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Functional Analysis

The Banach–Saks index of rearrangement invariant spaces on [0, 1]

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Abstract

The set of all rearrangement invariant function spaces on $[0,1]$ having the p-Banach–Saks property has a unique maximal element for all $p \in (1,2]$. For $p=2$ this is L_2, for $p \in (1,2)$ this is $L^0_{p,\infty}$. We compute the Banach–Saks index for the families of Lorentz spaces $L_{p,q},\ 1<p<\infty, 1 \leqslant q \leqslant \infty$, and Lorentz–Zygmund spaces $L(p,\alpha), 1 \leqslant p < \infty,\ \alpha \in \mathbb{R}$, extending the classical results of Banach–Saks and Kadec–Pelczynski for L_p-spaces. Our results show that the set of rearrangement invariant spaces with Banach–Saks index $p \in (1,2]$ is not stable with respect to the real and complex interpoltaion methods. ***To cite this article: E.M. Semenov, F.A. Sukochev, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Résumé

L'indice de Banach–Saks des espaces invariants par réarrangement sur $[0,1]$**.** L'ensemble des espaces invariants par réarrangement sur $[0,1]$ qui possèdent la propriété de p-Banach–Saks admet un unique élément maximal pour $p \in (1,2]$. Pour $p=2$ c'est L_2 ; pour $p \in (1,2)$ c'est $L^0_{p,\infty}$. Nous calculons l'indice de Banach–Saks de la famille des espaces de Lorentz $L_{p,q},\ 1<p<\infty, 1 \leqslant q \leqslant \infty$, et des espaces de Lorentz–Zygmund $L(p,\alpha), 1 \leqslant p < \infty,\ \alpha \in \mathbb{R}$, généralisant ainsi les résultats classiques de Banach–Saks et Kadec–Pelczynski pour les espaces L_p. Nous montrons que l'ensemble des espaces invariants par réarrangement qui ont $p \in (1,2]$ indice de Banach–Saks n'est pas stable par interpolation réelle ou complexe. ***Pour citer cet article : E.M. Semenov, F.A. Sukochev, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Version française abrégée

Soient E un espace de Banach et $p \geqslant 1$. Une suite bornée $\{x_n\} \subset E$ est appelée $p-BS$-suite (BS-suite) s'il existe une sous suite $\{y_{n_k}\} \subset \{x_k\}$ telle que

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$$\limsup_{m\to\infty} m^{-1/p}\left\|\sum_{k=1}^{m} y_k\right\|_E < \infty \quad \left(\lim_{m\to\infty}\frac{1}{m}\left\|\sum_{k=1}^{m} y_k\right\| = 0\right).$$

On dit que E possède la $p-BS$-propriété (BS-propriété) et on écrit $E\in\mathcal{BS}_p$ ($E\in\mathcal{BS}$) si toute suite $\{x_n\}\subset E$ faiblement nulle contient une $p-BS$-suite (BS-suite). Il est évident que tout espace de Banach a la $1-BS$-propriété. L'ensemble $\Gamma(E)=\{p\colon\ E\in\mathcal{BS}_p\}$ est égal à $[1,\alpha]$, ou bien à $[1,\alpha)$ pour un $\alpha\in[1,\infty]$. On écrit $\gamma(E)=\alpha$ si $\Gamma=[1,\alpha]$ et $\gamma(E)=\alpha-0$ si $\Gamma(E)=[1,\alpha)$. Le nombre $\gamma(E)$ est appelé l'indice de Banach–Saks. Pour les espaces L_p classiques (de suites ou de fonctions) ces indices sont bien connus. En effet il est facile de voir [6] que $\gamma(l_p)=p$ pour $1<p<\infty$ et $\gamma(l_1)=\gamma(c_0)=\infty$, tandis que les résultats classiques de Banach et Saks [2] et de Kadec et Pelczynski [4] donnent $\gamma(L_p)=\min(p,2)$ pour $p\in[1,\infty)$. Le but de la note présentée ici est d'étudier l'indice de Banach–Saks pour des espaces invariants par réarrangement sur l'intervalle $[0,1]$ [7]. Il découle de l'inégalité de Khintchine que, pour un espace E séparable invariant par réarrangement, on a $1\leqslant\gamma(E)\leqslant 2$. Le sous-ensemble de $\mathcal{BS}_p$ formé de tous les espaces invariants par réarrangement ayant la $p-BS$-propriété, ordonné par l'inclusion, possède un élément unique maximal. Si $p=2$, cet élément coïncide avec l'espace L_2, tandis que, pour $1<p<2$, il coïncide avec l'espace $L^0_{p,\infty}$ c'est à dire la « partie séparable » de l'espace $[L_1,L_\infty]_{1/p,\infty}$. Ici $[\cdot,\cdot]_{\theta,q}$, $0<\theta<1, 1\leqslant q\leqslant\infty$, désigne la méthode réelle d'interpolation [7,5]. En fait on calcule l'indice de Banach–Saks pour les espaces de la classe $[L_{p_1},L_{p_2}]_{\theta,q}$, $1\leqslant p_1<p_2\leqslant\infty$, qui coïncide avec la classe des $L_{p,q}$-espaces (de Lorentz), $1\leqslant p<\infty$, $1\leqslant q\leqslant\infty$, voir [5,7]. Nos résultats (Théoréme 2.6) montrent que dans la classe des $L_{p,q}$-espaces, les indices de Banach–Saks d'ependent de p et q. En particulier, $\gamma(L_{p,q})$ est une fonction discontinue en $q=1$ et le type (de Rademacher) de $L_{p,q}$ ne coïncide pas nécessairement avec son indice de Banach–Saks. De plus, il en découle que l'ensemble des espaces E invariants par réarrangement ayant $\gamma(E)\geqslant r$, $r\in(1,2]$, n'est pas stable par rapport aux méthodes complexe ou réelle d'interpolation.

1. Introduction

Let E be a Banach space and $p\geqslant 1$. A bounded sequence $\{x_n\}\subset E$ is called a $p-BS$-sequence (BS-sequence) if there exists a subsequence $\{y_{k_k}\}\subset\{x_k\}$ such that

$$\limsup_{m\to\infty} m^{-1/p}\left\|\sum_{k=1}^{m} y_k\right\|_E < \infty \quad \left(\lim_{m\to\infty}\frac{1}{m}\left\|\sum_{k=1}^{m} y_k\right\| = 0\right).$$

We shall say that E has the $p-BS$-property (the BS-property) and write $E\in\mathcal{BS}_p$ ($E\in\mathcal{BS}$) if every weakly null sequence $\{x_n\}\subset E$ contains a $p-BS$-sequence (BS-sequence). It is evident that every Banach space has $1-BS$-property. The set $\Gamma(E)=\{p\colon\ E\in\mathcal{BS}_p\}$ is either $[1,\alpha]$, or else $[1,\alpha)$ for some $\alpha\in[1,\infty]$. We shall write $\gamma(E)=\alpha$ if $\Gamma=[1,\alpha]$ and $\gamma(E)=\alpha-0$ if $\Gamma(E)=[1,\alpha)$. The number $\gamma(E)$ is called the Banach–Saks index. For classical (sequence and function) L_p-spaces these indices are well known. Indeed, it is easy to see [6] that $\gamma(l_p)=p$ for $1<p<\infty$ and $\gamma(l_1)=\gamma(c_0)=\infty$, whereas classical results of Banach and Saks [2] and Kadec and Pelczynski [4] yield $\gamma(L_p)=\min(p,2)$ for $p\in[1,\infty)$. The main objective of the present note is to study the Banach–Saks index in the class of rearrangement invariant spaces on $[0,1]$ [7,5]. A Banach function space E (with an order semicontinuous norm) on $[0,1]$ is said to be rearrangement invariant (r.i.) if

(1) $|x(t)|\leqslant|y(t)|$ a.e., $y\in E$ imply $x\in E$ and $\|x\|_E\leqslant\|y\|_E$,
(2) $x(t)$ and $y(t)$ are equimeasurable (i.e., if $x^*(t)=y^*(t)$ for all $t>0$, where x^* and y^* are right continuous non-increasing rearrangements of $|x|$ and $|y|$ respectively, see [5]) and $y\in E$, then $x\in E$ and $\|x\|_E=\|y\|_E$.

Denote by E^0 the closure of L_∞ in E. If $E\neq L_\infty$, then the space E^0 is a separable r.i. space. By [6], 1.a.7, any non-separable r.i. space E contains a subspace isomorphic to l_∞. Since l_∞ is a universal space it follows from Baernstein's result [1] that l_∞ contains a subspace without the *BS*-property. Consequently, E fails to have the *BS*-property and $\gamma(E) = 1$. Therefore, we shall investigate the *BS*-index of separable r.i. spaces only.

Recall that a Banach space E has the type $p \in (1, 2]$ if

$$\int_0^1 \left\| \sum_{k=1}^n r_n(s) x_k \right\|_E \mathrm{d}s \leqslant C \left(\sum_{k=1}^n \|x_k\|^p \right)^{1/p}$$

for any $x_1, x_2, \dots, x_n \in E$ and any $n \geqslant 1$, where $r_k(s) = \operatorname{sign}(\sin 2^k \pi s)$, $k = 1, 2, \dots,$ are the Rademacher functions. The Rademacher functions r_k form a weakly null sequence in any separable r.i. space E. Using the Khintchine inequality [6,7] and the continuity of embedding $E \subseteq L_1$ (which holds for any r.i. space E on $[0, 1]$, see [5]) we see that $1 \leqslant \gamma(E) \leqslant 2$ for any r.i. space E. Clearly, these bounds are exact.

The set of operators

$$\sigma_\tau x(t) = \begin{cases} x(t/\tau), & 0 \leqslant t \leqslant \min(\tau, 1), \\ 0, & \text{for all other } t \in [0, 1] \end{cases}$$

acts in any r.i. space and $\min(1, \tau) \leqslant \|\sigma_\tau\|_E \leqslant \max(1, \tau)$, $0 < \tau < \infty$. The numbers

$$\alpha_E = \lim_{\tau \to 0} \frac{\ln \|\sigma_\tau\|_E}{\ln \tau}, \qquad \beta_E = \lim_{\tau \to \infty} \frac{\ln \|\sigma_\tau\|_E}{\ln \tau}$$

are called the Boyd indices of r.i. space E. For every such E, we always have $0 \leqslant \alpha_E \leqslant \beta_E \leqslant 1$.

A Banach lattice E is called p-convex (q-concave) if there exist $C > 0$ such that

$$\left\| \left(\sum_{k=1}^n |x_k|^p \right)^{1/p} \right\|_E \leqslant C \left(\sum_{k=1}^n \|x_k\|_E^p \right)^{1/p} \quad \left(\left\| \left(\sum_{k=1}^n |x_k|^q \right)^{1/q} \right\|_E \geqslant \frac{1}{C} \left(\sum_{k=1}^n \|x_k\|_E^q \right)^{1/q} \right)$$

for any $x_1, x_2, \dots, x_n \in E$, $n \geqslant 1$. Here $p, q \in [1, \infty]$.

2. Results

Theorem 2.1. *Let E be a separable p-convex r.i. space for some $p > 1$ such that $\alpha_E > 0$. Then $\gamma(E) \geqslant \min(p, 2)$.*

In particular, if a separable r.i. space E is 2-convex and $\alpha_E > 0$, then $\gamma(E) = 2$. Theorem 2.1 may be applied to r.i. spaces with trivial type. For such spaces Rakov's result that any Banach space E of type p belongs to $\mathcal{BS}_p$ (see [9]) yields only the trivial estimate $\gamma(E) \geqslant 1$. The assumption $\alpha_E > 0$ in Theorem 2.1 is essential. Indeed, consider Orlicz space $\exp L_p := L_{M_p}(0, 1)$ with the Orlicz function $M_p(u) := \mathrm{e}^{u^p} - 1$, $p \geqslant 1$ (see, e.g., [7]). It is shown in [3] that $(\exp L_p)^0$ fails the *BS*-property and consequently $\gamma((\exp L)^0) = 1$. On the other hand, it is easy to see that $(\exp L_p)^0$ is p-convex for any $p \geqslant 1$.

Theorem 2.2. *If the separable r.i. space E satisfies the following properties*:

(1) *E is q-concave for some $q < \infty$*;
(2) $\beta_E < 1/2$.
(3) $2 -$ *BS-property holds for disjointly supported sequences from E, i.e., for every weakly null sequence $\{x_k\} \subset E$ of disjointly supported elements there exist an increasing sequence k_i of positive integers and a constant $C > 0$ such that*

$$\left\| \sum_{i=1}^n x_{k_i} \right\|_E \leqslant C\sqrt{n}, \quad n \geqslant 1;$$

then $\gamma(E) = 2$.

The Banach–Saks and Kadec–Pelczynski theorems show that the maximal index 2 in the scale L_p is attained if $2 \leqslant p < \infty$. There is a partial converse to this statement.

Theorem 2.3. *If E is a separable r.i. space and $\gamma(E) = 2$, then $L_p \subset E \subset L_2$ for some $p \in [2, \infty)$.*

Hence we have the following:

Corollary 2.4. *If E is a separable r.i. space and $\gamma(E) = \gamma((E^*)^0) = 2$, then $E = L_2$ up to the norm equivalence.*

For r.i. spaces with index $p \in (1, 2)$, we have different embeddings. To formulate the result, we first recall the definition of the Lorentz spaces $L_{p,q}$: $x \in L_{p,q}$ if and only if the quasi-norm

$$\|x\|_{p,q} = \begin{cases} \dfrac{q}{p}\left(\displaystyle\int_0^1 \left(x^*(t)t^{1/p}\right)^q \frac{\mathrm{d}t}{t}\right)^{1/q}, & q < \infty, \\ \sup x^*(t)t^{1/p}, & q = \infty \end{cases}$$

is finite. $L_{p,q}$-spaces play a significant role in the interpolation theory [5,7]. The expression $\|\cdot\|_{p,q}$ is a norm if $1 \leqslant q \leqslant p$ and is equivalent to a (Banach) norm if $q > p$.

Theorem 2.5. *Let E be a separable r.i. space and $1 < p < 2$. If $\gamma(E) = p$, then $\exp L_q \subset E \subset L^0_{p,\infty}$, where $q > p/(p-1)$.*

It follows from Theorems 2.3,2.5 and 2.6 (below) that the subset of $\mathcal{BS}_p$ formed by all rearrangement invariant spaces with the $p - BS$-property ordered by inclusion has a unique maximal element. If $p = 2$ this element coincides with the space L_2, whereas for $1 < p < 2$ it is given by the space $L^0_{p,\infty}$. When $p = 2$, such a set does not have any minimal element. Indeed, this follows from Theorem 2.3 and the Kadec–Pelczynski theorem [4].

Theorem 2.6. *Let $1 < p < \infty,\ 1 \leqslant q < \infty$. Then*

$$\gamma(L_{p,q}) = \begin{cases} \min(p, q, 2), & p \neq 2,\ q \neq 1 \text{ or } p = 2,\ 1 < q \leqslant 2, \\ \min(p, 2), & p \neq 2,\ q = 1, \\ 2 - 0, & p = 2,\ q \in \{1\} \cup (2, \infty) \end{cases}$$

and

$$\gamma\left(L^0_{p,\infty}\right) = \begin{cases} \min(p, 2), & p \neq 2, \\ 2 - 0, & p = 2. \end{cases}$$

It easily follows from Theorem 2.6 that (i) the function $(p, q) \to \gamma(L_{p,q})$ is discontinuous at $q = 1$ (and continuous at $q = \infty$); (ii) if $r \in (1, 2]$, then the set of r.i. spaces E such that $\gamma(E) \geqslant r$ is not stable with respect to the complex or real interpolation methods; (iii) the type of $L_{p,q}$ coincides with its *BS*-index if and only if $1 < q < \infty$, in other words if and only if $L_{p,q}$ is reflexive.

Let M be an Orlicz function satisfying the Δ_2-condition at ∞ and let L_M be the corresponding Orlicz space (see [7]). Denote

$$a_M = \sup\left\{p\colon\ p \geqslant 1 \inf_{\lambda, t \geqslant 1} M(\lambda t)/M(\lambda)t^p > 0\right\}.$$

Theorem 2.7.

(1) $\gamma(L_M) \leqslant \min(a_M, 2)$.

(2) *If* $M(u^{1/p})$ *is convex up to equivalence for some* $p \in (1,2]$, *then* $\gamma(L_M) \geqslant p$.

Consider the set of functions

$$M_{p,\alpha}(u) = u^p \ln^\alpha(\mathrm{e}+u), \quad u \geqslant 0,$$

where $1 < p < \infty$, $\alpha \in \mathbb{R}$. The function $M_{p,\alpha}$ is convex for $\alpha \geqslant 0$ and is convex up to equivalence for $\alpha < 0$. Denote by $L(p,\alpha)$ the corresponding Orlicz space (which is frequently called Lorentz–Zygmund space).

Corollary 2.8. *If* $1 < p < \infty$, $\alpha \in \mathbb{R}$, *then*

$$\gamma\big(L(p,\alpha)\big) = \begin{cases} \min(p,2), & \text{if } p > 2 \text{ or } \alpha \geqslant 0, \\ p-0, & \text{if } p \leqslant 2 \text{ and } \alpha < 0. \end{cases}$$

Theorem 2.6 and Corollary 2.8 extend to the Lorentz spaces $L_{p,q}$ and Orlicz spaces $L(p,\alpha)$ classical results concerning the Banach–Saks index given in [2,4].

All of the results announced here are contained in [8].

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