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Non-existence of minimizers for a nonlinear membrane plate under compression

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Abstract

The classical equations of a nonlinearly elastic membrane plate, made of Saint Venant–Kirchhoff material, have been justified by Fox et al. (Arch. Rational Mech. Anal. 124 (2) (1993) 157–199). We show that, under compression, the associated minimization problem admits no solution. The proof is based on a result of non-existence of minimizers of non-convex functionals due to Dacorogna and Marcellini (Arch. Rational Mech. Anal. 131 (4) (1995) 359–399). We generalize the application of their result from plane elasticity to membrane plates. ***To cite this article: K. Trabelsi, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Résumé

Non existence de minimiseurs pour une plaque membranaire non linéaire comprimée. Les équations classiques de plaques membranaires non linéairement élastique, constituées d'un matériau de Saint Venant–Kirchhoff ont été justifiées par Fox et al. (Arch. Rational Mech. Anal. 124 (2) (1993) 157–199). On montre que le problème de minimisation associé à une telle plaque comprimée n'admet pas de solution. La preuve fait appel à un résultat de non existence de minimiseurs de fonctionnelles non convexes dû à Dacorogna et Marcellini (Arch. Rational Mech. Anal. 131 (4) (1995) 359–399). On généralise l'application de leur résultat de l'élasticité plane aux plaques membranaires. ***Pour citer cet article : K. Trabelsi, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Version française abrégée

Soit $\mathbb{R}^{3\times 2}$ l'espace des matrices réelles 3×2 muni de la norme euclidienne usuelle $|F| = (\operatorname{tr}(F^{\mathrm T}F))^{1/2}$. La densité d'énergie bidimensionnelle non linéaire classique pour une plaque élastique membranaire est la suivante :

$$W(\xi) = \frac{E\nu}{2(1-\nu^2)}\bigl(\operatorname{tr}(\xi^t\xi - I)\bigr)^2 + \frac{E}{2(1+\nu)}\operatorname{tr}(\xi^t\xi - I)^2 \quad \forall \xi \in M_{3\times 2}, \tag{1}$$

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où ξ représente le gradient de la déformation bidimensionnelle, $E > 0$ est le module de Young et $0 \leqslant \nu < \frac{1}{2}$ est le coefficient de Poisson. La fonction ci-dessus exprime la différence exacte entre le tenseur métrique de la surface déformée et celui de l'état de référence.

Soit $\omega \subset \mathbb{R}^2$ un ouvert borné. Notons $\varphi : \omega \to \mathbb{R}^3$ la déformation et soient $f \in L^2(\omega; \mathbb{R}^3)$ et $\xi_0 \in M_{3\times 2}$. Le problème de minimisation considéré est le suivant

$$(P) \quad \inf\left\{ I(\varphi) = \int_\omega W(\nabla\varphi)\,\mathrm{d}x - \int_\omega f \cdot \varphi\,\mathrm{d}x\colon \varphi \in \varphi_0 + W_0^{1,4}\big(\omega; \mathbb{R}^3\big)\right\}, \tag{2}$$

où $\varphi_0 = \xi_0 x,\ \ x \in \overline{\omega}$. Les équations bi-dimensionnelles classiques d'une plaque membranaire non linéairement élastique associées au problème (P), telles qu'elles sont décrites par la littérature en mécanique (voir Green and Zerna [8], par exemple), ont été justifiées par Fox, Raoult and Simo [7] grâce à la méthode du développement asymptotique formel, introduite par Ciarlet and Destuynder [3], qui a été appliquée aux équations de l'élasticité tridimensionnelle pour un matériau de Saint Venant–Kirchhoff ; on renvoit le lecteur à Ciarlet [2] pour une revue des différentes théories non linéaires de plaques. Si la fonctionnelle I ci-dessus était semi-continue inférieurement faible sur $W^{1,4}(\omega; \mathbb{R}^3)$, alors elle le serait pour la topologie faible-$\star$ sur $W^{1,\infty}(\omega; \mathbb{R}^3)$ et par conséquent elle serait quasiconvexe par rapport au gradient de la déformation. Or ceci n'est pas vrai puisque on voit facilement que I n'est même pas rang-1 convexe ; voir Dacorogna [6] pour un résumé de la théorie de la relaxation en calcul des variations et pour les références en la matière.

Dans cette Note, on utilise une théorème de non-existence général relevant de fonctionnelles non convexes finies pour montrer que le problème (P) n'admet pas de solution si la condition au bord φ_0 exerce une pression sur la plaque (Théorème 3.1). Ce résultat dû à Dacorogna et Marcellini [5], a été exploité par les auteurs pour montrer un résultat de non-existence semblable dans le cas théorique de l'élasticité non linéaire bidimensionnelle pour un matériau de Saint Venant–Kirchhoff. Ici, la difficulté réside dans le fait que les matrices des gradients des déformations ne sont pas carrées. Ainsi, on a besoin d'un lemme technique algèbrique (Lemme 3.2) pour arriver à appliquer le résultat de non-existence qu'on étend au cas de forces extérieures appliquées mortes.

1. Introduction

Let $\mathbb{R}^{3\times 2}$ be the space of real 3×2 matrices endowed with the usual Euclidian norm $|F| = (\mathrm{tr}(F^{\mathrm{T}}F))^{1/2}$. The classical two-dimensional stored energy function for a nonlinearly elastic plane membrane is the following:

$$W(\xi) = \frac{E\nu}{2(1-\nu^2)}\big(\mathrm{tr}(\xi^t\xi - I)\big)^2 + \frac{E}{2(1+\nu)}\,\mathrm{tr}(\xi^t\xi - I)^2 \quad \forall \xi \in M_{3\times 2}, \tag{3}$$

where ξ stands for the two-dimensional deformation gradient, $E > 0$ is the Young modulus and $0 \leqslant \nu < \frac{1}{2}$ is the Poisson ratio. The above function expresses the exact difference between the metric tensor of the unknown surface and that of the reference configuration.

Let $\omega \subset \mathbb{R}^2$ be a bounded open set. Let $\varphi : \omega \to \mathbb{R}^3$ denote the deformation. Finally, let $f \in L^2(\omega; \mathbb{R}^3)$ and $\xi_0 \in M_{3\times 2}$. Then the associated minimization problem is

$$(P) \quad \inf\left\{ I(\varphi) = \int_\omega W(\nabla\varphi)\,\mathrm{d}x - \int_\omega f \cdot \varphi\,\mathrm{d}x\colon \varphi \in \varphi_0 + W_0^{1,4}\big(\omega; \mathbb{R}^3\big)\right\}, \tag{4}$$

where $\varphi_0 = \xi_0 x,\ \ x \in \overline{\omega}$. The classical two-dimensional equations of a nonlinearly elastic membrane plate associated to the problem (P), as found in the mechanical litterature (see Green and Zerna [8], for instance), have been justified by Fox, Raoult and Simo [7] by means of the method of formal asymptotic expansions, introduced by Ciarlet and Destuynder [3], who then applied it to the three-dimensional equations of nonlinear elasticity for a Saint Venant–Kirchhoff material; we refer to Ciarlet [2] for an extensive survey of the different nonlinear plate theories. If the above functional I was weakly lower semicontinuous on $W^{1,4}(\omega;\mathbb{R}^3)$, then it would be weakly-$\star$ lower semicontinuous on $W^{1,\infty}(\omega;\mathbb{R}^3)$ and therefore quasiconvex with respect to the gradient of the deformation. However, it is easily seen that I is not even rank-one convex; we refer to Dacorogna [6] for an account of the relaxation theory in the calculus of variations and for an extensive list of related references.

In this paper, we use a general non-existence theorem dealing with the minimization of non-convex functionals to show that problem (P) has no solution if the prescribed function on the border φ_0 exerts a pressure on the plate. This result, due to Dacorogna and Marcellini [5], was applied by the authors to show a similar non-existence result, albeit in the theoretical case of the two-dimensional nonlinear Saint Venant–Kirchhoff elastic energy. Here, the difficulty lies in the fact that the gradient matrices are not square. Hence, we need a technical algebraic lemma to be able to use the non-existence result, which we extend to the case of applied external dead loadings.

2. Preliminaries

We recall a definition of strict convexity from Dacorogna and Marcellini [5], which is central to their non-existence theorem.

Definition 2.1. A convex function $h:\mathbb{R}^{n\times N}\to\mathbb{R}$ is said to be strictly convex at $\xi_0=(\xi_0^\alpha)_{1\leqslant\alpha\leqslant N}\in\mathbb{R}^{n\times N}$ in at least N directions if there exists $\lambda=(\lambda^\alpha)_{1\leqslant\alpha\leqslant N}\in\mathbb{R}^{n\times N}$ such that $\lambda^\alpha\neq 0$ and $\langle\lambda^\alpha;\xi^\alpha-\xi_0^\alpha\rangle_{\mathbb{R}^n}=0$ for all $\alpha=1,2,\dots,N$, whenever $\xi=(\xi^\alpha)_{1\leqslant\alpha\leqslant N}$ satisfies the condition $\frac{1}{2}(h(\xi)+h(\xi_0))=h(\frac{\xi+\xi_0}{2})$.

Let us now consider the more general minimization problem

$$(Q)\quad \inf\left\{J(u)=\int_\Omega F(\nabla u)\,\mathrm{d}x-\int_\Omega f\cdot u\,\mathrm{d}x\colon\ u\in u_0+W_0^{1,4}\big(\Omega;\mathbb{R}^N\big)\right\},\tag{5}$$

where $\Omega\subset\mathbb{R}^n$ is a bounded open set, $F:\mathbb{R}^{n\times N}\to\mathbb{R}$ is a lower semicontinuous function, $f\in L^2(\Omega;\mathbb{R}^3)$ and $u_0(x)=A_0x$ with $A_0\in\mathbb{R}^{n\times N}$. Then, under appropriate growth and coercivity conditions, we have

Theorem 2.2. *Let F^{**} denote the bipolar function of F and QF denote its quasiconvex envelope. Let $A_0\in\mathbb{R}^{n\times N}$ be such that*

(i) $F^{**}(A_0)=QF(A_0)<F(A_0)$,
(ii) *F^{**} is strictly convex in at least N directions at A_0;*

then problem (Q) has no solution.

The proof is identical to that of Dacorogna and Marcellini [5] once it is noticed that a solution to problem (Q) is also a solution to the associated relaxed problem; we send the reader back to Trabelsi [11] for a complete proof.

We recall the expression of the quasiconvex envelope of the plate membrane stored energy function W in terms of the singular values $s_1(\xi)$ and $s_2(\xi)$ of the deformation gradient, denoted ξ (i.e., the eigenvalues of the matrix $(\xi^t\xi)^{1/2}$). For $\xi\in\mathbb{R}^{3\times 2}$ and $0\leqslant s_1(\xi)\leqslant s_2(\xi)$, we have

$$QW(\xi)=W^{**}(\xi)=\begin{cases}0 & \text{if } \xi\in D_1=\{\xi\in M_{3\times 2}\colon\ s_2\leqslant 1 \text{ and } s_1^2+\nu s_2^2<1+\nu\},\\ \dfrac{E}{2}(s_2^2-1)^2 & \text{if } \xi\in D_2=\{\xi\in M_{3\times 2}\colon\ s_2>1 \text{ and } s_1^2+\nu s_2^2<1+\nu\},\\ W(\xi) & \text{if } \xi\notin D_1\cup D_2.\end{cases}\tag{6}$$

The computation of the above envelope can be deduced from the computation of the quasiconvex envelope of the Saint Venant–Kirchhoff stored energy function performed by Le Dret and Raoult [10]; see also Le Dret and Raoult [9], where the authors compute the limit membrane model obtained for a Saint Venant–Kirchhoff material.

3. Main result

We can now announce our non-existence result

Theorem 3.1. *If* $s_1^2(\xi_0) + \nu s_2^2(\xi_0) < 1 + \nu$, *then problem* (P) *has no solution.*

Remark 1. Note that if $f = 0$ a.e., then ξ_0 is a minimizer of problem (P) if $s_1^2(\xi_0) + \nu s_2^2(\xi_0) \geqslant 1 + \nu$. This is a direct consequence of the definition of quasiconvexity.

Remark 2. Practically, the above result means that a membrane plate 'behaves badly' under compression. This fact was already observed by Le Dret and Raoult [9] in their asymptotic analysis. As a matter of fact, their two-dimensional limit model is a relaxed problem. In another context, Coutand [4] showed that the solution he found to the local boundary-value problem of the membrane plate under slight compression via the implicit function theorem is not even a local minimizer.

To prove the above theorem, we need a technical algebraic lemma that overcomes the obstacle of dealing with non-square deformation gradients. Namely, we prove the following

Lemma 3.2. *Let* $\zeta_0 \in M_{3\times 2}$ *such that* $s_1(\zeta_0) < s_2(\zeta_0)$. *There exists* $\lambda = (\lambda^\alpha)_{1\leqslant\alpha\leqslant N} \in \mathbb{R}^{3\times 2}$ *such that* $\lambda^\alpha \neq 0$ *and* $\langle \lambda^\alpha, \zeta^\alpha - \zeta_0^\alpha\rangle_{\mathbb{R}^3} = 0$ *for all* $\alpha = 1, 2$, *whenever*

$$\zeta \in E = \left\{\zeta \in M_{3\times 2}\colon\ \frac{s_2(\xi) + s_2(\xi_0)}{2} = s_2\left(\frac{\xi + \xi_0}{2}\right)\right\}. \tag{7}$$

Remark 3. Note that the above lemma does not say that the function s_2 is strictly convex in at least two directions at ζ_0 since function s_2 is not even convex.

Sketch of the proof of Lemma 3.2. The first step of the proof is to show that

$$E = \left\{\zeta \in M_{3\times 2}\colon\ \sup_{|u|=1} |\zeta u| = |\zeta e_2| \text{ and } \exists\, \lambda \in \mathbb{R}_+\colon\ \zeta e_2 = \lambda \zeta_0 e_2\right\}. \tag{8}$$

We first remark that

$$\sup_{|u|=1} |\zeta u| = s_2(\zeta) \quad \forall \zeta \in M_{3\times 2}. \tag{9}$$

Then, choosing $\xi \in E$ and $w \in \mathbb{R}^2$ such that

$$|w| = 1 \quad \text{and} \quad \bigl|(\zeta + \zeta_0)w\bigr| = \sup_{|u|=1}\bigl|(\zeta + \zeta_0)u\bigr|, \tag{10}$$

we show that $w = e_2$ by exploiting (9). Hence, rewriting (10) yields the announced result.

The second step of the proof is to define an orthonormal basis $\{f_1, f_2, f_3\}$ of $\mathbb{R}^3$, where $f_\alpha = \zeta e_\alpha / |\zeta e_\alpha|$. Then we write that

$$\zeta - \zeta_0 = \begin{pmatrix} p - b_0 & 0 \\ 0 & s_2(\zeta) - s_2(\zeta_0) \\ q & 0 \end{pmatrix}, \tag{11}$$

where $b_0 = |\zeta_0 e_1|$, $p = \langle \zeta e_1, f_1\rangle$ and $q = \langle \zeta e_1, f_3\rangle$. Thus, the proof is complete. □

We turn now to the proof of Theorem 3.1

Sketch of proof. Here we follow Dacorogna and Marcellini [5]. If we assume that $s_1^2(\xi_0) + \nu s_2^2(\xi_0) < 1 + \nu$, then ξ_0 is either in D_1 or in D_2 (see (6)). We study these two cases separately.

Step 1. Assume that $\xi_0 \in D_2$. We prove that W^{**} is strictly convex at ξ_0 in at least two directions. Therefore we consider $\xi_0 \in D_2$ such that

$$\frac{W^{**}(\xi) + W^{**}(\xi_0)}{2} = W^{**}\left(\frac{\xi + \xi_0}{2}\right). \tag{12}$$

Consider now the function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = \frac{E}{2}(x^2 - 1)^2$. The function h is strictly convex as long as $x > 1$, and we can write $W^{**}(\xi) = h(s_2(\xi))$. Since $s_2(\xi) > 1$, we deduce that

$$\xi \in E = \left\{\zeta \in M_{3\times 2}\colon \frac{s_2(\xi) + s_2(\xi_0)}{2} = s_2\left(\frac{\xi + \xi_0}{2}\right)\right\}. \tag{13}$$

We conclude by Lemma 3.2.

Step 2. Assume that $\xi_0 \in D_1$. According to (6), we have $QW(\xi_0) = W^{**}(\xi_0) = 0$. Assume now that (P) has a solution $\varphi \in \varphi_0 + W_0^{1,4}(\omega; \mathbb{R}^3)$. Then, recalling that the infinimums of problems (P) and (QP) coincide, we necessarily have $W(\nabla\varphi) = 0$ a.e. in ω. From (3), we deduce that

$$s_1(\nabla\varphi) = s_2(\nabla\varphi) = 1 \quad \text{a.e. in } \omega \quad \text{and} \quad \det \nabla\varphi^t \nabla\varphi = 1 \quad \text{a.e. in } \omega. \tag{14}$$

On the one hand, this implies that

$$\int_\omega \det \nabla\varphi^t \nabla\varphi \, \mathrm{d}x = \operatorname{meas} \omega, \tag{15}$$

and on the other hand, the boundary data and the fact that $\varphi \in W^{1,4}(\omega; \mathbb{R}^3)$ guarantee that

$$\int_\omega \det \nabla\varphi^t \nabla\varphi \, \mathrm{d}x = \frac{1}{2} \int_{\partial\omega} \left\{(\operatorname{Cof} \nabla\varphi_0{}^t \nabla\varphi_0) n(x)\right\} \cdot \left\{(\nabla\varphi_0{}^t \nabla\varphi_0) x\right\} \mathrm{d}a. \tag{16}$$

For a proof of (16), we refer to Ciarlet [1], Theorem 2.7-1. Next, combining (15) and (16), we get

$$\frac{2 \operatorname{meas} \omega}{\det \nabla\varphi_0{}^t \nabla\varphi_0} = \int_{\partial\omega} \left\{(\nabla\varphi_0{}^t \nabla\varphi_0)^{-1} n(x)\right\} \cdot \left\{(\nabla\varphi_0{}^t \nabla\varphi_0 x)\right\} \mathrm{d}a = \int_{\partial\omega} n(x) \cdot x \, \mathrm{d}a = \int_\omega \operatorname{div} x \, \mathrm{d}x = 2 \operatorname{meas} \omega. \tag{17}$$

Hence, $\det \xi_0^t \xi_0 = s_1(\xi_0) s_2(\xi_0) = 1$ and since $\xi_0 \in D_1$, we conclude that $s_1(\xi_0) = s_2(\xi_0) = 1$. However, this contradicts the data since $s_1(\xi_0)^2 + \nu s_2(\xi_0)^2 < 1 + \nu$ and $0 \leqslant s_1(\xi_0) \leqslant s_2(\xi_0) \leqslant 1$ if $\xi_0 \in D_1$, and the proof is complete. □

Remark 4. We refer to Trabelsi [11] for a detailed proof of all the announced results.

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