Available online at www.sciencedirect.com

SCIENCE DIRECT®

C. R. Acad. Sci. Paris, Ser. I 337 (2003) 587–592

COMPTES RENDUS MATHEMATIQUE

Partial Differential Equations

Energy concentration and Sommerfeld condition for Helmholtz and Liouville equations

Benoît Perthame [a], Luis Vega [b]

[a] *Département de mathématiques et applications, UMR 8553, École normale supérieure, 45, rue d'Ulm, 75230 Paris cedex 05, France*

[b] *Universidad del Pais Vasco, Apdo. 644, 48080 Bilbao, Spain*

Received 25 August 2003; accepted 15 September 2003

Presented by Gilles Lebeau

Abstract

We consider the Helmholtz equation with a variable index of refraction $n(x)$, which is not necessarily constant at infinity but can have an angular dependency like $n(x) \to n_\infty(x/|x|)$ as $|x| \to \infty$. We prove that the Sommerfeld condition at infinity still holds true under the weaker form

$$\frac{1}{R}\int_{|x|\leqslant R}\left|\nabla u - \mathrm{i}n_\infty^{1/2}\left(\frac{x}{|x|}\right)u\,\frac{x}{|x|}\right|^2\mathrm{d}x \to 0,\quad \text{as } R\to\infty.$$

Our approach consists in proving this estimate in the framework of the limiting absorbtion principle. We use Morrey–Campanato type of estimates and a new inequality on the energy decay, namely

$$\int_{\mathbb{R}^d}\left|\frac{\partial}{\partial\omega}n_\infty(\omega)\right|^2\frac{|u|^2}{|x|}\,\mathrm{d}x\leqslant C,\quad \omega=\frac{x}{|x|}.$$

It is a striking feature that the index n_∞ appears in this formula and not the phase gradient, in apparent contradiction with existing literature. ***To cite this article: B. Perthame, L. Vega, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Résumé

Concentration de l'énergie et condition de Sommerfeld pour les équations de Helmholtz et de Liouville. Nous considérons l'équation de Helmholtz avec un indice de réfraction $n(x)$ qui peut varier à l'infini en fonction de l'angle, $n(x) \to n_\infty(x/|x|)$ lorsque $|x| \to \infty$. Nous prouvons que la condition de radiation de Sommerfeld reste valable sous la forme faible

$$\frac{1}{R}\int_{|x|\leqslant R}\left|\nabla u - \mathrm{i}n_\infty^{1/2}\left(\frac{x}{|x|}\right)u\,\frac{x}{|x|}\right|^2\mathrm{d}x \to 0,\quad \text{lorsque } R\to\infty.$$

Nous démontrons cette estimation via le principe d'absorbtion limite. Nous utilisons des inégalités de type Morrey–Campanato et une nouvelle estimation a priori sur le contrôle de l'énergie à l'infini

E-mail addresses: Benoit.Perthame@ens.fr (B. Perthame), mtpvegol@lg.ehu.es (L. Vega).

doi:10.1016/j.crma.2003.09.006

$$\int_{\mathbb{R}^d} \left| \frac{\partial}{\partial \omega} n_\infty(\omega) \right|^2 \frac{|u|^2}{|x|} \, \mathrm{d}x \leqslant C, \quad \omega = \frac{x}{|x|}.$$

Le point surprenant de la condition de Sommerfeld ci-dessus est que l'indice n_∞ y apparaît et non le gradient de la phase, ce qui contredit apparamment la littérature existante. ***Pour citer cet article : B. Perthame, L. Vega, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Version française abrégée

Nous considéron le principe d'absorption limite pour l'équation de Helmholtz

$$\mathrm{i}\varepsilon u_\varepsilon + \Delta u_\varepsilon + n(x) u_\varepsilon = -f(x), \quad \varepsilon > 0, \tag{1}$$

avec un indice de réfraction $n(x)$ qui peut varier à l'infini en fonction de l'angle $n \approx n_\infty(x/|x|)$. La théorie de perturbation du cas n constant ne s'applique plus et nous développons ici une méthode spécifique.

Le premier résultat de cette note est que l'on peut caractériser la limite $\varepsilon \to 0$ grâce à une condition de radiation de Sommerfeld classique

$$\frac{1}{R} \int_{|x| \leqslant R} \left| \nabla u - \mathrm{i} n_\infty^{1/2} \frac{x}{|x|} u \right|^2 \mathrm{d}x \to 0, \quad \text{lorsque } R \to \infty. \tag{2}$$

Celle-ci est surprenante car on s'attend plutôt à voir apparaître la phase $\varphi(x)$ suivant (6) comme cela est établi dans [1,10,11], en apparente contradiction avec notre résultat puisqu'en général

$$\nabla \varphi(x) \neq n_\infty^{1/2} \left(\frac{x}{|x|} \right) \frac{x}{|x|}.$$

Ceci peut s'expliquer grâce au deuxième résultat de cette note : une nouvelle estimation, énoncée dans le Théorème 2.1 qui borne a priori la quantité

$$\int_{|x| \geqslant R} \left| \nabla_\omega n_\infty \left(\frac{x}{|x|} \right) \right|^2 \frac{|u_\varepsilon|^2}{|x|} \, \mathrm{d}x.$$

Cette borne montre que l'énergie se concentre à l'infini suivant les rayons correspondant aux extrema de n_∞ qui sont justement les points où $\nabla \varphi(x) = n_\infty^{1/2}(\frac{x}{|x|}) \frac{x}{|x|}$. En effet la quantité $\int_{|x| \geqslant R} \frac{|u_\varepsilon|^2}{|x|} \, \mathrm{d}x$ n'est pas bornée en général, car selon [8], la borne naturelle pour l'équation de Helmholtz avec ce type d'indice est (suivant le cas à coefficient constant)

$$\sup_{R > 0} \frac{1}{R} \int_{|x| \leqslant R} |u_\varepsilon|^2 \, \mathrm{d}x < \infty.$$

La même méthode se transpose aux équations de Liouville stationnaires dans tout l'espace

$$\xi \cdot \nabla_x f + \frac{1}{2} \nabla_x n(x) \cdot \nabla_\xi f = \sigma(x) \delta\big(\xi = n(x)\big).$$

Nous montrons également pour ce problème l'existence d'une solution dans des espaces de Morrey–Campanato avec une condition de radiation « sortante » à l'infini.

1. The Helmholtz equation

We consider the Helmholtz equation with a variable index of refraction $n(x)$, with a slow, and only radial decay to a constant $n_\infty(x/|x|)$ at infinity (long range potential)

$$\mathrm{i}\varepsilon u_\varepsilon + \Delta u_\varepsilon + n(x)u_\varepsilon = -f(x), \quad \varepsilon > 0. \tag{1}$$

Our main interest is the so called limiting absorbtion principle (i.e., to study the limit when $\varepsilon > 0$ approaches to 0 in (1)) and the validity of the Sommerfeld radiation condition at infinity. One of the main results in this note is to prove that, in the limit $\varepsilon \to 0$, we have

$$\frac{1}{R}\int\limits_{|x|\leqslant R}\left|\nabla u - \mathrm{i}n_\infty^{1/2}\frac{x}{|x|}u\right|^2 \mathrm{d}x \to 0, \quad \text{as } R \to \infty. \tag{2}$$

A direct consequence of this condition is the more classical setting

$$\liminf \int\limits_{|x|=r}\left|\nabla u - \mathrm{i}n_\infty^{1/2}\left(\frac{x}{|x|}\right)\frac{x}{|x|}u\right|^2 \mathrm{d}\sigma(x) \to 0, \quad \text{as } r \to \infty, \tag{3}$$

where $\mathrm{d}\sigma$ denotes the Lebesgue measure on the sphere. It is a striking feature that the index n_∞ appears in this formula and not the phase gradient, in apparent contradiction with existing literature. Indeed, in [1,10,11] for instance, the Sommerfeld condition is stated with the phase, defined by $|\nabla\varphi| = n^{1/2}$, as

$$\lim \int\limits_{|x|=r} |\nabla u - \mathrm{i}\nabla\varphi\, u|^2 \,\mathrm{d}\sigma(x) \to 0, \quad \text{as } r \to \infty. \tag{4}$$

It turns out that, in general, $\nabla\varphi$ is not equal to $n_\infty^{1/2}\,\frac{x}{|x|}$ with our assumptions. This phenomenon, as well as the proof of (3), can be explained by a new and fundamental energy estimate that we state later on (Theorem 12 below) and express that the energy at infinity is concentrated on the extrema of n_∞, the only points where the equality $\nabla\varphi(x) = n_\infty^{1/2}(\frac{x}{|x|})\frac{x}{|x|}$ holds, thus explaining how (3) and (4) can both hold true.

In the next section we state precisely our assumptions and results and we give the steps of the proof in a third section. Similar results hold for steady Liouville equation and are presented in a last section.

2. Assumptions and main results

We consider an index $n(x)$ that satisfies the assumptions

$$n \in L^\infty, \quad n > 0; \tag{5}$$

$$2\sum_{j\in\mathbb{Z}} \sup_{C(j)} \frac{(x\cdot\nabla n(x))_-}{n(x)} := 2\sum_{j\in\mathbb{Z}} \beta_j = \beta < +\infty, \tag{6}$$

where $C(j) = \{x \in \mathbb{R}^d \text{ s.t. } 2^j \leqslant |x| \leqslant 2^{j+1}\}$. Then, we assume that there are $0 < \delta < 1/2$, $\Gamma > 0$ and a function $n_\infty(\frac{x}{|x|})$ such that

$$\begin{cases} n_\infty(\frac{x}{|x|}) \in C^3(S^{d-1}), \quad n_\infty(\frac{x}{|x|}) \geqslant n_0 > 0, \\ |n(x) - n_\infty(\frac{x}{|x|})| \leqslant n_0\frac{\Gamma}{|x|^\delta} \quad \text{for } |x| \text{ large enough}, \end{cases} \tag{7}$$

and one of the following two conditions (with the notation $\omega = \frac{x}{|x|}$),

$$\left|n(x) - n_\infty\left(\frac{x}{|x|}\right)\right| \leqslant n(x)\,\frac{\widetilde{\Gamma}}{|x|}, \quad \widetilde{\Gamma} > 0; \tag{8}$$

$$\begin{cases} \text{there exists } \tilde\beta < 1,\ \delta > 0 \text{ and } \overline{\Gamma} > 0 \text{ such that} \\ \left(\frac{\partial(n-n_\infty)}{\partial\omega} \cdot \frac{\partial n_\infty}{\partial\omega}\right)_- \leqslant \tilde\beta \left|\frac{\partial n_\infty}{\partial\omega}\right|^2 + n(x)\frac{\overline{\Gamma}}{|x|^\delta}. \end{cases} \tag{9}$$

With these assumptions we are ready to state our main results. We define

$$|||u|||_{R_0}^2 := \sup_{R>R_0} \frac{1}{R} \int_{|x|\leqslant R} |u|^2\,\mathrm{d}x, \tag{10}$$

$$N_{R_0}(f) := \sum_{j>J}\left[2^{j+1}\int_{C(j)} |f|^2\,\mathrm{d}x\right]^{1/2} + \left[R_0 \int_{B_{R_0}} |f|^2\,\mathrm{d}x\right]^{1/2}, \tag{11}$$

with J defined by $2^J < R_0 < 2^{J+1}$, and we drop the index R_0 if $R_0 = 0$. Then we have

Theorem 2.1 (Energy decay at infinity). *For dimensions $d \geqslant 2$, we assume* (5)–(7) *and either* (8) *or* (9)*. Then the solution to the Helmholtz equation* (1) *satisfies, for some $R(n)$ and R large enough*

$$\overline{M} := \int_{|x|\geqslant R} \left|\nabla_\omega n_\infty\left(\frac{x}{|x|}\right)\right|^2 \frac{|u_\varepsilon|^2}{|x|}\,\mathrm{d}x \leqslant C\left[\left(\varepsilon + \|n\|_\infty\right) N_{R(n)}\left(\frac{f}{n^{1/2}}\right) M + M^2\right], \tag{12}$$

for some constant C independent of ϵ and

$$M^2 := |||\nabla u_\varepsilon|||_{R(n)}^2 + |||\,n^{1/2} u_\varepsilon\,|||_{R(n)}^2 + \int_{|x|\geqslant R(n)} \frac{|\nabla_\tau u_\varepsilon|^2}{|x|}\,\mathrm{d}x.$$

The norm involved here is natural for dispersive equations because it has the correct space homogeneity (see [6] for instance). We recall that for solutions to Helmholtz equation, the above quantity M is also naturally controlled. For the constant coefficient it is known to be an optimal norm (see [2,5] for instance). In [7], we were able to provide a priori bounds on M (under similar but weaker assumptions as above), which are compatible with the high frequency limit [3] because they have the correct space homogeneity. In particular, it is known that, in general, the solutions to Helmholtz equation (with $\varepsilon = 0$) are not bounded in the space $L^2(\frac{\mathrm{d}x}{|x|})$. This fact shows the interest of the energy estimate in Theorem 12; it expresses that the energy is concentrated on the directions of extrema of $n_\infty(\frac{x}{|x|})$ at infinity. Several arguments indicate that the energy should be concentrated on local maxima of $n_\infty(\frac{x}{|x|})$ but this is still an open question for Helmholtz equation; the trajectories of the bicharacteristics system converge to local maxima (see [4] for instance), and for the corresponding Liouville equation some characteristic quantities exhibit behaviors compatible with this scenario [9].

The energy control in Theorem 12 leads to the Sommerfeld condition. Namely, we have

Theorem 2.2 (Sommerfeld condition). *For dimensions $d \geqslant 2$, assume* (5)–(7) *and either* (8) *or* (9)*. Then, with $R(n)$ given in Theorem* 12*, there exists a unique solution to the Helmholtz equation with $\varepsilon = 0$ which satisfies $M < \infty$ and for $R \geqslant R(n)$,*

$$\begin{aligned} &\frac{1}{R}\int_{|x|\leqslant R} \left|\nabla u - \mathrm{i}\frac{x}{|x|}\,n_\infty^{1/2}\,u(x)\right|^2 \mathrm{d}x \\ &\quad\leqslant C\left(\int_{|x|\geqslant 1} \kappa_R(x)\frac{|\nabla_\tau u|^2}{|x|}\,\mathrm{d}x\right)^{1/2} \mathcal{N}(f) + C\left(\frac{\Gamma + \|n\|_{L^\infty}}{R^{\delta/2}} + \sum_{2^j > R(n)} \beta_j \kappa_R\left(2^j\right)\right)\mathcal{N}(f)^2. \end{aligned} \tag{13}$$

Here $\kappa_R(x) = \min\{\frac{|x|}{R}, 1\}$, and $\mathcal{N}(f) := [N_{R(n)}(f)^2 + \|n\|_{L^\infty} N_{R(n)}(\frac{f}{n^{1/2}})^2]^{1/2}$.

3. Sketch of the proofs

The proofs are given in detail in [8]. We indicate here the key ideas.

The proof of the energy decay estimate in Theorem 12 is based on a multiplier method that uses the specific form of $n_\infty(\frac{x}{|x|})$ for x large. The multiplier is, as classical in this theory, of the form $\nabla\Psi\cdot\nabla\bar{u}$, and the specific choice $\Psi(x)=q(\frac{|x|}{R})\,n_\infty(\frac{x}{|x|})$ leads to the result.

The proof of Theorem 2.2 uses several steps. The first step is to derive a Sommerfeld type condition with absorbtion and it is then necessary to include an additional term in the radiation inequality. We prove the inequality

$$\begin{aligned}
&\frac{1}{R}\int\limits_{R(n)\leqslant|x|\leqslant R}\left|\nabla u_\varepsilon - \mathrm{i}\frac{x}{|x|}\,n_\infty^{1/2}\,u_\varepsilon\right|^2\mathrm{d}x+\varepsilon\int\limits_{R(n)\leqslant|x|}\frac{\kappa_R}{n_\infty^{1/2}}\left|\nabla u_\varepsilon - \mathrm{i}\frac{x}{|x|}\,n_\infty^{1/2}\,u_\varepsilon\right|^2\mathrm{d}x\\
&\quad\leqslant CM\left[N_{R(n)}(\kappa_R f)+\frac{1}{R\,n_0^{1/2}}N_{R(n)}(f)+\frac{1}{R}N_{R(n)}\left(\frac{f}{n^{1/2}}\right)\right]+M^2\left(\sum_{2^j>R(n)}\beta_j\,\kappa_R\big(2^j\big)+C\varepsilon\right)\\
&\qquad+C\frac{1}{R^{\delta/2}}\left[M^2+MN_{R(n)}\left(\frac{f}{n^{1/2}}\right)\right]+C\left(\int\limits_{|x|\geqslant R(n)}\kappa_R\frac{|\nabla_\tau u_\varepsilon|^2}{|x|}\,\mathrm{d}x\right)^{1/2}\overline{M}.
\end{aligned}\tag{14}$$

The second step consists in proving a uniqueness theorem for the Helmholtz equation with $\varepsilon=0$ and for solutions which are bounded only in the norm $|||\cdot|||$ as defined by the quantity in M. This follows from methods of Carleman's type that have been carried out in this context by [12] for instance. The third step consists in proving a priori bounds on the remaining terms in the right hand side and typically M. This follows from [7] under the smallness assumption on the variation of the index n, $\beta<1$. In order to avoid this smallness argument, we argue through a compactness argument. If M blows up as ε vanishes, then after renormalizing, we obtain a solution to Helmholtz equation (with $\varepsilon=0$) that is bounded in the $|||\cdot|||$ norm and satisfies the Sommerfeld inequality with a vanishing source. The uniqueness theorem applies in such a situation and leads to a contradiction.

4. Stationary Liouville equation

Similar results can be proved on the Liouville equation. This is natural because it appears as the high frequency limit of Helmholtz equation through its Wigner transform [3]. We consider the stationary Liouville equation on the density $f_\alpha(x,\xi)$ with $x,\xi\in\mathbb{R}^d$,

$$\alpha f_\alpha+\xi\cdot\nabla_x f_\alpha+\frac{1}{2}\nabla_x n(x)\cdot\nabla_\xi f_\alpha=\sigma(x)\delta\big(\xi=n(x)\big),\tag{15}$$

with σ a nonnegative L^1 function. For the sake of simplicity, we assume that $n=n_\infty(\frac{x}{|x|})\in C^2(S^{d-1})$ and n_∞ is positive, but more general assumptions could be used following Section 2.

Eq. (15) admits nonnegative measure valued solutions which are supported in the set $\{\xi=n(x)\}$ and satisfy local energy controls which are uniform in $\alpha>0$,

$$\frac{1}{R}\int\limits_{\{|x|\leqslant R\}}\int\limits_{\xi\in\mathbb{R}^d}|\xi|^2 f_\alpha(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi\leqslant\|n\|_{L^\infty}^{1/2}\int\limits_{\mathbb{R}^d}\sigma(x)\,\mathrm{d}x,\quad\forall R>0,\tag{16}$$

$$\int\limits_{\mathbb{R}^{2d}}\frac{|\nabla_\omega n|^2}{|x|}f_\alpha(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi\leqslant C\big(n,D^2n\big)\int\limits_{\mathbb{R}^d}\sigma(x)\,\mathrm{d}x.\tag{17}$$

These estimates are enough to pass to the limit as α vanishes but uniqueness does not hold and some outgoing condition at infinity has to be added. This uniqueness condition at infinity is similar to Sommerfeld radiation condition. It expresses more intuitively than for Helmholtz equation that no rays are incoming, i.e., we have $f(x,\xi)=0$ for $x\cdot\xi\leqslant 0$ and $|x|$ large. The question is to give a precise meaning to this statement. We have

Theorem 4.1. *The limiting (nonnegative measure valued) solution f to (15) with $\alpha=0$ satisfies*

$$\frac{1}{R}\int_{\{|x|\leqslant R\}}\int_{\xi\in\mathbb{R}^d}\left|\xi-\frac{x}{|x|}n^{1/2}\right|^2 f(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi\to 0\quad \text{as } R\to\infty. \tag{18}$$

We refer to [9] for details and proofs of these results. Additional informations on the limit $\alpha\to 0$ and on the behavior for large x are also available, such as general estimates related to the concentration of trajectories on the maximum of n.

References

[1] S. Agmon, J. Cruz-Sampedro, I. Herbst, Generalized Fourier transform for Schrödinger operators with potentials of order zero, J. Funct. Anal. 167 (1999) 345–369.

[2] S. Agmon, L. Hörmander, Asymptotic properties of solutions of differential equations with simple characteristics, J. Anal. Math. 30 (1976) 1–37.

[3] J.D. Benamou, F. Castella, T. Katsaounis, B. Perthame, High frequency limit of the Helmholtz equations, Rev. Mat. Iberoamericana (2002).

[4] I. Herbst, Spectral and scattering for Schrödinger operators with potentials independent of $|x|$, Amer. J. Math. 113 (3) (1991) 509–565.

[5] L. Hörmander, The Analysis of Linear Partial Differential Operators, in: Grundlehren Math. Wiss., Vol. 257, Springer, 1990.

[6] C. Kenig, G. Ponce, L. Vega, Small solutions to nonlinear Schrödinger equations, Ann. Inst. H. Poincaré Anal. Non Lineaire 10 (1993) 255–288.

[7] B. Perthame, L. Vega, Morrey–Campanato estimates for Helmholtz equation, J. Funct. Anal. 164 (2) (1999) 340–355.

[8] B. Perthame, L. Vega, Energy decay and Sommerfeld condition for Helmholtz equation with variable index at infinity, Preprint.

[9] B. Perthame, L. Vega, Sommerfeld condition for a Liouville equation and concentration of trajectories, Bull. Braz. Math. Soc. (N.S.) 34 (1) (2003) 1–15.

[10] Y. Saito, Schrödinger operators with a nonspherical radiation condition, Pacific J. Math. 126 (2) (1987) 331–359.

[11] Bo Zhang, Radiation condition and limiting amplitude principle for acoustic propagators with two unbounded media, Proc. Roy. Soc. Edinburgh Sect. A 128 (1998) 173–192.

[12] Bo Zhang, On transmission problems for wave propagation in two locally perturbed half spaces, Math. Proc. Cambridge Philos. Soc. 115 (1994) 545–558.