+1)\big)/Div \tag{6}$$

is an isomorphism for any L' (cf. Remark 7, p. 295, [2]). Passing to the direct limit in (6) over all finite extensions L'/L such that $L' \subset L_v^h$ we immediately get that the right vertical arrow in the diagram is an isomorphism. The equality (5) follows from Theorem 5.2(a) [3]. $\square$

3. Proof of the Theorem 1.1

Proposition 3.1. *For any number field L, any $m > 0$ and any l odd we have equality*

$$\frac{\sharp H^2(G_{L,S_l};\mathbb{Z}_l(n+1))}{\sharp D_{m+1}(L)_l} = \frac{|\prod_{v|l} w_n(L_v)|_l^{-1}}{|w_n(L)|_l^{-1}}. \tag{7}$$

In particular if l does not divide the number (7) then $H^2(G_{L,S_l};\mathbb{Z}_l(n+1)) = D_{m+1}(L)_l$.

Proof. The proof of Proposition 3.1 follows upon chasing in the following commutative diagram with exact rows (see [10] and the proof of Theorem 4, [2])

$$\begin{array}{ccccccc}
0 & \longrightarrow & D_{m+1}(L)_l & \longrightarrow & H^1(G_L;\mathbb{Q}_l/\mathbb{Z}_l(n+1))/Div & \longrightarrow & \\
 & & \downarrow & & \downarrow {\scriptstyle =} & & \\
0 & \longrightarrow & H^1(G_{L,S_l};\mathbb{Q}_l/\mathbb{Z}_l(n+1))/Div & \longrightarrow & H^1(G_L;\mathbb{Q}_l/\mathbb{Z}_l(n+1))/Div & \longrightarrow & \\
 & \longrightarrow & \bigoplus_v H^1(G_{L_v^h};\mathbb{Q}_l/\mathbb{Z}_l(n+1))/Div & \longrightarrow & H^0(G_L;\mathbb{Q}_l/\mathbb{Z}_l(-n))^* & \longrightarrow & 0 \\
 & & {\scriptstyle proj}\downarrow & & & & \\
 & \longrightarrow & \bigoplus_{v\nmid l} H^1(G_{L_v^h};\mathbb{Z}_l(n+1))/Div & \longrightarrow & 0 & &
\end{array}$$

where Div denotes the maximal divisible subgroup in a given group. Note that $H^2(G_{L,S_l};\mathbb{Z}_l(n+1)) \cong H^1(G_{L,S_l};\mathbb{Q}_l/\mathbb{Z}_l(n+1))/Div$ by Proposition 2.3, [14]. $\square$

Proposition 3.2. *Let n be odd and l be an odd prime number such that l does not divide $\sharp G(E/F)$. For any abelian extension E/F of totally real field F by a CM field E, the Stickelberger's ideal $S_n(E/F,l)$ annihilates $D_{n+1}(E)_l$ if $l\nmid n$ and $nS_n(E/F,l)$ annihilates $D_{n+1}(E)_l$ if $l|n$.*

Proof. See Theorem 1 and Corollary 1 of Chapter IV of [1], especially comments in Section 4 of this chapter. We need to take l relatively prime to $\sharp G(E/F)$ since the Wiles' proof of the Brummer Conjecture [15] is not completely correct for primes l dividing $\sharp G(E/F)$ (see [9], Section 4.1). $\square$

Proof of the Theorem 1.1. Under the assumptions of the theorem one gets

$$S_n(E/F,l) \subset Ann_{\mathbb{Z}_l[G(E/F)]} H^2\left(\mathcal{O}_E\left[\frac{1}{l}\right];\mathbb{Z}_l(n+1)\right)$$

by Propositions 3.1 and 3.2. On the other hand observe that Quillen–Lichtenbaum conjecture states that the natural maps defined by Soulé [13] and Dwyer and Friedlander [7],

$$K_{2n}(\mathcal{O}_E)\otimes\mathbb{Z}_l \to H^2\big(G_{E,S_l};\mathbb{Z}_l(n+1)\big),$$
$$K_{2n+1}(\mathcal{O}_E)\otimes\mathbb{Z}_l \to H^1\big(G_{E,S_l};\mathbb{Z}_l(n+1)\big)$$

are isomorphisms. Hence under the assumptions of the theorem one gets

$$S_n(E/F,l) \subset Ann_{\mathbb{Z}_l[G(E/F)]} K_{2n}(\mathcal{O}_E)_l$$

by Propositions 3.1 and 3.2 or equivalently by Propositions 2.2 and 3.2. $\square$

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