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Partial Differential Equations

On the Cauchy problem for the generalized Benjamin–Ono equation with small initial data

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Abstract

We prove global well-posedness results for small initial data in $H^s(\mathbb{R}),\ s > s_k$, and in $\dot{\mathcal{B}}_2^{s_k,1}(\mathbb{R})$, $s_k = 1/2 - 1/k$, for the generalized Benjamin–Ono equation $\partial_t u + \mathcal{H}\partial_x^2 u + \partial_x(u^{k+1}) = 0,\ k \geqslant 4$. We also consider the cases $k = 2, 3$. ***To cite this article: L. Molinet, F. Ribaud, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Résumé

Sur le problème de Cauchy pour l'équation de Benjamin–Ono généralisée avec données initiales petites. Nous montrons que l'équation de Benjamin–Ono généralisée $\partial_t u + \mathcal{H}\partial_x^2 u + \partial_x(u^{k+1}) = 0,\ k \geqslant 4$, est globalement bien posée dans $H^s(\mathbb{R}),\ s > s_k$, et dans $\dot{\mathcal{B}}_2^{s_k,1}(\mathbb{R}),\ s_k = 1/2 - 1/k$, pour les données petites. Nous considérons également les cas $k = 2, 3$. ***Pour citer cet article : L. Molinet, F. Ribaud, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

1. Introduction and main results

We consider the Cauchy problem for the generalized Benjamin–Ono equation

$$\begin{cases} \partial_t u + \mathcal{H}\partial_x^2 u + \partial_x(u^{k+1}) = 0, & k \geqslant 2,\ (t,x) \in \mathbb{R} \times \mathbb{R}, \\ u(0,x) = u_0(x), \end{cases} \tag{GBO}$$

where $\mathcal{H}$ denotes the Hilbert transform. The Benjamin–Ono equation ($k = 1$) arises as a model for long internal gravity waves in deep stratified fluids. For $k \geqslant 2$, the local well-posedness of (GBO) is known in $H^s(\mathbb{R}),\ s > 3/2$, see [2]. Moreover, in the case of small initial data, (GBO) is locally well-posed (in time) in $H^s(\mathbb{R})$ as soon as

$$s > 1 \quad \text{if } k = 2, \quad s > 5/6 \quad \text{if } k = 3 \quad \text{and} \quad s \geqslant 3/4 \quad \text{if } k \geqslant 4, \tag{1}$$

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see [2]. Note also that the global well-posedness for small initial data in $H^s(\mathbb{R})$ is only known for $k \geqslant 4$ and $s \geqslant 1$, see [2] again. Up to now these results are the best ones concerning (GBO) with small initial data. On the other hand, as noticed in [1], by scaling considerations one could expect (GBO) to be locally well-posed in $H^s(\mathbb{R})$ for $s > s_k$ and ill-posed for $s < s_k$. In this direction, it is proved in [1] that the flow map $u_0 \mapsto u(t)$ (if it exists) is not locally uniformly continuous in $\dot{H}^{s_k}(\mathbb{R})$. Hence for the (GBO) equation there exists a large gap between positive and negative available results.

In this Note our aim is to prove that for small initial data, (GBO) is locally well-posed in $H^s(\mathbb{R})$ as soon as

$$s > 1/2 \quad \text{if } k = 2, \quad s > 1/3 \quad \text{if } k = 3 \quad \text{and} \quad s > s_k \quad \text{if } k \geqslant 4, \tag{2}$$

and globally well-posed as soon as

$$s \geqslant 1/2 \quad \text{if } k = 3 \quad \text{and} \quad s > s_k \quad \text{if } k \geqslant 4. \tag{3}$$

Actually we prove that for $k \geqslant 4$, (GBO) is globally well-posed for small initial data in the homogeneous Besov space $\dot{\mathcal{B}}_2^{s_k,1}(\mathbb{R})$. We prove also that (GBO) is locally well-posed in the nonhomogeneous Besov space $\mathcal{B}_2^{1/2,1}(\mathbb{R})$ if $k = 2$ and in $\mathcal{B}_2^{1/3,1}(\mathbb{R})$ if $k = 3$. More precisely, for $k \geqslant 4$, we have the following results (see below for the definition of the spaces X and X_s).

Theorem 1.1. *Let $k \geqslant 4$. There exists $\delta = \delta(k) > 0$ such that for all $u_0 \in \dot{\mathcal{B}}_2^{s_k,1}(\mathbb{R})$ with $\|u_0\|_{\dot{\mathcal{B}}_2^{s_k,1}} \leqslant \delta$, there exists a unique solution u of* (GBO) *in*

$$X \cap C_b\big(\mathbb{R}, \dot{\mathcal{B}}_2^{s_k,1}(\mathbb{R})\big).$$

Moreover, for any $T > 0$ and any $r \in [k, 3k]$, u belongs to $L^r_{t,x}([-T,+T],\mathbb{R})$ and the flow-map is smooth from $\dot{\mathcal{B}}_2^{s_k,1}(\mathbb{R})$ to $C_b(\mathbb{R}, \dot{\mathcal{B}}_2^{s_k,1}(\mathbb{R}))$ near the origin.

Theorem 1.2. *Let be $k \geqslant 4$ and $s > s_k$. There exists $\delta = \delta(k) > 0$ such that for all $u_0 \in H^s(\mathbb{R})$ with $\|u_0\|_{\dot{\mathcal{B}}_2^{s_k,1}} \leqslant \delta$, there exists a unique solution u of* (GBO) *in*

$$X_s \cap C_b\big(\mathbb{R}, H^s(\mathbb{R})\big).$$

Moreover, for any $T > 0$ and any $r \in [2, 3k]$, u belongs to $L^r_{t,x}([-T,+T],\mathbb{R})$ and the flow-map is smooth from $H^s(\mathbb{R})$ to $C([-T,+T], H^s(\mathbb{R}))$ near the origin.

Remark 1. This approach seems to be quite general. Clearly the same results hold for the 1-D derivative Schrödinger equations. Also, in a forthcoming paper, we prove by the same way well-posedness results for the generalized KdV equation in larger spaces than the critical homogeneous Sobolev spaces $\dot{H}^{s_k}$, $s_k = (k-4)/(2k)$, $k \geqslant 4$, see [4]. See also [5] for applications to nonlinear wave equation.

1.1. *Sketch of the proofs*

We solve (GBO) via the contraction method applied to the integral equation

$$u = V(t)u_0 - \int_0^t V(t-t')\partial_x\big(u^{k+1}(t')\big)\,\mathrm{d}t', \tag{4}$$

where $V(t)$ denotes the operator with symbol $\mathrm{e}^{\mathrm{i}t|\xi|\xi}$. We assume that $k \geqslant 4$ and that $u_0 \in \dot{\mathcal{B}}_2^{s_k,1}(\mathbb{R})$, this case being the most interesting, see [3] for details and further results. We work in the space–time Lebesgue spaces $L^r_x L^q_t$ and $L^q_t L^r_x$. Sometimes we also use their local in time versions $L^r_x L^q_T$ and $L^q_T L^r_x$. Next we need to consider Δ_j and

S_j the two convolution operators associated with a Littlewood–Paley decomposition. Recall that the homogeneous Besov space $\dot{\mathcal{B}}_2^{s_k,1}(\mathbb{R})$ denotes the completion of $\mathcal{S}(\mathbb{R})$ with respect to the norm $\|f\|_{\dot{\mathcal{B}}_2^{s_k,1}} = \|(2^{js_k}\|\Delta_j(f)\|_{L^2})\|_{l^1(\mathbb{Z})}$ and that the usual H^s-norm is equivalent to the norm $\|(2^{js}\|\Delta_j(f)\|_{L^2})\|_{l^2(\mathbb{Z})}$.

First we prove some linear estimates for the free and forcing terms when applied to phase localized functions. Recall the sharp Kato smoothing effect, the maximal in time inequality and the following lemma, see [2].

$$\left\|D_x^{1/2}V(t)f\right\|_{L_x^\infty L_t^2} \leqslant C\|f\|_{L^2}, \qquad \left\|V(t)f\right\|_{L_x^4 L_t^\infty} \leqslant C\left\|D_x^{1/4}f\right\|_{L^2}. \tag{5}$$

Lemma 1.3.

$$\left\|V(t)\Delta_j f\right\|_{L_t^\infty L_x^2} \leqslant C\|\Delta_j f\|_{L^2}, \qquad 2^{j/2}\left\|V(t)\Delta_j f\right\|_{L_x^\infty L_t^2} \leqslant C\|\Delta_j f\|_{L^2}, \tag{6}$$

$$\left\|\int_0^t V(t-t')\partial_x\Delta_j h(t')\,\mathrm{d}t'\right\|_{L_x^\infty L_t^2} \leqslant C\|\Delta_j h\|_{L_x^1 L_t^2}. \tag{7}$$

We state now new linear estimates for phase localized functions.

Lemma 1.4. *Let $p_i \in [4,+\infty[$ and $q_i \in [2,+\infty]$ with $1/q_i \leqslant 1/2 - 2/p_i$, or $p_1 \in [4,+\infty[$ and $q_1 \in [2,+\infty[$ with $1/q_1 \leqslant 1/2 - 2/p_1$ and $(p_2,q_2) = (+\infty,2)$. Then,*

$$\left\|V(t)\Delta_j f\right\|_{L_x^{p_i} L_t^{q_i}} \leqslant C2^{j(1/2-1/p_i-2/q_i)}\|\Delta_j f\|_{L^2}, \tag{8}$$

$$2^{-j(1/2-1/p_1-2/q_1)}\left\|\int_0^t V(t-t')\partial_x\Delta_j h(t')\,\mathrm{d}t'\right\|_{L_x^{p_1} L_t^{q_1}} \leqslant C2^j 2^{j(1/2-1/p_2-2/q_2)}\|\Delta_j h\|_{L_x^{\bar{p}_2} L_t^{\bar{q}_2}}. \tag{9}$$

Moreover

$$\left\|\int_0^t V(t-t')\partial_x\Delta_j h(t')\,\mathrm{d}t'\right\|_{L_t^\infty L_x^2} \leqslant C2^{j/2}\|\Delta_j h\|_{L_x^1 L_t^2}. \tag{10}$$

Proof. From (5) together with Riesz–Thorin theorem we obtain

$$\left\|\partial_x^\alpha V(t)\Delta_j f\right\|_{L_x^{4/(1-\theta)} L_t^{2/\theta}} \leqslant C2^{j(\alpha+(1-3\theta)/4)}\|f\|_{L^2}, \quad 0 \leqslant \theta \leqslant 1,$$

and (8) follows from Sobolev embedding theorems. By duality (8) yields

$$2^{-j(1/2-1/p_1-2/q_1)}\left\|\int_{-\infty}^{+\infty} V(t-t')\partial_x\Delta_j h(t')\,\mathrm{d}t'\right\|_{L_x^{p_1} L_t^{q_1}} \leqslant C2^j 2^{j(1/2-1/p_2-2/q_2)}\|\Delta_j h\|_{L_x^{\bar{p}_2} L_t^{\bar{q}_2}}. \tag{11}$$

Then a suitable modification of the Christ–Kiselev lemma enables us to deduce (9) from (11). Next, by (6), $\|\partial_x V(t)\Delta_j f\|_{L_x^\infty L_t^2} \leqslant C2^{j/2}\|f\|_{L^2}$ and by duality

$$\left\|\int_{-\infty}^{+\infty} V(t-t')\partial_x\Delta_j h(t')\,\mathrm{d}t'\right\|_{L_t^\infty L_x^2} \leqslant C2^{j/2}\|\Delta_j h\|_{L_x^1 L_t^2}. \tag{12}$$

This proves (10) since $V(t)$ is a unitary group in $L^2(\mathbb{R})$.

Let us introduce our resolution spaces. Consider the following norms: $N(u)=\sum_{-\infty}^{+\infty}2^{js_k}\|\Delta_j u\|_{L^\infty_t L^2_x}$, $T(u)=\sum_{-\infty}^{+\infty}2^{j/2}2^{js_k}\|\Delta_j u\|_{L^\infty_x L^2_t}$, $M(u)=\sum_{-\infty}^{+\infty}\|\Delta_j u\|_{L^k_x L^\infty_t}$, $\|u\|_X=N(u)+T(u)+M(u)$ and let X be the completion of $\mathcal{S}(\mathbb{R}^2)$ with respect to $\|\cdot\|_X$. From (6) and (8) with $(p_i,q_i)=(k,+\infty)$,

$$\|V(t)u_0\|_X\leqslant C\|u_0\|_{\dot{\mathcal{B}}^{s_k,1}_2}. \tag{13}$$

Now from (7), (9) with $(p_1,q_1)=(k,+\infty)$ and $(p_2,q_2)=(\infty,2)$ and (10),

$$\left\|\int_0^t V(t-t')\partial_x u^{k+1}(t')\,\mathrm{d}t'\right\|_X\leqslant C\sum_{j=-\infty}^{+\infty}2^{j/2}2^{js_k}\|\Delta_j u^{k+1}\|_{L^1_x L^2_t}. \tag{14}$$

Using a standard argument we can assume that $\Delta_j u^{k+1}=\sum_{r\geqslant j}\Delta_r u(S_r u)^k$ and by Hölder inequality this allows to bound the right-hand side of (14) by

$$\sum_{j=-\infty}^{+\infty}2^{j/2}2^{js_k}\left(\sum_{r\geqslant j}\|\Delta_r u\|_{L^\infty_x L^2_t}\|S_r u\|^k_{L^k_x L^\infty_t}\right). \tag{15}$$

Note that $\|\Delta_r u\|_{L^\infty_x L^2_t}\leqslant C2^{-r(s_k+1/2)}\gamma_r$ with $\|(\gamma_r)\|_{l^1}\leqslant C\,T(u)$ and that $\|S_r u\|_{L^k_x L^\infty_t}\leqslant C\sum_{p\leqslant r}\|\Delta_p u\|_{L^k_x L^\infty_t}\leqslant C\,M(u)$. Hence from (15) and discrete Young inequalities we obtain

$$\left\|\int_0^t V(t-t')\partial_x u^{k+1}(t')\,\mathrm{d}t'\right\|_X\leqslant CT(u)M(u)^k. \tag{16}$$

Once estimates (13) and (16) have been derived the proof of the existence and uniqueness part of Theorem 1.1 easily follows. In the same way, according to estimate (9), $(2^{j(s_k-1/2+1/p+2/q)}\|\Delta_j u\|_{L^p_x L^q_t})\in l^1(\mathbb{Z})$ for $1/q\leqslant 1/2-2/p$. Hence the low frequencies part of u belongs to $L^r_{T,x}$ for $r\geqslant k$ and the high frequencies part of u belongs to $L^r_{T,x}$ for $r\leqslant 3k$. Thus u belongs to $L^r_{T,x}$, $r\in[k,3k]$. For $u_0\in H^s(\mathbb{R})$ small enough in $\dot{\mathcal{B}}^{s_k,1}_2(\mathbb{R})$, we solve (GBO) in X_s defined trough the norm $\|\cdot\|_{X_s}=\|\cdot\|_X+\lambda_0\|\cdot\|_{Y_0}+\lambda_s\|\cdot\|_{Y_s}$ where $\lambda_\theta=\|u_0\|_{\dot{\mathcal{B}}^{s_k,1}_2}/\|u_0\|_{\dot{H}^\theta}$ and $\|u\|_{Y_\theta}=\|(2^{j\theta}\|\Delta_j u\|_{L^\infty_t L^2_x})\|_{l^2(\mathbb{Z})}+\|(2^{j/2}2^{j\theta}\|\Delta_j u\|_{L^\infty_x L^2_t})\|_{l^2(\mathbb{Z})}$. The proof is the same as previously up to some minor modifications.

For $k=2,3$ we use the estimates $\|V(t)\Delta_j u_0\|_{L^2_x L^\infty_T}\leqslant C(T)\|\Delta_j u_0\|_{H^{1/2}}$, $j\geqslant 0$, and $\|V(t)S_0u_0\|_{L^2_x L^\infty_T}\leqslant C(T)\|S_0u_0\|_{L^2}$ together with similar arguments to prove the local well-posedness. When $k=3$ and $s\geqslant 1/2$, the global well-posedness result follows then from the conservation of the energy, $E(u)=\int(|D_x^{1/2}u|^2-c_ku^{k+2})\,\mathrm{d}x$.

References

[1] H.A. Biagioni, F. Linares, Ill-posedness for the derivative Schrödinger and generalized Benjamin–Ono equations, Trans. Amer. Math. Soc. 353 (2001) 3649–3659.

[2] C.E. Kenig, G. Ponce, L. Vega, On the generalized Benjamin–Ono equations, Trans. Amer. Math. Soc. 342 (1994) 155–172.

[3] L. Molinet, F. Ribaud, On the generalized Benjamin–Ono equation with small initial data, Preprint.

[4] L. Molinet, F. Ribaud, On the Cauchy problem for the generalized Korteweg–de Vries equation, Comm. Partial Differential Equations, in press.

[5] F. Planchon, Self-similar solutions and semi-linear wave equations in Besov spaces, J. Math. Pures Appl. 79 (2000) 809–820.