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C. R. Acad. Sci. Paris, Ser. I 337 (2003) 609–614
COMPTES RENDUS MATHEMATIQUE

Probability Theory

# Smoothness of Wigner densities on the affine algebra

Uwe Franz [a], Nicolas Privault [b], René Schott [c]

[a] Institut für Mathematik und Informatik, Ernst-Moritz-Arndt-Universität Greifswald, Jahnstraße 15a, 17487 Greifswald, Germany
[b] Département de mathématiques, Université de La Rochelle, 17042 La Rochelle, France
[c] Institut Elie Cartan and LORIA, BP 239, Université H. Poincaré-Nancy I, 54506 Vandœuvre-lès-Nancy, France

Received 23 July 2003; accepted 17 September 2003

Presented by Paul Malliavin

**Abstract**

The non-commutative Malliavin calculus on the Heisenberg–Weyl algebra (see (i) C. R. Acad. Sci. Paris, Sér. I 328 (11) (1999) 1061–1066, (ii) Infin. Dimens. Anal. Quantum Probab. Relat. Top. 4 (1) (2001) 11–38) is extended to the affine algebra. A differential calculus is established, which generalizes the corresponding commutative integration by parts formulas. As an application we obtain sufficient conditions for the smoothness of Wigner type laws of non-commutative random variables with gamma and continuous binomial marginals. ***To cite this article: U. Franz et al., C. R. Acad. Sci. Paris, Ser. I 337 (2003).***
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**Résumé**

**Régularité de densités de Wigner sur l'algèbre affine.** Le calcul de Malliavin non-commutatif sur l'algèbre de Heisenberg–Weyl (voir (i) C. R. Acad. Sci. Paris, Sér. I 328 (11) (1999) 1061–1066, (ii) Infin. Dimens. Anal. Quantum Probab. Relat. Top. 4 (1) (2001) 11–38) est étendu à l'algèbre affine. Un calcul différentiel non-commutatif qui généralise les formules d'intégration par parties classiques est établi. Comme application nous obtenons des conditions suffisantes pour la régularité de lois de Wigner pour des variables aléatoires non-commutatives de lois marginales gamma et binomiale continue. ***Pour citer cet article : U. Franz et al., C. R. Acad. Sci. Paris, Ser. I 337 (2003).***
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## Version française abrégée

Dans [5] un calcul de Malliavin non-commutatif a été introduit sur l'algèbre de Heisenberg–Weyl $\{\mathbf{p}, \mathbf{q}, I\}$, avec $[\mathbf{p}, \mathbf{q}] = 2\mathrm{i}I$, en généralisant aux densités de Wigner le calcul de Malliavin par rapport aux variables gaussiennes. En particulier ceci permet de prouver la régularité de densités de Wigner [9] ayant des marginales gaussiennes. Dans cette Note nous traitons d'autres lois de probabilité dans un cadre plus général, voir [2] pour des références sur les applications de ces densités de Wigner généralisées. En particulier nous considérons des variables aléatoires non-commutatives ayant des marginales de loi gamma et binomiale continue. Pour cela nous utilisons la construction de telles variables aléatoires sur les algèbres de Lie, à partir de résultats généraux de [2]. En utilisant une représentation de l'algèbre affine sur un espace de Hilbert $\mathcal{H}$ donnée par $X_1 = -\frac{\mathrm{i}}{2}P$ et $X_2 = \mathrm{i}(Q + M)$ avec

*E-mail addresses:* franz@uni-greifswald.de (U. Franz), nprivaul@univ-lr.fr (N. Privault), schott@loria.fr (R. Schott).

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doi:10.1016/j.crma.2003.09.014

$[X_1, X_2] = X_2$, nous obtenons une expression de la densité jointe de $(P, Q+M)$ à l'aide de fonctions de Wigner, et calculons la fonction caractéristique :

$$\langle \phi, \mathrm{e}^{\mathrm{i}uP+\mathrm{i}v(Q+M)}\psi\rangle_{\mathcal{H}} = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}v\omega\,\mathrm{sinch}\,u}\bar{\phi}\big(\omega\,\mathrm{e}^{u}\big)\psi\big(\omega\,\mathrm{e}^{-u}\big)\,\mathrm{e}^{-|\omega|\cosh u}\frac{|\omega|^{\beta-1}}{\Gamma(\beta)}\,\mathrm{d}\omega, \quad \phi, \psi \in \mathcal{H}.$$

Nous montrons ensuite qu'un opérateur $O$ satisfaisant

$$O\big(\mathrm{e}^{-\mathrm{i}u\xi_1-\mathrm{i}v\xi_2}\big) = \mathrm{e}^{-\frac{\mathrm{i}}{2}uP+\mathrm{i}v(Q+M)}$$

peut être étendu par continuité à $L^2_{\mathbb{C}}(\mathbb{R}^2; \frac{\mathrm{d}\xi_1\,\mathrm{d}\xi_2}{2\pi|\xi_2|})$ avec

$$O(f) = \int_{\mathbb{R}^2} (\mathcal{F}f)(x_1, x_2)\,\mathrm{e}^{-\frac{\mathrm{i}}{2}x_1P+\mathrm{i}x_2(Q+M)}\,\mathrm{d}x_1\,\mathrm{d}x_2,$$

où $\mathcal{F}$ représente la transformée de Fourier. A l'aide de l'opérateur gradient

$$D_xF = -\frac{\mathrm{i}}{2}x_1[P, F] + \frac{\mathrm{i}}{2}x_2[Q+M, F], \quad x = (x_1, x_2) \in \mathbb{R}^2,$$

qui agit sur une classe d'opérateurs suffisamment réguliers de $\mathcal{H}$, nous obtenons la formule d'entrelacement

$$D_{(x_1,2x_2)}O(f) = O\big(x_1\xi_2\partial_1 f(\xi_1, \xi_2) - x_2\xi_2\partial_2 f(\xi_1, \xi_2)\big), \quad x_1, x_2 \in \mathbb{R},$$

qui permet d'établir la régularité de la loi jointe de $(P, Q+M)$ par intégration par parties non commutative. Nous définissons aussi un opérateur

$$\delta(F \otimes x) = \frac{x_1}{2}\big\{Q + \alpha(M-\beta), F\big\} + \frac{x_2}{2}\{P, F\} - D_xF, \quad x = (x_1, x_2) \in \mathbb{R}^2,$$

analogue de l'intégrale de Skorohod, et qui satisfait une formule d'intégration par parties.

## 1. Random variables on the affine algebra

Let $a^-$, $a^+$ denote the boson annihilation and creation operators and let $\mathbf{q} = a^- + a^+$, $\mathbf{p} = \mathrm{i}(a^- - a^+)$, with $[\mathbf{p}, \mathbf{q}] = 2\mathrm{i}I$. The joint law of $(\mathbf{p}, \mathbf{q})$ is called a Wigner law [9], and has Gaussian marginals in the vacuum state. Moreover, $\{\mathbf{p}, \mathbf{q}, I\}$, with $[\mathbf{p}, \mathbf{q}] = 2\mathrm{i}I$, yields a representation of the Heisenberg–Weyl algebra.

Let now $\tilde{a}^-_\tau = \tau\partial_\tau$, i.e., $\tilde{a}^-_\tau f(\tau) = \tau f'(\tau)$, $f \in \mathcal{C}^\infty_b(\mathbb{R})$. The adjoint $\tilde{a}^+_\tau$ of $\tilde{a}^-_\tau$ with respect to the gamma density $\gamma_\beta(\tau) = 1_{\{\tau\geqslant 0\}}\frac{\tau^{\beta-1}}{\Gamma(\beta)}\mathrm{e}^{-\tau}$ on $\mathbb{R}_+$ satisfies

$$\int_0^\infty g(\tau)\tilde{a}^-_\tau f(\tau)\gamma_\beta(\tau)\,\mathrm{d}\tau = \int_0^\infty f(\tau)\tilde{a}^+_\tau g(\tau)\gamma_\beta(\tau)\,\mathrm{d}\tau, \quad f, g \in \mathcal{C}^\infty_b(\mathbb{R}), \tag{1}$$

and is given by $\tilde{a}^+_\tau = (\tau-\beta) - \tilde{a}^-_\tau$. The multiplication operator $\tilde{a}^- + \tilde{a}^+ = \tau - \beta$ has a compensated gamma law (or spectral measure) in the vacuum state in $L^2_{\mathbb{C}}(\mathbb{R}_+, \gamma_\beta(\tau)\,\mathrm{d}\tau)$. In [8] it has been noticed that when $\beta = 1$, $\mathrm{i}(\tilde{a}^- - \tilde{a}^+)$ has the continuous binomial density $(2\cosh\pi\xi_1/2)^{-1}$, in relation to a representation of the subgroup of upper-triangular matrices of $\mathrm{sl}_2$. This type of law can be studied for all $\beta > 0$ in the general framework of [1], starting from a representation $(M, B^-, B^+)$ of $\mathrm{sl}_2$: $[B^-, B^+] = M$, $[M, B^-] = -2B^-$, $[M, B^+] = 2B^+$, which can be constructed as $M = \beta + 2\tilde{a}^\circ_\tau$, $B^- = \tilde{a}^-_\tau - \tilde{a}^\circ_\tau$, $B^+ = \tilde{a}^+_\tau - \tilde{a}^\circ_\tau$, with $\tilde{a}^\circ_\tau = \tilde{a}^+_\tau\partial_\tau = -(\beta-\tau)\partial - \tau\partial^2$. Letting $Q = B^- + B^+ = \tilde{a}^- + \tilde{a}^+ - 2\tilde{a}^\circ$ and $P = \mathrm{i}(B^- - B^+) = \mathrm{i}(\tilde{a}^- - \tilde{a}^+)$, we have $Q + M = \tau$ and more generally $Q + \alpha M$ has a gamma law when $\alpha = \pm 1$, whereas $P = \mathrm{i}(\tilde{a}^- - \tilde{a}^+)$ has a continuous binomial distribution with parameter $\beta$. The Heisenberg–Weyl Malliavin calculus of [4,5] relies on a functional calculus which allows to define the composition of a function on $\mathbb{R}^2$ with a couple of non-commutative random variables, and on a covariance identity which plays the role of integration by parts formula. Here, $\{-\frac{\mathrm{i}}{2}P, \mathrm{i}(Q+M)\}$ form a representation of the affine algebra: $[-\frac{\mathrm{i}}{2}P, \mathrm{i}(Q+M)] = \mathrm{i}(Q+M)$. In order to extend the construction of [4, 5] to the gamma and continuous binomial laws we will use the formalism of [2] which provides in particular a functional calculus on the affine algebra.

## 2. Functional calculus on the affine algebra

The affine group can be constructed as the group of $2\times 2$ matrices of the form

$$g = \begin{pmatrix} \mathrm{e}^{x_1} & x_2\,\mathrm{e}^{x_1/2}\operatorname{sinch}\frac{x_1}{2} \\ 0 & 1 \end{pmatrix} = \mathrm{e}^{x_1X_1+x_2X_2}, \quad x_1, x_2 \in \mathbb{R},$$

where $\operatorname{sinch} x = (\sinh x)/x$, $x \in \mathbb{R}$, and $X_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $X_2 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, generate the affine algebra, with $[X_1, X_2] = X_2$. Consider the representation of the affine group on $\mathcal{H} = L^2_{\mathbb{C}}(\mathbb{R}, \gamma_\beta(|\tau|)\,\mathrm{d}\tau)$ defined by

$$\big(\widehat{U}(g)\phi\big)(\tau) = \phi(a\tau)\,\mathrm{e}^{\mathrm{i}b\tau}\,\mathrm{e}^{-(a-1)|\tau|/2}a^{\beta/2}, \quad \phi \in L^2_{\mathbb{C}}\big(\mathbb{R}, \gamma_\beta\big(|\tau|\big)\,\mathrm{d}\tau\big),\ g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}.$$

We have $\widehat{U}(X_1) = -\frac{\mathrm{i}}{2}P$ and $\widehat{U}(X_2) = \mathrm{i}(Q+M)$. Given $\phi, \psi \in \mathcal{H}$, let

$$W_{|\phi\rangle\langle\psi|}(\xi_1, \xi_2) = \int_{\mathbb{R}} \phi\left(\frac{\xi_2\,\mathrm{e}^{-x/2}}{\operatorname{sinch} x/2}\right)\frac{|\xi_2|\,\mathrm{e}^{-\mathrm{i}x\xi_1}}{\operatorname{sinch} x/2}\bar{\psi}\left(\frac{\xi_2\,\mathrm{e}^{x/2}}{\operatorname{sinch} x/2}\right)\mathrm{e}^{-|\xi_2|\frac{\cosh x/2}{\operatorname{sinch} x/2}}\left(\frac{|\xi_2|}{\operatorname{sinch} x/2}\right)^{\beta-1}\frac{\mathrm{d}x}{\Gamma(\beta)}, \tag{2}$$

$\xi_1, \xi_2 \in \mathbb{R}$, denote the Wigner function on the affine algebra, cf. (102) of [2]. The next two propositions are obtained by computing the action of $\mathrm{e}^{-\frac{\mathrm{i}}{2}uP+\mathrm{i}v(Q+M)} = \widehat{U}(\mathrm{e}^{uX_1+vX_2})$ in two different ways, using results of [2], see [6].

**Proposition 2.1.** *Let $\phi, \psi \in \mathcal{H}$. We have*

$$\big\langle\psi\,|\,\mathrm{e}^{\frac{\mathrm{i}}{2}uP-\mathrm{i}v(Q+M)}\phi\big\rangle_{\mathcal{H}} = \int_{\mathbb{R}^2} \mathrm{e}^{\mathrm{i}u\xi_1+\mathrm{i}v\xi_2}W_{|\phi\rangle\langle\psi|}(\xi_1, \xi_2)\frac{\mathrm{d}\xi_1\,\mathrm{d}\xi_2}{2\pi|\xi_2|}, \quad u, v \in \mathbb{R}.$$

As a consequence, the joint density of $(\frac{1}{2}P, -(Q+M))$ in the state $|\phi\rangle\langle\psi|$ is given as

$$\widetilde{W}_{|\phi\rangle\langle\psi|}(\xi_1, \xi_2) = \frac{1}{2\pi|\xi_2|}W_{|\phi\rangle\langle\psi|}(\xi_1, \xi_2), \quad \xi_1, \xi_2 \in \mathbb{R}.$$

**Proposition 2.2.** *The characteristic function of $(P, Q+M)$ in the state $|\phi\rangle\langle\psi|$ is given by*

$$\big\langle\psi, \mathrm{e}^{\mathrm{i}uP+\mathrm{i}v(Q+M)}\phi\big\rangle_{\mathcal{H}} = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}v\omega\operatorname{sinch} u}\bar{\psi}(\omega\,\mathrm{e}^{u})\phi(\omega\,\mathrm{e}^{-u})\,\mathrm{e}^{-|\omega|\cosh u}\frac{|\omega|^{\beta-1}}{\Gamma(\beta)}\,\mathrm{d}\omega, \quad u, v \in \mathbb{R}.$$

In the vacuum state $\Omega = 1_{\mathbb{R}_+}$ we have

$$\big\langle\Omega, \mathrm{e}^{\mathrm{i}uP+\mathrm{i}v(Q+M)}\Omega\big\rangle_{\mathcal{H}} = \frac{1}{(\cosh u - \mathrm{i}v\operatorname{sinch} u)^\beta}.$$

Note that $\widetilde{W}_{|\psi\rangle\langle\phi|}$ has the correct marginals, since:

$$\int_{\mathbb{R}} \widetilde{W}_{|\phi\rangle\langle\psi|}(\xi_1, \xi_2)\,\mathrm{d}\xi_1 = \gamma_\beta(|\xi_2|)\bar{\phi}(\xi_2)\psi(\xi_2), \quad \xi_2 \in \mathbb{R},$$

and

$$\int_{\mathbb{R}} \widetilde{W}_{|\psi\rangle\langle\phi|}(\xi_1,\xi_2)\,\mathrm{d}\xi_2 = \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{e}^{-\mathrm{i}\xi_1 x}\bar{\phi}\big(\omega\,\mathrm{e}^{x/2}\big)\psi\big(\omega\,\mathrm{e}^{-x/2}\big)\,\mathrm{e}^{-|\omega|\cosh x/2}\frac{|\omega|^{\beta-1}}{\Gamma(\beta)}\,\mathrm{d}x\,\mathrm{d}\omega,$$

$\xi_1 \in \mathbb{R}$. In the vacuum state $\Omega = 1_{\mathbb{R}_+}$, this yields respectively a Gamma law and the density

$$\int_{\mathbb{R}} W_{|\Omega\rangle\langle\Omega|}(\xi_1,\xi_2)\frac{\mathrm{d}\xi_2}{2\pi\xi_2} = c\left|\Gamma\left(\frac{\beta}{2}+\frac{\mathrm{i}}{2}\xi_1\right)\right|^2,$$

where $c$ is a normalization constant and $\Gamma$ is the Gamma function, which gives the expected hyperbolic cosine density when $\beta = 1$.

**Definition 2.3.** For $f$ in the Schwartz space $\mathcal{S}(\mathbb{R}^2)$, let the operator $O(f)$ be defined as

$$O(f) = \int_{\mathbb{R}^2} (\mathcal{F}f)(x_1,x_2)\,\mathrm{e}^{-\frac{\mathrm{i}}{2}x_1 P + \mathrm{i}x_2(Q+M)}\,\mathrm{d}x_1\,\mathrm{d}x_2.$$

The following proposition extends the definition of $O$ by continuity to a map from $L^2_{\mathbb{C}}(\mathbb{R}^2; \frac{\mathrm{d}\xi_1\,\mathrm{d}\xi_2}{2\pi|\xi_2|})$ into the space $\mathcal{B}_2(\mathcal{H})$ of Hilbert–Schmidt operators on $\mathcal{H}$. It is obtained from the isometry given by the representation theorem of square-integrable representations of [3].

**Proposition 2.4.** *We have the bound*

$$\big\|O(f)\big\|_{\mathcal{B}_2(\mathcal{H})} \leqslant \|f\|_{L^2_{\mathbb{C}}(\mathbb{R}^2; \frac{\mathrm{d}\xi_1\,\mathrm{d}\xi_2}{2\pi|\xi_2|})}.$$

Note that we have

$$\big\langle\psi, O(f)\phi\big\rangle_{\mathcal{H}} = \int_{\mathbb{R}^2} \widetilde{W}_{|\phi\rangle\langle\psi|}(\xi_1,\xi_2) f(\xi_1,\xi_2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2, \quad f \in L^2_{\mathbb{C}}\left(\mathbb{R}^2; \frac{\mathrm{d}\xi_1\,\mathrm{d}\xi_2}{2\pi|\xi_2|}\right),$$

and $O(\mathrm{e}^{-\mathrm{i}u\xi_1 - \mathrm{i}v\xi_2}) = \mathrm{e}^{-\frac{\mathrm{i}}{2}uP + \mathrm{i}v(Q+M)} = \widehat{U}(\mathrm{e}^{uX_1+vX_2}),\ u, v \in \mathbb{R}$.

## 3. Malliavin calculus on the affine algebra

We define a gradient operator which will be useful in showing the smoothness of Wigner densities. Let $\mathcal{S}_{\mathcal{H}}$ denote the algebra of operators on $\mathcal{H}$ that leave $\mathcal{S}(\mathbb{R})$ invariant.

**Definition 3.1.** Fix $\kappa \in \mathbb{R}$ and let $x = (x_1, x_2) \in \mathbb{R}^2$. The gradient operator $D_x$ is defined as

$$D_x F = -\frac{\mathrm{i}}{2}x_1[P,F] + \frac{\mathrm{i}}{2}x_2[Q+\kappa M, F], \quad F \in \mathcal{S}_{\mathcal{H}}.$$

The following intertwining relation is the non-commutative analog of the integration by parts (1), and is proved using the covariance identity of [2].

**Proposition 3.2.** *Let* $x_1, x_2 \in \mathbb{R}$. *We have for* $\kappa = 1$:

$$D_{(x_1,2x_2)}O(f) = \big[x_1X_1 + x_2X_2, O(f)\big] = O\big(x_1\xi_2\partial_1 f(\xi_1,\xi_2) - x_2\xi_2\partial_2 f(\xi_1,\xi_2)\big).$$

We turn to showing the smoothness of the Wigner density $\widetilde{W}_{|\psi\rangle\langle\phi|}(\xi_1,\xi_2)$. Let $H^\sigma_{1,2}(\mathbb{R}\times(0,\infty))$ denote the Sobolev space with respect to the norm

$$\|f\|^2_{H^\sigma_{1,2}(\mathbb{R}\times(0,\infty))} = \int_0^\infty \frac{1}{\xi_2}\int_{\mathbb{R}} \left|f(\xi_1,\xi_2)\right|^2 \mathrm{d}\xi_1\,\mathrm{d}\xi_2 + \int_0^\infty \xi_2 \int_{\mathbb{R}} \left(\left|\partial_1 f(\xi_1,\xi_2)\right|^2 + \left|\partial_2 f(\xi_1,\xi_2)\right|^2\right)\mathrm{d}\xi_1\,\mathrm{d}\xi_2.$$

**Theorem 3.3.** *Let $\phi,\psi\in\mathrm{Dom}\,X_1\cap\mathrm{Dom}\,X_2$. Then*

$$1_{\mathbb{R}\times(0,\infty)} W_{|\phi\rangle\langle\psi|} \in H^\sigma_{1,2}\big(\mathbb{R}\times(0,\infty)\big).$$

**Proof.** We have, for $f\in\mathcal{C}_c^\infty(\mathbb{R}\times(0,\infty))$ and $x_1,x_2\in\mathbb{R}$:

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2} \big(x_1\partial_1 f(\xi_1,\xi_2)+x_2\partial_2 f(\xi_1,\xi_2)\big)\overline{W}_{|\phi\rangle\langle\psi|}(\xi_1,\xi_2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\right| \\
&\quad = 2\pi\left|\big\langle\phi\big|O\big(x_1\xi_2\partial_1 f(\xi_1,\xi_2)-x_2\xi_2\partial_2 f(\xi_1,\xi_2)\big)\psi\big\rangle_{\mathcal{H}}\right| = 2\pi\left|\big\langle\phi\big|\big[x_1X_1+x_2X_2,\,O(f)\big]\psi\big\rangle_{\mathcal{H}}\right| \\
&\quad \leqslant \sqrt{2\pi}\,\|\phi\|_{\mathcal{H}}\big\|(x_1X_1+x_2X_2)\psi\big\|\,\|f\|_{L^2_{\mathbb{C}}(\mathbb{R}^2;\frac{\mathrm{d}\xi_1\,\mathrm{d}\xi_2}{|\xi_2|})}. \qquad \square
\end{aligned} \tag{3}$$

Under the same hypothesis we can show that $1_{\mathbb{R}\times(-\infty,0)}W_{|\phi\rangle\langle\psi|}$ belongs to the Sobolev space $H^\sigma_{1,2}(\mathbb{R}\times(-\infty,0))$ which is defined in a way similar to (3). We now define the analog of a Skorohod integral operator.

**Definition 3.4.** Fix $\alpha\in\mathbb{R}$ and let for $F\in\mathcal{S}_{\mathcal{H}}$:

$$\delta(F\otimes x) = \frac{x_1}{2}\big\{Q+\alpha(M-\beta),F\big\} + \frac{x_2}{2}\{P,F\} - D_x F,$$

with $x=(x_1,x_2)\in\mathbb{R}^2$.

Given $F,U,V\in\mathcal{S}_{\mathcal{H}}$, let

$$U\overleftarrow{D}^F_x = -\frac{\mathrm{i}}{2}x_1[P,U]F + \frac{\mathrm{i}}{2}x_2[Q,U]F, \qquad \overrightarrow{D}^F_x V = -\frac{\mathrm{i}}{2}x_1F[P,V] + \frac{\mathrm{i}}{2}x_2F[Q,V],$$

and define a two-sided gradient as $U\overleftrightarrow{D}^F_x V = U\overleftarrow{D}^F_x V + U\overrightarrow{D}^F_x V$. Let $E[X]=\langle\Omega,X\Omega\rangle_{\mathcal{H}}$ denote the expectation of $X$ when $\Omega=1_{\mathbb{R}_+}$ is the vacuum state in $\mathcal{H}$. The integration by parts formulas given below generalizes the classical integration by parts formula (1) on $\mathbb{R}$. It follows from the relations

$$E[D_xF] = \tfrac{1}{2}E\big[x_1\{Q,F\}+x_2\{P,F\}\big], \quad \text{and} \quad E\big[\delta(F\otimes x)\big]=0, \quad x=(x_1,x_2)\in\mathbb{R}^2.$$

Although $\delta$ is not the adjoint of $D$, we have the following analog of the commutative integration by parts formula.

**Proposition 3.5.** *Let $x=(x_1,x_2)\in\mathbb{R}$. Assume that $x_1(Q+\alpha M)+x_2P$ commutes with $U,V\in\mathcal{S}_{\mathcal{H}}$. We have*

$$E\big[U\overleftrightarrow{D}^F_x V\big] = E\big[U\delta(F\otimes x)V\big], \quad F\in\mathcal{S}_{\mathcal{H}}.$$

We also have the commutation formula, for $\kappa=0$:

$$D_x\delta(F\otimes y) - \delta(D_xF\otimes y) = \frac{1}{2}x_1y_1\{M+\alpha Q,F\} + \mathrm{i}\frac{y_1-\mathrm{i}y_2}{2}x_2[M,F] - \frac{\mathrm{i}}{2}x_1y_2\{M,F\} + \frac{\alpha}{2}x_2y_1\{P,F\},$$

and

$$\delta(GF\otimes x) = G\delta(F) - G\overleftarrow{D}_F - \frac{x_1}{2}[Q+\alpha M,G]F - \frac{x_2}{2}[P,G]F,$$

$$\delta(FG\otimes x)=\delta(F)G-\vec{D}_F G-\frac{x_1}{2}F[Q+\alpha M,G]-\frac{x_2}{2}F[P,G].$$

By standard arguments, the operators $D$ and $\delta$ can be shown to be closable for the topology of weak convergence in the space of bounded operators on $\mathcal{H}$.

## 4. Relation to the commutative case

In the Gaussian interpretation of Fock space, $\mathbf{q}=a_x^- + a_x^+ = x$ is multiplication by $x\in\mathbb{R}$. Taking $\beta=1/2$ and writing $\tau=\frac{1}{2}x^2$, we have the relations

$$\tilde{a}_\tau^- = \tfrac{1}{2}\mathbf{q}a_x^-,\qquad \tilde{a}_\tau^+ = \tfrac{1}{2}a_x^+\mathbf{q},\qquad \tilde{a}_\tau^\circ = \tfrac{1}{2}a_x^+a_x^-,$$

i.e.,

$$\tilde{a}_\tau^- f(\tau)=\tfrac{1}{2}\mathbf{q}a_x^- f\left(\frac{x^2}{2}\right),\qquad \tilde{a}_\tau^+ f(\tau)=\tfrac{1}{2}a_x^+\mathbf{q} f\left(\frac{x^2}{2}\right),\qquad \tilde{a}_\tau^\circ f(\tau)=\tfrac{1}{2}a_x^+a_x^- f\left(\frac{x^2}{2}\right),$$

see, e.g., [7]. The representation $(M,B^-,B^+)$ of $\mathfrak{sl}_2$ can be constructed as

$$M=\tfrac{1}{2}\big(a_x^-a_x^+ + a_x^+a_x^-\big),\qquad B^-=\tfrac{1}{2}\big(a_x^-\big)^2,\qquad B^+=\tfrac{1}{2}\big(a_x^+\big)^2.$$

We have

$$Q+\alpha M=\frac{\alpha+1}{2}\frac{\mathbf{p}^2}{2}+\frac{\alpha-1}{2}\frac{\mathbf{q}^2}{2},\quad\text{and}\quad M+\alpha Q=\frac{\alpha+1}{2}\frac{\mathbf{p}^2}{2}+\frac{1-\alpha}{2}\frac{\mathbf{q}^2}{2}.$$

The commutative case is obtained with $\alpha=1$ when considering functionals of $\mathbf{q}^2/2$ only, or with $\alpha=-1$ when considering functionals of $\mathbf{p}^2/2$ only. For example the analogs of the classical integration by parts formula (1) are written as

$$E[D_{(1,0)}F]=\tfrac{1}{2}E\left[\left\{\frac{\mathbf{p}^2}{2},F\right\}-F\right],\qquad E[D_{(1,0)}F]=\tfrac{1}{2}E\left[F-\left\{\frac{\mathbf{q}^2}{2},F\right\}\right],$$

when $\alpha=1$ and $\alpha=-1$ respectively.

## Acknowledgement

Work supported in part by the European Community's Human Potential Programme under contract HPRN-CT-2002-00279, RTN QP-Applications, and by a DAAD-EGIDE Procope cooperation.

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