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Jensen's inequality for g-expectation: part 1

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Abstract

Briand et al. (Electron. Comm. Probab. 5 (2000) 101–117) gave a counterexample and proposition to show that given g, g-expectations usually do not satisfy Jensen's inequality for most of convex functions. This yields a natural question, under which conditions on g, do g-expectations satisfy Jensen's inequality for convex functions? In this paper, we shall deal with this question in the case that g is convex and give a necessary and sufficient condition on g under which Jensen's inequality holds for convex functions. ***To cite this article: Z. Chen et al., C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Résumé

L'inégalité de Jensen pour la g-espérance. Briand et al. (Electron. Comm. Probab. 5 (2000) 101–117) ont donné un contre-exemple et une proposition qui démontrent que donné g, les g-espérances ne satisfont pas l'inégalité de Jensen pour la majorité des fonctions convexes. Ceci mène donc de façon naturelle à la question : sous quelles conditions sur g les g-espérances satisfont l'inégalité de Jensen pour les fonctions convexes ? Dans cet article, nous obtenons une solution pour un g convexe et donnons une condition nécessaire et suffisante sur g sous laquelle l'inégalité de Jensen est satisfaite pour tout les fonctions convexes. ***Pour citer cet article : Z. Chen et al., C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Version française abrégée

Pardoux et Peng [3] ont démontré que avec des hypothèses sur la valeur finale et les coefficients, une équation différentielle stochastique rétrograde possède comme solution un couple unique. À partir de telles équations stochastiques, Peng [5] introduit la notion de g-espérance. Il démontra que de nombreuses propriétés de l'espérance mathématique classique sont préservées par la g-espérance, toutefois la g-espérance n'est pas linéaire et donc est

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une sorte d'espérance mathématique non linéaire. Peng [4] introduit les notions de g-espérance conditionelle et de g-martingale. De plus, Peng [4], Chen et Peng [2], et Briand, Coquet, Hu, Mémin, et Peng [1] (dorénavant BCHMP) ont étudié des propriétés des g-espérances et des g-martingales : à savoir le théorème de décomposition de Doob–Meyer pour les g-martingales et l'inégalité pour les montées des g-martingales. BCHMP [1] ont aussi étudié l'inégalité pour la g-espérance. Ils donnent un contre-exemple et une proposition indiquant que même pour une fonction linéaire, l'inégalité de Jensen ne sera pas satisfaite pour certaines g-espérances. Tout ceci nous mène donc à la question : sous quelles conditions sur g, l'inégalité de Jensen concernant les g-espérance est-elle satisfaite pour une fonction convexe quelconque ? Dans cet article, nous étudions cette question et donnons la condition nécessaire et suffisante sur g sous laquelle l'inégalité de Jensen est satisfaite.

1. Introduction

Pardoux and Peng [3] showed that under some suitable assumptions on the terminal value and coefficients, a backward stochastic differential equation (BSDE) has a unique pair solution. Based on such BSDE, Peng [5] introduced the notion of g-expectations. He proved that the g-expectation preserves many of properties of the classical mathematical expectation, but not the linearity property, and thus the g-expectation is a type of nonlinear mathematical expectation. Peng [4] introduced the notion of conditional g-expectation and g-martingale. Furthermore, Peng [4], Chen and Peng [2] and Briand, Coquet, Hu, Mémin and Peng [1] (hereafter referred to as BCHMP) studied some properties of g-expectations, and of g-martingales; such as Doob–Meyer decomposition theorem for g-martingales, upcrossing inequality for g-martingales and inverse comparison theorem for BSDEs. BCHMP [1] also studied Jensen's inequality for g-expectations and gave a counterexample and a proposition to indicate that even for a linear function, Jensen's inequality might fail for some g-expectations. This suggests a natural question: under which conditions on g in the g-expectation, does Jensen's inequality hold for any convex function? In this paper, we study this question and give a necessary and sufficient condition on g under which Jensen's inequality holds.

2. Notation and assumptions

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(B_t)_{t\geqslant 0}$ be a d-dimensional standard Brownian motion with respect to filtration $(\mathcal{F}_t)_{t\geqslant 0}$ generated by the Brownian motion and all P-null subsets, i.e.,

$$\mathcal{F}_t = \sigma\left\{B_s, s \in [0,t]\right\} \vee \mathcal{N}, \quad t \in [0,T],$$

where $\mathcal{N}$ is the set of all P-null subsets. Fix a real number $T > 0$. We assume that $\mathcal{F}_T = \mathcal{F}$. Denote

1. $L^2(\Omega, \mathcal{F}_t, P) := \{\xi\colon \xi \text{ is a } \mathcal{F}_t\text{-measurable and } E\xi^2 < \infty\},\ t \in [0,T]$;
2. $L^2(0,T) := \{\psi\colon \psi \text{ is a progressively measurable process with } \mathbf{E}[\int_0^T |\psi_t|^2\,\mathrm{d}t] < \infty\}$.

Let $g(\omega, t, y, z) : \Omega \times [0,T] \times \mathbf{R} \times \mathbf{R}^d \to \mathbf{R}$ satisfy the following conditions:

(A1) (Lipschitz condition) There exists a constant $K \geqslant 0$, such that $\forall (y_1, z_1), (y_2, z_2) \in \mathbf{R}^{1+d}$

$$\left|g(\omega,t,y_1,z_1) - g(\omega,t,y_2,z_2)\right| \leqslant K\left(|y_1-y_2| + |z_1-z_2|\right), \quad t \geqslant 0;$$

(A2) g is continuous in t and $\forall (t,y) \in [0,T] \times \mathbf{R}$,

$$g(t,y,0) \equiv 0,$$

where $|z|$ is the norm of $z \in \mathbf{R}^d$.

Under the above assumptions on g, by Pardoux and Peng's theorem [3], for each $\xi \in L^2(\Omega, \mathcal{F}, P)$, the BSDE

$$y_t = \xi + \int_t^T g(y_s, z_s, s)\,\mathrm{d}s - \int_t^T z_s \cdot \mathrm{d}B_s, \quad 0 \leqslant t \leqslant T, \tag{1}$$

has a unique solution (y_t, z_t), which depends on the terminal value ξ and generator g, and where $x \cdot y$ is the inner product of $x, y \in \mathbf{R}^d$. Peng [4] proposed the following notions:

Definition 2.1 (Peng [4]). Suppose g satisfies (A1) and (A2). For any $\xi \in L^2(\Omega, \mathcal{F}, P)$, let (y_t, z_t) be the solution of BSDE (1), define

$$\mathcal{E}_g[\xi] := y_0.$$

$\mathcal{E}_g[\xi]$ is called the g-expectation of the random variable ξ with respect to g.

Immediately, Peng [4] showed that for each $t \in [0, T]$, there is a unique $\mathcal{F}_t$-measurable random variable $\eta \in L^2(\Omega, \mathcal{F}_t, P)$ such that

$$\mathcal{E}_g[\xi 1_A] = \mathcal{E}_g[\eta 1_A], \quad \text{for all } A \in \mathcal{F}_t,\ 0 \leqslant t \leqslant T.$$

Peng [4] called η the conditional g-expectation of random variable ξ with respect to the σ-algebra $\mathcal{F}_t$ and denoted η by

$$\mathcal{E}_g[\xi|\mathcal{F}_t] := \eta.$$

He found that conditional g-expectation $\mathcal{E}_g[\xi|\mathcal{F}_t]$ actually is the value of $\{y_t\}$, the solution of BSDE (1) at time t, that is

$$\mathcal{E}_g[\xi|\mathcal{F}_t] = y_t, \quad t \in [0, T]. \tag{2}$$

If $g \equiv 0$ then obviously $\mathcal{E}_g[\xi|\mathcal{F}_t] = E[\xi|\mathcal{F}_t]$, $\mathcal{E}_g[\xi] = E[\xi]$.

BCHMP [1] discuss Jensen's inequality for g-expectations and obtained the following proposition when $g(t, y, z)$ is independent of y and convex in z for all $t \geqslant 0$:

Proposition 2.2. *Suppose g satisfies* (A1) *and* (A2) *and that $\varphi : \mathbf{R} \to \mathbf{R}$ is a convex function. Suppose also that $\xi, \varphi(\xi) \in L^2(\Omega, \mathcal{F}, P)$ and that*

$$\partial\varphi\big(\mathcal{E}_g[\xi|\mathcal{F}_t]\big) \cap \,]0, 1[^c \neq \emptyset,$$

where $\partial\varphi$ is the derivative of φ, and $]0, 1[^c$ is the complement set of $]0, 1[= (0, 1)$. Then

$$\varphi\big(\mathcal{E}_g[\xi|\mathcal{F}_t]\big) \leqslant \mathcal{E}_g\big[\varphi(\xi)|\mathcal{F}_t\big], \quad 0 \leqslant t \leqslant T. \tag{3}$$

In particular, if $t = 0$,

$$\varphi\big(\mathcal{E}_g[\xi]\big) \leqslant \mathcal{E}_g\big[\varphi(\xi)\big].$$

If we choose $\varphi(x) = x/2,\ \forall x \in \mathbf{R}$, then $\partial\varphi = \frac{1}{2}$, and Proposition 2.2 does not apply. In fact, Proposition 2.2 and the counterexample in [1] shows that Jensen's inequality usually is not true for g-expectations even when the convex function applied to φ is a linear function. In Section 3 we obtain a necessary and sufficient condition on g so that Jensen's inequality (3) holds for all convex functions whenever $\xi,\ \varphi(\xi) \in L^2(\Omega, \mathcal{F}, P)$.

The following result [2, Proposition 2.3] is needed in the proof of our Theorem 3.1. We rewrite it in the following form:

Proposition 2.3 (Proposition 2.3 [1]). *Suppose $\{X_t\}$ is the following process*:

$$\mathrm{d}X_t = a_t\,\mathrm{d}t + b_t\,\mathrm{d}B_t,$$

where a and b are two continuous, bounded adapted processes. Then

$$\lim_{s\to t}\frac{\mathcal{E}_g[X_s|\mathcal{F}_t]-EX_s}{s-t}=g(t,a_t,b_t),$$

where the limit is in the sense of $L^2(\Omega,\mathcal{F}_t,P)$.

3. Main result

As is done in [1], in this paper we shall consider the case where $g(t,z)$ does not depend on y. This is not a serious restriction since if g is convex and satisfies assumptions (A1) and (A2) then it does not depend on y; see the remark following [1, Lemma 4.5]. The function g is said to be positively homogeneous if for each $z\in\mathbf{R}^d$ and any positive real number $\lambda\geqslant 0$, then $g(t,\lambda z)=\lambda g(t,z)$, $\forall t\in[0,T]$.

In order to simplify the proof of our main result. Theorem 3.1 considers the case of $g:\mathbf{R}^d\to\mathbf{R}$, a function of z only.

Now consider $g:\mathbf{R}^d\to\mathbf{R}$. We introduce our main result on Jensen's inequality for g-expectations.

Theorem 3.1. *Let g be a convex function and satisfy* (A1) *and* (A2) *and $\xi\in L^2(\Omega,\mathcal{F},P)$. Suppose $\varphi:\mathbf{R}\to\mathbf{R}$ is a convex function such that $\varphi(\xi)\in L^2(\Omega,\mathcal{F},P)$. Then*

- (i) *Jensen's inequality* (3) *holds if and only if g is positively homogeneous*;
- (ii) *If $d=1$, the necessary and sufficient condition for Jensen's inequality* (3) *to hold is that there exist two constants $a\geqslant 0$ and b such that $g(z)=a|z|+bz$.*

Proof. *Sufficient condition*: Since g is a convex and positively homogeneous function on $\mathbf{R}^d$, then by the well-known Hahn–Banach extension theorem in finite-dimensional real space $\mathbf{R}^d$ (see Yosida [6, pp. 102, 108]), there exists a convex and closed subset D denoted by

$$D=\left\{b\in\mathbf{R}^d:\ b\cdot z\leqslant g(z),\ \forall z\in\mathbf{R}^d\right\}$$

such that

- (Bi) $g(z)=\sup_{b\in D}b\cdot z$, $\forall z\in\mathbf{R}^d$;
- (Bii) for each $z\in\mathbf{R}^d$, there exists $b(z)\in D$, such that $b(z)\cdot z=g(z)$.

Clearly $b(z)$ is bounded. Indeed under assumptions (A1) and (A2), we have $|g(z)|\leqslant K|z|$, which implies that $|b(z)\cdot z|\leqslant K|z|$ and hence $|b(z)|\leqslant K$.

Given $\xi\in L^2(\Omega,\mathcal{F},P)$, let $(\hat{y}_t,\hat{z}_t)$ be the solution of the following BSDE:

$$y_t=\xi+\int_t^T g(z_s)\,\mathrm{d}s-\int_t^T z_s\cdot\mathrm{d}B_s,\quad 0\leqslant t\leqslant T,$$

which can be written as

$$\hat{y}_t=\xi+\int_t^T b(\hat{z}_s)\cdot\hat{z}_s\,\mathrm{d}s-\int_t^T\hat{z}_s\cdot\mathrm{d}B_s=\xi-\int_t^T\hat{z}_s\,\mathrm{d}\overline{W}_s,\tag{4}$$

where $\overline{W}_t := B_t - \int_0^t b(\hat{z}_s)\,\mathrm{d}s$. Set

$$\frac{\mathrm{d}Q}{\mathrm{d}P} = \exp\left(-\frac{1}{2}\int_0^T \left|b(\hat{z}_s)\right|^2 \mathrm{d}s - \int_0^T b(\hat{z}_s)\,\mathrm{d}B_s\right).$$

Note that since $b(\cdot)$ is bounded, thus $\{\overline{W}_t\}$ is a Q-Brownian motion.

Taking conditional expectation $E_Q[\cdot|\mathcal{F}_t]$ on the both sides of BSDE (4), and since $\mathcal{E}_g[\xi|\mathcal{F}_t] = \hat{y}_t$ by (2), we obtain

$$\mathcal{E}_g[\xi|\mathcal{F}_t] = \mathbf{E}_Q[\xi|\mathcal{F}_t].$$

Applying the classical Jensen's inequality yields

$$\varphi\big(\mathcal{E}_g[\xi|\mathcal{F}_t]\big) = \varphi\big(E_Q[\xi|\mathcal{F}_t]\big) \leqslant E_Q\big[\varphi(\xi)|\mathcal{F}_t\big].$$

For the given $b(\hat{z}_t)$ and probability measure Q, let us now consider $\bar{y}_t := E_Q[\varphi(\xi)|\mathcal{F}_t]$. It is easy to check, by Pardoux and Peng's theorem [3] and recalling Eq. (1), that there exists $\{\bar{z}\} \in L^2(0,T)$ such that $(\bar{y}_t, \bar{z}_t)$ is the solution of the following BSDE:

$$\bar{y}_t = \varphi(\xi) + \int_t^T b(\hat{z}_s)\cdot \bar{z}_s\,\mathrm{d}s - \int_t^T \bar{z}_s \cdot \mathrm{d}B_s, \quad 0 \leqslant t \leqslant T. \tag{5}$$

Also consider

$$y_t = \varphi(\xi) + \int_t^T g(z_s)\,\mathrm{d}s - \int_t^T z_s \cdot \mathrm{d}B_s, \quad 0 \leqslant t \leqslant T. \tag{6}$$

Comparing BSDEs (5) and (6), and noting that from (Bi) $b(\hat{z}_t)\cdot z \leqslant g(z),\ \forall z \in \mathbf{R}^d$, and applying the comparison theorem of BSDEs (see Peng [4]), we have

$$E_Q\big[\varphi(\xi)|\mathcal{F}_t\big] = \bar{y}_t \leqslant y_t = \mathcal{E}_g\big[\varphi(\xi)|\mathcal{F}_t\big], \quad \forall t \in [0,T].$$

This completes the proof of the sufficient condition part.

Necessary condition: For each $z \in \mathbf{R}^d$ choose $X_s := z\cdot(B_s - B_t),\ t < s \leqslant T$. Then $E[X_s] = 0$.

For any $\lambda > 0$, consider $\varphi(x) := \lambda x$. Clearly $\varphi(\cdot)$ is convex and $X_s,\ \varphi(X_s) \in L^2(\Omega, \mathcal{F}, P)$. If Jensen's inequality (3) holds then

$$\lambda\mathcal{E}_g[X_s|\mathcal{F}_t] \leqslant \mathcal{E}_g[\lambda X_s|\mathcal{F}_t], \quad t \leqslant s \leqslant T.$$

This and the fact that $E[X_s] = 0$ then implies

$$\frac{\lambda\mathcal{E}_g[X_s|\mathcal{F}_t] - \lambda E[X_s]}{s-t} \leqslant \frac{\mathcal{E}_g[\lambda X_s|\mathcal{F}_t] - E[\lambda X_s]}{s-t}, \quad t < s \leqslant T.$$

Letting $s \to t$ and applying Proposition 2.3 yields

$$\lambda g(z) \leqslant g(\lambda z), \quad \forall \lambda \geqslant 0, \quad z \in \mathbf{R}^d. \tag{7}$$

Since g is convex with $g(0) = 0$, then for any $z \in \mathbf{R}^d$ and $0 \leqslant \lambda \leqslant 1$ we have $g(\lambda z) \leqslant \lambda g(z)$. This with inequality (7) implies

$$g(\lambda z) = \lambda g(z), \quad \lambda \in [0,1]. \tag{8}$$

We still need to show that the equality (8) is true for any $\lambda > 1$. If $\lambda > 1$, then $0 < 1/\lambda < 1$ and hence by (8)

$$g(\lambda z) = \lambda \times \frac{1}{\lambda} g(\lambda z) = \lambda g\left(\frac{1}{\lambda} \times \lambda \times z\right), \quad \forall z \in \mathbf{R}^d.$$

Since z is arbitrary in $\mathbf{R}^d$ then (8) holds for any $\lambda \geqslant 0$.

The proof of Theorem 3.1 part (i) is now complete.

We now prove part (ii). We only need to show that if $d = 1$, then a positively homogeneous function $g(z)$ is of the form $g(z) = a|z| + bz$. Indeed, if $d = 1$ and g is a positively homogeneous function on $\mathbf{R}$, then

$$g(z) = g(z)I_{[z\geqslant 0]} + g(z)I_{[z\leqslant 0]} = g(1)zI_{[z\geqslant 0]} + g(-1)(-z)I_{[z\leqslant 0]}. \tag{9}$$

Note that $zI_{[z\geqslant 0]} = z^+,\ (-z)I_{[z\leqslant 0]} = z^-$, but

$$z^+ = \frac{|z| + z}{2}, \qquad z^- = \frac{|z| - z}{2}.$$

Thus from (9)

$$g(z) = \frac{g(1) + g(-1)}{2}|z| + \frac{g(1) - g(-1)}{2}z.$$

Defining $a := \frac{g(1)+g(-1)}{2},\ b := \frac{g(1)-g(-1)}{2}$. Obviously $a \geqslant 0$, since the convexity of g yields

$$\frac{g(1) + g(-1)}{2} \geqslant g(0) = 0.$$

The proof of Theorem 3.1 part (ii) is now complete. □

Part 2 of this Note will consider some applications of Jensen's inequality for g-expectations.

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