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TORSION p-ADIC GALOIS REPRESENTATIONS AND A CONJECTURE OF FONTAINE

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ABSTRACT. – Let p be a prime, K a finite extension of $\mathbb{Q}_p$ and T a finite free $\mathbb{Z}_p$-representation of $\mathrm{Gal}(\bar{K}/K)$. We prove that $T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is semi-stable (resp. crystalline) with Hodge–Tate weights in $\{0, \ldots, r\}$ if and only if, for all n, T/p^nT is torsion semi-stable (resp. crystalline) with Hodge–Tate weights in $\{0, \ldots, r\}$.

RÉSUMÉ. – Soient p un nombre premier, r un entier positif, K une extension finie de $\mathbb{Q}_p$ et T une $\mathbb{Z}_p$-représentation de $\mathrm{Gal}(\bar{K}/K)$ libre de rang fini en tant que $\mathbb{Z}_p$-module. On montre que $T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ est semi-stable (resp. cristalline) à poids de Hodge–Tate dans $\{0, \ldots, r\}$ si et seulement si, pour tout entier n, la représentation T/p^nT est le quotient de deux réseaux dans une représentation semi-stable (resp. cristalline) à poids de Hodge–Tate dans $\{0, \ldots, r\}$.

1. Introduction

Let k be a perfect field of characteristic p, $W(k)$ its ring of Witt vectors, $K_0 = W(k)[\frac{1}{p}]$, K/K_0 a finite totally ramified extension and $e = e(K/K_0)$ the absolute ramification index. For many technical reasons, we are interested in understanding the universal deformation ring of a fixed residual representation of $G := \mathrm{Gal}(\bar{K}/K)$. In particular, it is important to study those deformations that are semi-stable (resp. crystalline). In [12], Fontaine conjectured that there exists a quotient of the universal deformation ring parameterizing semi-stable (resp. crystalline) representations. To prove the conjecture, it suffices to prove the following:

CONJECTURE 1.0.1 [12]. – *Fix an integer $r > 0$. Let T be a finite free $\mathbb{Z}_p$-representation of G. Then $T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is semi-stable (resp. crystalline) with Hodge–Tate weights in $\{0, \ldots, r\}$ if and only if, for all n, $T_n := T/p^nT$ is torsion semi-stable (resp. torsion crystalline) with Hodge–Tate weights in $\{0, \ldots, r\}$, in the sense that there exist G-stable $\mathbb{Z}_p$-lattices $L'_{(n)} \subset L_{(n)}$ inside a semi-stable (resp. crystalline) Galois representation $V_{(n)}$ with Hodge–Tate weights in $\{0, \ldots, r\}$ such that $T_n \simeq L_{(n)}/L'_{(n)}$ as $\mathbb{Z}_p[G]$-modules.*

If T/p^nT comes from the generic fiber of a finite flat group scheme over $\mathcal{O}_K$, i.e., in the case that $r = 1$ and $V_{(n)}$ is crystalline for all n, the conjecture has been proved by Ramakrishna ([21]). The case that $e = 1$ and $V_{(n)}$ is crystalline has been proved by L. Berger ([2]), and the case that $e = 1$ and $r < p - 1$ was shown by Breuil ([6]). In this paper, we give a complete proof of Conjecture 1.0.1 without any restriction. Our main input is from [14], where Kisin proved that any G-stable $\mathbb{Z}_p$-lattice in a semi-stable Galois representation is of finite $E(u)$-height. More

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precisely, fix a uniformiser $\pi \in K$ with Eisenstein polynomial $E(u)$. Let $K_\infty = \bigcup_{n\geqslant 1} K(\sqrt[p^n]{\pi})$, $G_\infty = \mathrm{Gal}(\bar{K}/K_\infty)$ and $\mathfrak{S} = W(k)[\![u]\!]$. We equip $\mathfrak{S}$ with the endomorphism φ which acts via Frobenius on $W(k)$, and sends u to u^p. For every positive integer r, let $\mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$ denote the category of finite free $\mathfrak{S}$-modules $\mathfrak{M}$ equipped with a φ-semi-linear map $\varphi: \mathfrak{M} \to \mathfrak{M}$ such that the cokernel of $\varphi^* = 1 \otimes \varphi : \mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \to \mathfrak{M}$ is killed by $E(u)^r$. Such modules with φ-structure are called φ-modules of finite $E(u)$-height. For any $\mathfrak{M} \in \mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$, one associates a finite free $\mathbb{Z}_p$-representation $T_{\mathfrak{S}}(\mathfrak{M})$ of G_∞ ([8]). Kisin ([14]) proved that any G_∞-stable $\mathbb{Z}_p$-lattice L in a semi-stable Galois representation arises from a φ-module of finite $E(u)$-height, i.e., there exists $\mathfrak{L} \in \mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$ such that $T_{\mathfrak{S}}(\mathfrak{L}) \simeq L$. In particular, this result implies that if a $\mathbb{Z}_p[G]$-module M is torsion semi-stable with Hodge–Tate weights in $\{0, \ldots, r\}$ then there exists a (p-power) torsion φ-module $\mathfrak{M}$ of height r (see §2 for precise definitions) such that $T_{\mathfrak{S}}(\mathfrak{M}) \simeq M$ as G_∞-modules. Therefore, we can use torsion φ-modules of finite $E(u)$-height to study torsion representations of G_∞. If $p > 2$ and $r = 1$, Breuil and Kisin proved that there exists an anti-equivalence between the category of finite flat group schemes over $\mathcal{O}_K$ and torsion φ-modules of height 1 ([14], [4]). Thus, torsion φ-modules of finite $E(u)$-height can be seen as a natural extension of finite flat group schemes over $\mathcal{O}_K$. In particular, we extend many results on finite flat group schemes over $\mathcal{O}_K$ to torsion φ-modules of finite $E(u)$-height. For example, under the hypotheses of Conjecture 1.0.1, we prove that the $\mathbb{Z}_p$-representation T in Conjecture 1.0.1 must arise from a φ-module of finite $E(u)$-height, i.e., there exists $\mathfrak{M} \in \mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$ such that $T_{\mathfrak{S}}(\mathfrak{M}) \simeq T$ as G_∞-modules. To prove this result, we extend Tate's isogeny theorem on p-divisible groups to finite level as in [19] and [3], i.e., we show that the functor $T_{\mathfrak{S}}$ is "weakly" fully faithful on torsion objects. (See Theorem 2.4.2 for details.)

So far, only the G_∞-action on T has been used. To fully use the G-action on T, we construct an A_{cris}-linear injection (in §5)

$$\iota : \mathfrak{M} \otimes_{\mathfrak{S},\varphi} A_{\mathrm{cris}} \to T^\vee \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}} \tag{1.0.1}$$

such that ι is compatible with Frobenius and G_∞-action (cf. Lemma 5.3.4). Note that T is a representation of G. There is a natural G-action on the right-hand side of (1.0.1). However, it is not clear if $\mathfrak{M} \otimes_{\mathfrak{S},\varphi} A_{\mathrm{cris}}$ is G-stable (viewed as a submodule of $T^\vee \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}}$ via ι). In §6 we prove that $\mathfrak{M} \otimes_{\mathfrak{S},\varphi} B^+_{\mathrm{cris}}$ is stable under the G-action after very carefully analyzing "G-action" on $\mathfrak{M}/p^n\mathfrak{M} \otimes_{\mathfrak{S},\varphi} A_{\mathrm{cris}}$ for each n. In fact, we show that $G(\mathfrak{M})$ lies in $\mathfrak{M} \otimes_{\mathfrak{S},\varphi} \mathcal{R}_{K_0}$ for a subring $\mathcal{R}_{K_0}$ of B^+_{cris}. Finally, we prove that $\mathcal{R}_{K_0}$ is small enough to show that $\dim_{K_0}(T^\vee \otimes_{\mathbb{Z}_p} B^+_{\mathrm{st}})^G \geqslant \mathrm{rank}_{\mathbb{Z}_p}(T)$ and thus prove Conjecture 1.0.1. Let us apply our theorem to the universal deformation ring of Galois representations. Let $E/\mathbb{Q}_p$ be a finite extension with finite residue field $\mathbb{F}$. Denote by $\mathscr{C}$ the category of local Noetherian complete $\mathcal{O}_E$-algebras with residue field $\mathbb{F}$. For $A \in \mathscr{C}$, an A-representation T of G is an A-module of finite type equipped with a linear and continuous action of G. Fix a finite free $\mathbb{F}$-representation $\bar{\rho}$ which is torsion semi-stable (resp. crystalline) with Hodge–Tate weights in $\{0, \ldots, r\}$. Let $D(A)$ be the set of isomorphism classes of finite free A-representations T such that $T/\mathfrak{m}_{\mathcal{O}_E} T \simeq \bar{\rho}$ and $D^{\mathrm{ss},r}(A)$ (resp. $D^{\mathrm{cris},r}(A)$) the subset of $D(A)$ consisting of isomorphism classes of those representations that are torsion semi-stable (resp. crystalline) with Hodge–Tate weights in $\{0, \ldots, r\}$. By [20] and [21], if $\mathrm{H}^0(G, \mathrm{GL}(\bar{\rho})) = \mathbb{F}$, then $D(A)$, $D^{\mathrm{ss},r}(A)$ and $D^{\mathrm{cris},r}(A)$ are pro-representable by complete local Noetherian rings $R_{\bar{\rho}}$, $R^{\mathrm{ss},r}_{\bar{\rho}}$ and $R^{\mathrm{cris},r}_{\bar{\rho}}$, respectively. $R^{\mathrm{ss},r}_{\bar{\rho}}$ and $R^{\mathrm{cris},r}_{\bar{\rho}}$ are quotients of $R_{\bar{\rho}}$.

THEOREM 1.0.2. – *For any finite K_0-algebra B, a map $x : R_{\bar{\rho}} \to B$ factors through $R^{\mathrm{ss},r}_{\bar{\rho}}$* (*resp. $R^{\mathrm{cris},r}_{\bar{\rho}}$*) *if and only if the induced B-representation V_x of G is semi-stable* (*resp. crystalline*) *with Hodge–Tate weights in* $\{0, \ldots, r\}$.

In fact, the existence of such a quotient of $R_{\bar\rho}$ satisfying the property in the above theorem has been known by Kisin (cf., Theorem in [16]). Here we reprove the Theorem in [16] and further show that such quotient is just $R_{\bar\rho}^{\mathrm{ss},r}$ (or $R_{\bar\rho}^{\mathrm{cris},r}$). As explained in the introduction of [16], it will be useful to distinguish four flavors of the statement that some property **P** (e.g., being crystalline, semi-stable etc.) of p-adic Galois representations cuts out a closed subspace of the generic fiber of $\operatorname{Spec} R_{\bar\rho}$.

(1) Let $E/\mathbb{Q}_p$ be a finite extension and $x_i : R_{\bar\rho} \to E$ $(i \geqslant 1)$ a sequence of points converging p-adically to a point $x : R_{\bar\rho} \to E$. Write V_{x_i} and V_x for the corresponding E-representations. If the V_{x_i} have **P**, then V_x has **P**.
(2) The set $\{x \in \operatorname{Hom}_E(R_{\bar\rho}, \mathbb{C}_p) \mid x \text{ has } \mathbf{P}\}$ cuts out a closed analytic subspace in the rigid analytic space associated to $R_{\bar\rho}$ (see [1] for the more precise statement).
(3) There is a quotient $R_{\bar\rho}^{\mathbf{P}}$ of $R_{\bar\rho}$ such that $R_{\bar\rho} \to E$ factors through $R_{\bar\rho}^{\mathbf{P}}$ if and only if V_x has **P**.
(4) Let V be a finite dimensional E-representation of G, and $L \subset V$ a G-stable $\mathbb{Z}_p$-lattice. Suppose that for each n, $L/p^n L$ is a subquotient of lattices in a representation having **P**. Then V has **P**.

It is not hard to see that we have the implications (4) $\Longrightarrow$ (3) $\Longrightarrow$ (2) $\Longrightarrow$ (1). Conjecture 1.0.1 is just (4) for **P** the property of being semi-stable or crystalline with bounded Hodge–Tate weights. For the same condition **P**, (3) is established in [16], which is sufficient for applications to modularity theorems as in [15] (whereas (1) is not). Recently, Berger and Colmez proved (2) for **P** the property of being de Rham, crystalline or semi-stable with bounded Hodge–Tate weights via the theory of (φ, Γ)-modules in [1].

Convention 1.0.3. – We will deal with many p-power torsion modules. To simplify our notations, if M is a $\mathbb{Z}$-module, then we denote $M/p^n M$ by M_n. We also have to consider various Frobenius structures on different modules. To minimize possible confusion, we sometimes add a subscript to φ to indicate over which module the Frobenius is defined. For example, $\varphi_{\mathfrak{M}}$ indicates the Frobenius defined over $\mathfrak{M}$. We often drop the subscript if no confusion will arise. We use *contravariant* functors (almost) everywhere. So removing the " * " from the notations for those functors will be more convenient. For example, the notation V_{st} as used in this paper is denoted by V_{st}^* in [7]. If V is a finite $\mathbb{Z}_p$-representation of G_∞, we denote by $V^\vee$ the dual representation of V, i.e., $V^\vee = \operatorname{Hom}_{\mathbb{Z}_p}(V, \mathbb{Q}_p/\mathbb{Z}_p)$ if V is killed by some power of p and $V^\vee = \operatorname{Hom}_{\mathbb{Z}_p}(V, \mathbb{Z}_p)$ if V is a finite free $\mathbb{Z}_p$-module. Finally, if X is a matrix, X^t denotes its transpose. We always denote the identity map by Id.

2. φ-modules of finite $E(u)$-height and representations of G_∞

This paper consists of 2 parts. §2–§4 is the first part, where we mainly discuss the theory of φ-modules of finite $E(u)$-height over $\mathfrak{S}$ and their associated $\mathbb{Z}_p$-representations of G_∞. The key results to be proved are Theorem 2.4.2, Theorem 3.2.2 and Theorem 2.4.1 and its refinement Corollary 4.4.1. The second part (§5–§8) of this paper will combine the inputs from the first part and Kisin's result (Theorem 5.4.1) to prove Conjecture 1.0.1.

2.1. Preliminaries

Throughout this paper we fix a positive integer r and a uniformiser $\pi \in K$ with Eisenstein polynomial $E(u)$. Recall that $\mathfrak{S} = W(k)[\![u]\!]$ is equipped with a Frobenius endomorphism φ via $u \mapsto u^p$ and the natural Frobenius on $W(k)$. A *φ-module* (over $\mathfrak{S}$) is an $\mathfrak{S}$-module $\mathfrak{M}$ equipped with a φ-semi-linear map $\varphi : \mathfrak{M} \to \mathfrak{M}$. A morphism between two objects $(\mathfrak{M}_1, \varphi_1)$, $(\mathfrak{M}_2, \varphi_2)$

is a $\mathfrak{S}$-linear morphism compatible with the φ_i. Denote by $\mathrm{Mod}^r_{/\mathfrak{S}}$ the category of φ-modules *of finite $E(u)$-height r* in the sense that $\mathfrak{M}$ is of finite type over $\mathfrak{S}$ and the cokernel of φ^* is killed by $E(u)^r$, where φ^* is the $\mathfrak{S}$-linear map $1\otimes\varphi:\mathfrak{S}\otimes_{\varphi,\mathfrak{S}}\mathfrak{M}\to\mathfrak{M}$. Let $\mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$ be the sub-category of $\mathrm{Mod}^r_{/\mathfrak{S}}$ consisting of finite $\mathfrak{S}$-modules $\mathfrak{M}$ which are killed by some power of p and have projective dimension 1 in the sense that $\mathfrak{M}$ has a two term resolution by finite free $\mathfrak{S}$-modules. We give $\mathrm{Mod}^r_{/\mathfrak{S}}$ the structure of an exact category induced by that on the Abelian category of $\mathfrak{S}$-modules. We denote by $\mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$ the subcategory of $\mathrm{Mod}^r_{/\mathfrak{S}}$ consisting of finite free $\mathfrak{S}$-modules. Let $R=\varprojlim\mathcal{O}_{\bar K}/p$ where the transition maps are given by Frobenius. By the universal property of the Witt vectors $W(R)$ of R, there is a unique surjective projection map $\theta:W(R)\to\widehat{\mathcal{O}}_{\bar K}$ to the p-adic completion of $\mathcal{O}_{\bar K}$, which lifts the projection $R\to\mathcal{O}_{\bar K}/p$ onto the first factor in the inverse limit. Let $\pi_n\in\bar K$ be a p^n-root of π, such that $(\pi_{n+1})^p=\pi_n$; write $\underline{\pi}=(\pi_n)_{n\geqslant 0}\in R$ and let $[\underline{\pi}]\in W(R)$ be the Teichmüller representative. We embed the $W(k)$-algebra $W(k)[u]$ into $W(R)$ by the map $u\mapsto[\underline{\pi}]$. This embedding extends to an embedding $\mathfrak{S}\hookrightarrow W(R)$, and, as $\theta([\underline{\pi}])=\pi$, $\theta|_{\mathfrak{S}}$ is the map $\mathfrak{S}\to\mathcal{O}_K$ sending u to π. This embedding is compatible with Frobenius endomorphisms. Denote by $\mathcal{O}_{\mathcal{E}}$ the p-adic completion of $\mathfrak{S}[\frac{1}{u}]$. Then $\mathcal{O}_{\mathcal{E}}$ is a discrete valuation ring with residue field the Laurent series ring $k((u))$. We write $\mathcal{E}$ for the field of fractions of $\mathcal{O}_{\mathcal{E}}$. If $\mathrm{Fr}\,R$ denotes the field of fractions of R, then the inclusion $\mathfrak{S}\hookrightarrow W(R)$ extends to an inclusion $\mathcal{O}_{\mathcal{E}}\hookrightarrow W(\mathrm{Fr}\,R)$. Let $\mathcal{E}^{\mathrm{ur}}\subset W(\mathrm{Fr}\,R)[\frac{1}{p}]$ denote the maximal unramified extension of $\mathcal{E}$ contained in $W(\mathrm{Fr}\,R)[\frac{1}{p}]$, and $\mathcal{O}^{\mathrm{ur}}$ its ring of integers. Since $\mathrm{Fr}\,R$ is easily seen to be algebraically closed, the residue field $\mathcal{O}^{\mathrm{ur}}/p\mathcal{O}^{\mathrm{ur}}$ is the separable closure of $k((u))$. We denote by $\widehat{\mathcal{E}^{\mathrm{ur}}}$ the p-adic completion of $\mathcal{E}^{\mathrm{ur}}$, and by $\widehat{\mathcal{O}^{\mathrm{ur}}}$ its ring of integers. $\widehat{\mathcal{E}^{\mathrm{ur}}}$ is also equal to the closure of $\mathcal{E}^{\mathrm{ur}}$ in $W(\mathrm{Fr}\,R)[\frac{1}{p}]$. We write $\mathfrak{S}^{\mathrm{ur}}=\widehat{\mathcal{O}^{\mathrm{ur}}}\cap W(R)\subset W(\mathrm{Fr}\,R)$. We regard all these rings as subrings of $W(\mathrm{Fr}\,R)[\frac{1}{p}]$. Recall that $K_\infty=\bigcup_{n\geqslant 0}K(\pi_n)$, and $G_\infty=\mathrm{Gal}(\bar K/K_\infty)$. G_∞ acts continuously on $\mathfrak{S}^{\mathrm{ur}}$ and $\mathcal{E}^{\mathrm{ur}}$ and fixes the subring $\mathfrak{S}\subset W(R)$. Finally, we denote by $\mathrm{Rep}_{\mathbb{Z}_p}(G_\infty)$ the category of continuous $\mathbb{Z}_p$-linear representations of G_∞ on finite $\mathbb{Z}_p$-modules and by $\mathrm{Rep}^{\mathrm{tor}}_{\mathbb{Z}_p}(G_\infty)$ the subcategory consisting of those representations killed by some power of p.

2.2. Fontaine's theory on finite $\mathbb{Z}_p$-representations of G_∞

Recall ([8], A, §1.1.4) that a finite $\mathcal{O}_{\mathcal{E}}$-module M is called *étale* if M is equipped with a φ-semi-linear map $\varphi_M:M\to M$, such that the induced $\mathcal{O}_{\mathcal{E}}$-linear map $\varphi_M^*:\mathcal{O}_{\mathcal{E}}\otimes_{\varphi,\mathcal{O}_{\mathcal{E}}}M\to M$ is an isomorphism. We denote by $\mathbf{\Phi M}_{\mathcal{O}_{\mathcal{E}}}$ the category of étale modules with the obvious morphisms. An argument in [4], §2.1.1, shows that K_∞/K is a strictly APF extension in the sense of [24]. Then Proposition A 1.2.6 in [8] implies that the functor

$$T^\vee:\mathbf{\Phi M}_{\mathcal{O}_{\mathcal{E}}}\to\mathrm{Rep}_{\mathbb{Z}_p}(G_\infty);\quad M\mapsto(M\otimes_{\mathcal{O}_{\mathcal{E}}}\widehat{\mathcal{O}^{\mathrm{ur}}})^{\varphi=1}\tag{2.2.1}$$

is an equivalence of Abelian categories and the inverse of $T^\vee$ is given by

$$\mathrm{Rep}_{\mathbb{Z}_p}(G_\infty)\to\mathbf{\Phi M}_{\mathcal{O}_{\mathcal{E}}};\quad V\mapsto(V\otimes_{\mathbb{Z}_p}\widehat{\mathcal{O}^{\mathrm{ur}}})^{G_\infty}.$$

In particular, for any $M\in\mathbf{\Phi M}_{\mathcal{O}_{\mathcal{E}}}$, we have the following natural $\widehat{\mathcal{O}^{\mathrm{ur}}}$-linear isomorphism compatible with φ-structures.

$$\tilde\iota:M\otimes_{\mathcal{O}_{\mathcal{E}}}\widehat{\mathcal{O}^{\mathrm{ur}}}\simeq T^\vee(M)\otimes_{\mathbb{Z}_p}\widehat{\mathcal{O}^{\mathrm{ur}}}.\tag{2.2.2}$$

We frequently use the contravariant version of $T^\vee$ in this paper. For $M \in \mathbf{\Phi M}_{\mathcal{O}_\mathcal{E}}$, define

$$T(M) = \mathrm{Hom}_{\mathcal{O}_\mathcal{E},\varphi}(M, \widehat{\mathcal{O}^{\mathrm{ur}}}) \quad \text{if } M \text{is } p\text{-torsion free} \tag{2.2.3}$$

and (recall $\mathcal{O}^{\mathrm{ur}}_n = \mathcal{O}^{\mathrm{ur}}/p^n\mathcal{O}^{\mathrm{ur}}$)

$$T(M) = \mathrm{Hom}_{\mathcal{O}_\mathcal{E},\varphi}(M, \mathcal{O}^{\mathrm{ur}}_n) \quad \text{if } M \text{ is killed by } p^n. \tag{2.2.4}$$

It is easy to show that $T^\vee(M)$ is the dual representation of $T(M)$. See for example §1.2.7 in [8], where Fontaine uses $V^*_\mathcal{E}(M)$ to denote $T(M)$.

Recall that a $\mathfrak{S}$-module $\mathfrak{M}$ is called *p'-torsion free* ([8], B 1.2.5) if for all nonzero $x \in \mathfrak{M}$, $\mathrm{Ann}(x) = 0$ or $\mathrm{Ann}(x) = p^n\mathfrak{S}$ for some n. This is equivalent to the natural map $\mathfrak{M} \to \mathfrak{M} \otimes_\mathfrak{S} \mathcal{O}_\mathcal{E}$ being injective. If $\mathfrak{M}$ is killed by some power of p, then $\mathfrak{M}$ is p'-torsion free if and only if $\mathfrak{M}$ is u-torsion free. A φ-module $\mathfrak{M}$ over $\mathfrak{S}$ is called *étale* if $\mathfrak{M}$ is p'-torsion free and $\mathfrak{M} \otimes_\mathfrak{S} \mathcal{O}_\mathcal{E}$ is an étale $\mathcal{O}_\mathcal{E}$-module. Since $E(u)$ is a unit in $\mathcal{O}_\mathcal{E}$, we see that for any $\mathfrak{M} \in \mathrm{Mod}^r_{/\mathfrak{S}}$, $\mathfrak{M}$ is étale if and only if $\mathfrak{M}$ is p'-torsion free. Obviously, any object in $\mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$ is étale. In the next subsection, we will show that any object in $\mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$ is also étale. For any étale $\mathfrak{M} \in \mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$, we can associate a $\mathbb{Z}_p[G_\infty]$-module via

$$T_\mathfrak{S}(\mathfrak{M}) = \mathrm{Hom}_{\mathfrak{S},\varphi}\big(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}}[1/p]/\mathfrak{S}^{\mathrm{ur}}\big). \tag{2.2.5}$$

Similarly, for any $\mathfrak{M} \in \mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$, we define

$$T_\mathfrak{S}(\mathfrak{M}) = \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}}). \tag{2.2.6}$$

There is a natural injection $T_\mathfrak{S}(\mathfrak{M}) \hookrightarrow T(M)$ where $M := \mathfrak{M} \otimes_\mathfrak{S} \mathcal{O}_\mathcal{E}$. In fact, this injection is an isomorphism by the following Proposition 2.2.1 below. Let Λ be a φ-module over $\mathfrak{S}$. We denote by $F_\mathfrak{S}(\Lambda)$ the set of $\mathfrak{S}$-submodules $\mathfrak{M}$ such that $\mathfrak{M}$ is of $\mathfrak{S}$-finite type, stable under φ and étale. Define $j_*(\Lambda) = \bigcup_{\mathfrak{M} \in F_\mathfrak{S}(\Lambda)} \mathfrak{M}$. If A is a ring of characteristic p, we denote by A^{sep} the separable closure of A.

PROPOSITION 2.2.1 (Fontaine). – *For all $n \geqslant 1$, we have*
(1) $j_*(\mathrm{Fr}\, R) = k(\!(\underline{\pi})\!)^{\mathrm{sep}} \cap R = k[\![\underline{\pi}]\!]^{\mathrm{sep}}$,
(2) $j_*(W_n(\mathrm{Fr}\, R)) = \mathfrak{S}^{\mathrm{ur}}_n$,
(3) $j_*(W(\mathrm{Fr}\, R)) \subset \mathfrak{S}^{\mathrm{ur}}$ *and* $j_*(W(\mathrm{Fr}\, R))$ *is dense in* $\mathfrak{S}^{\mathrm{ur}}$,
(4) $\mathfrak{S}^{\mathrm{ur}}_n = W_n(R) \cap \mathcal{O}^{\mathrm{ur}}_n \subset W_n(\mathrm{Fr}\, R)$.

Proof. – Proposition 1.8.3 in [8]. Note that Fontaine uses $A^+_{S,n}$ to denote $\mathfrak{S}^{\mathrm{ur}}_n$. □

COROLLARY 2.2.2. – *Let $\mathfrak{M} \in \mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$ be étale or $\mathfrak{M} \in \mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$. Then $T_\mathfrak{S}(\mathfrak{M}) = T(\mathfrak{M} \otimes_\mathfrak{S} \mathcal{O}_\mathcal{E})$.*

Proof. – Let $M := \mathfrak{M} \otimes_\mathfrak{S} \mathcal{O}_\mathcal{E}$. It suffices to show that the natural injection $T_\mathfrak{S}(\mathfrak{M}) \hookrightarrow T(M)$ is a surjection. Suppose that $\mathfrak{M}$ is killed by p^n. For any $f \in T(M) = \mathrm{Hom}_{\mathfrak{S},\varphi}(M, \mathcal{O}^{\mathrm{ur}}_{\mathcal{E},n})$, $f(\mathfrak{M}) \subset \mathcal{O}^{\mathrm{ur}}_{\mathcal{E},n}$ is obviously a $\mathfrak{S}$-module of $\mathfrak{S}$-finite type, stable under φ. Since $\mathcal{O}^{\mathrm{ur}}_{\mathcal{E},n}$ is obviously p'-torsion free, $f(\mathfrak{M})$ is p'-torsion free. By Lemma 2.3.1 below, we see that $f(\mathfrak{M})$ is étale. Therefore $f(\mathfrak{M}) \in F_\mathfrak{S}(\mathcal{O}^{\mathrm{ur}}_{\mathcal{E},n})$ and $f(\mathfrak{M}) \subset \mathfrak{S}^{\mathrm{ur}}$. Thus $f \in \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}}_n) = T_\mathfrak{S}(\mathfrak{M})$. The above proof also works if $\mathfrak{M}$ is $\mathfrak{S}$-finite free by replacing $\mathfrak{S}^{\mathrm{ur}}_n$ with $\mathfrak{S}^{\mathrm{ur}}$, and $\mathcal{O}^{\mathrm{ur}}_{\mathcal{E},n}$ with $\mathcal{O}^{\mathrm{ur}}_\mathcal{E}$. □

COROLLARY 2.2.3. – *For all $n \geqslant 1$, $\mathfrak{S}^{\mathrm{ur}}_n[\frac{1}{u}] = \mathfrak{S}^{\mathrm{ur}}_n \otimes_\mathfrak{S} \mathcal{O}_\mathcal{E} \simeq \mathcal{O}^{\mathrm{ur}}_n$.*

Proof. – It is clear if $n = 1$ by Proposition 2.2.1 (1). The more general case can be proved by a standard *dévissage* argument; the details are left to the readers. □

2.3. Some properties of $\mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$

By §2.3 in [14] and [4], if $p \geqslant 3$, then $\mathrm{Mod}^{1,\mathrm{tor}}_{/\mathfrak{S}}$ is anti-equivalent to the category of finite flat group schemes over $\mathcal{O}_K$ (also see [17]). It is thus expected that modules in $\mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$ have similar properties to those of finite flat group schemes over $\mathcal{O}_K$. In this subsection, we extend some basic properties of finite flat group schemes over $\mathcal{O}_K$ to $\mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$.

LEMMA 2.3.1. – *Let $0 \to \mathfrak{M}' \to \mathfrak{M} \to \mathfrak{M}'' \to 0$ be an exact sequence of φ-modules over $\mathfrak{S}$. Suppose that $\mathfrak{M}'$, $\mathfrak{M}$ and $\mathfrak{M}''$ are p'-torsion free and $\mathfrak{M} \in \mathrm{Mod}^{r}_{/\mathfrak{S}}$. Then $\mathfrak{M}'$ and $\mathfrak{M}''$ are étale and in $\mathrm{Mod}^{r}_{/\mathfrak{S}}$.*

Proof. – See Proposition 1.3.5 in [8]. □

PROPOSITION 2.3.2. – *Let $\mathfrak{M} \in \mathrm{Mod}^{r}_{/\mathfrak{S}}$ be killed by p^n. The following statements are equivalent*:
(1) $\mathfrak{M} \in \mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$,
(2) $\mathfrak{M}$ *is u-torsion free,*
(3) $\mathfrak{M}$ *is étale,*
(4) $\mathfrak{M}$ *is a successive extension of finite free $k[\![u]\!]$-modules $\mathfrak{M}_i$ with $\mathfrak{M}_i \in \mathrm{Mod}^{r}_{/\mathfrak{S}}$,*
(5) $\mathfrak{M}$ *is a quotient of two finite free $\mathfrak{S}$-modules $\mathfrak{N}'$ and $\mathfrak{N}''$ with $\mathfrak{N}'$, $\mathfrak{N}'' \in \mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$.*

Proof. – (1) $\Longrightarrow$ (2) By the definition of $\mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$, there exist finite free $\mathfrak{S}$-modules $\mathfrak{N}'$ and $\mathfrak{N}''$ such that

$$0 \to \mathfrak{N}'' \to \mathfrak{N}' \to \mathfrak{M} \to 0$$

is exact. Let $\alpha_1, \ldots, \alpha_d$ and $\beta_1, \ldots, \beta_d$ be bases for $\mathfrak{N}''$ and $\mathfrak{N}'$ respectively and let A be the transition matrix; that is, $(\alpha_1, \ldots, \alpha_d) = (\beta_1, \ldots, \beta_d)A$. Since $\mathfrak{M}$ is killed by p^n for some n, there exists a matrix B with coefficients in $\mathfrak{S}$ such that $AB = p^n I$. Now suppose that $\bar{x} \in \mathfrak{M}$ is killed by u^m with $x = \sum_{i=1}^{d} x_i \alpha_i$. Then we have

$$u^m(x_1, \ldots, x_d) = (y_1, \ldots, y_d)A^{\mathrm{t}}$$

for some $y_i \in \mathfrak{S}$, $i = 1, \ldots, d$. Since $AB = p^n I$, we have

$$(y_1, \ldots, y_d) = u^m(p^n)^{-1}(x_1, \ldots, x_d)B^{\mathrm{t}}.$$

Let $(z_1, \ldots, z_d) = (p^n)^{-1}(x_1, \ldots, x_d)B^{\mathrm{t}}$. Since $y_i \in \mathfrak{S}$, it is not hard to see that $z_i \in \mathfrak{S}$ for all $i = 1, \ldots, d$. Then we see that $(x_1, \ldots, x_d) = (z_1, \ldots, z_d)A^{\mathrm{t}}$, $x \in \mathfrak{N}''$ and $\bar{x} = 0$.

(2) $\Longleftrightarrow$ (3) Since $E(u)$ is a unit in $\mathcal{O}_{\mathcal{E}}$, any $\mathfrak{M} \in \mathrm{Mod}^{r}_{/\mathfrak{S}}$ is étale if and only if $\mathfrak{M}$ is p'-torsion free. If $\mathfrak{M}$ is killed by some power of p, then this is equivalent to $\mathfrak{M}$ being u-torsion free.

(3) $\Longrightarrow$ (4) We proceed by induction on n. The case $n = 1$ is obvious. For $n > 1$, consider the exact sequence of étale $\mathcal{O}_{\mathcal{E}}$-modules

$$0 \longrightarrow pM \longrightarrow M \xrightarrow{\mathrm{pr}} M/pM \longrightarrow 0,$$

where $M := \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$ and pr is the natural projection. Let $\mathfrak{M}'' = \mathrm{pr}(\mathfrak{M})$ and $\mathfrak{M}' = \mathrm{Ker}(\mathrm{pr})$, then we get an exact sequence of φ-modules over $\mathfrak{S}$

$$0 \longrightarrow \mathfrak{M}' \longrightarrow \mathfrak{M} \xrightarrow{\mathrm{pr}} \mathfrak{M}'' \longrightarrow 0. \tag{2.3.1}$$

By induction, it suffices to show that $\mathfrak{M}'$ and $\mathfrak{M}''$ are étale and belong to $\mathrm{Mod}^r_{/\mathfrak{S}}$. But $\mathfrak{M}'$ and $\mathfrak{M}''$ are obviously u-torsion free and hence p'-torsion free. Then by Lemma 2.3.1, we have $\mathfrak{M}', \mathfrak{M}'' \in \mathrm{Mod}^r_{/\mathfrak{S}}$.

(4) $\Longrightarrow$ (5) Fontaine has proved this result (Theorem 1.6.1 in B, [8]) for the case $e = 1$. In particular, Fontaine's argument for reducing the problem to the case that $\mathfrak{M}$ is killed by p also works here. Therefore, without loss of generality, we may assume that $\mathfrak{M}$ is killed by p. In this case, $\mathfrak{M}$ is a finite free $k[\![u]\!]$-module. Let $\alpha_1, \dots, \alpha_d$ be a basis of $\mathfrak{M}$ and $\varphi(\alpha_1, \dots, \alpha_d) = (\alpha_1, \dots, \alpha_d)X$, where X is a $d \times d$ matrix with coefficients in $k[\![u]\!]$. Since the cokernel of $\varphi^*_{\mathfrak{M}}$ is killed by u^{er}, there exists a matrix Y with coefficients in $k[\![u]\!]$ such that $XY = u^{er}I$, where I is the identity matrix. Let $\mathfrak{N}$ be a finite free $k[\![u]\!]$-module with basis $\beta_1, \dots, \beta_d, \beta'_1, \dots, \beta'_d$ and a φ-structure defined by

$$\varphi_{\mathfrak{N}}(\beta_1, \dots, \beta_d, \beta'_1, \dots, \beta'_d) = (\beta_1, \dots, \beta_d, \beta'_1, \dots, \beta'_d) \begin{pmatrix} I & 0 \\ 0 & u^{er}I \end{pmatrix} \begin{pmatrix} A & I \\ I & uI \end{pmatrix},$$

where $A = (I - uY)^{-1}(\varphi(E) - Y)$ and $E = X - u^{er+1}I$. It is obvious that $(\mathfrak{N}, \varphi_{\mathfrak{N}})$ belongs to $\mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$. We construct a $\mathfrak{S}$-linear map $f : \mathfrak{N} \to \mathfrak{M}$ defined by:

$$f(\beta_1, \dots, \beta_d, \beta'_1, \dots, \beta'_d) = (\alpha_1, \dots, \alpha_d)(E, I). \tag{2.3.2}$$

It is obvious that f is surjective. To check that f is compatible with φ-structures, it suffices to check $f \circ \varphi_{\mathfrak{N}} = \varphi_{\mathfrak{M}} \circ f$ on the basis. This is equivalent to verifying the following matrix equation:

$$X\big(\varphi(E), I\big) = (E, I) \begin{pmatrix} I & 0 \\ 0 & u^{er}I \end{pmatrix} \begin{pmatrix} A & I \\ I & uI \end{pmatrix},$$

which is a straightforward computation. So let $\mathfrak{N}'$ be a finite free $\mathfrak{S}$-module with basis $\hat{\beta}_1, \dots, \hat{\beta}_d, \hat{\beta}'_1, \dots, \hat{\beta}'_d$ and a φ-structure defined by

$$\varphi_{\mathfrak{N}'}(\hat{\beta}_1, \dots, \hat{\beta}_d, \hat{\beta}'_1, \dots, \hat{\beta}'_d) = (\hat{\beta}_1, \dots, \hat{\beta}_d, \hat{\beta}'_1, \dots, \hat{\beta}'_d) \begin{pmatrix} I & 0 \\ 0 & E(u)^r I \end{pmatrix} \begin{pmatrix} \hat{A} & I \\ I & uI \end{pmatrix}$$

with $\hat{A}$ any lift of A. It is easy to check that $\mathfrak{N} = \mathfrak{N}'/p\mathfrak{N}'$ and $\mathfrak{N} \in \mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$. Thus we have a φ-module morphism $g : \mathfrak{N}' \to \mathfrak{M}$ with g surjective. Let $\mathfrak{N}'' = \mathrm{Ker}(g)$. Using the explicit definition (2.3.2) of f, we can easily find a $\mathfrak{S}$-basis for $\mathfrak{N}''$. Thus $\mathfrak{N}''$ is $\mathfrak{S}$-finite free. Finally, using Lemma 2.3.1 for $0 \to \mathfrak{N}'' \to \mathfrak{N}' \xrightarrow{g} \mathfrak{M} \to 0$, we see that $\mathfrak{N}'' \in \mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$.

(5) $\Longrightarrow$ (1) Trivial. □

COROLLARY 2.3.3. – *Let* $f : \mathfrak{M} \to \mathfrak{M}'$ *be a morphism in* $\mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$. *Then* $\mathrm{Ker}(f)$ *belongs to* $\mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$.

Proof. – Lemma 2.3.1 shows that $\mathrm{Ker}(f) \in \mathrm{Mod}^r_{/\mathfrak{S}}$ and $\mathrm{Ker}(f)$ is obviously u-torsion free. □

In general, $\mathrm{Cok}(f)$ is not necessarily in $\mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$. See Example 2.3.5.

By the above lemma, any object $\mathfrak{M} \in \mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$ is étale. Thus Corollary 2.2.2 implies that $T_{\mathfrak{S}}(\mathfrak{M}) = T(\mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}) = \mathrm{Hom}_{\mathcal{O}_{\mathcal{E}},\varphi}(\mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}, \mathcal{O}^{\mathrm{ur}})$. Therefore, the functor $T_{\mathfrak{S}}$ defined in (2.2.5) is well defined on $\mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$. In summary, we have

COROLLARY 2.3.4. – *The contravariant functor $T_{\mathfrak{S}}$ from $\mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$ to $\mathrm{Rep}^{\mathrm{tor}}_{\mathbb{Z}_p}(G_\infty)$ is well defined and exact.*

If $r=1$ and $p>2$, [4] and §2.3 in [14] proved that there exists an anti-equivalence $\mathcal{G}$ between $\mathrm{Mod}^{1,\mathrm{tor}}_{\mathfrak{S}}$ (resp. $\mathrm{Mod}^{1,\mathrm{fr}}_{\mathfrak{S}}$) and the category of finite flat group schemes over $\mathcal{O}_K$ (resp. p-divisible groups over $\mathcal{O}_K$). Furthermore, for any $\mathfrak{M}\in\mathrm{Mod}^{1,\mathrm{tor}}_{\mathfrak{S}}$ (resp. $\mathrm{Mod}^{1,\mathrm{fr}}_{\mathfrak{S}}$), there exists a natural isomorphism of $\mathbb{Z}_p[G_\infty]$-modules

$$\mathcal{G}(\mathfrak{M})(\bar{K})|_{G_\infty}\simeq T_{\mathfrak{S}}(\mathfrak{M}).$$

In general, $T_{\mathfrak{S}}$ is not fully faithful if $er\geqslant p-1$.

Example 2.3.5. – Let $\mathfrak{S}^\star:=\mathfrak{S}\cdot\alpha$ be the rank-1 free $\mathfrak{S}$-module equipped with $\varphi(\alpha)={c_0}^{-1}E(u)\cdot\alpha$ where pc_0 is the constant coefficient of $E(u)$. By Example 2.2.3 in [4], if $p>2$, $\mathcal{G}(\mathfrak{S}^\star)=\mu_{p^\infty}$. In particular, $T_{\mathfrak{S}}(\mathfrak{S}^\star)=\mu_{p^\infty}(\bar{K})|_{G_\infty}=\mathbb{Z}_p(1)$. If $p=2$, Theorem (2.2.7) in [14] shows that $\mathcal{G}(\mathfrak{S}^\star)$ is isogenous to μ_{2^∞}. Thus $T_{\mathfrak{S}}(\mathfrak{S}^\star)$ is a G_∞-stable $\mathbb{Z}_2$-lattice in $\mathbb{Q}_2(1)$. So we still have $T_{\mathfrak{S}}(\mathfrak{S}^\star)\simeq\mathbb{Z}_2(1)$. Suppose $e=p-1$. Consider the map $\mathfrak{f}:\mathfrak{S}_1^\star\to\mathfrak{S}_1$ given by $\alpha\mapsto c_0^{-1}u^e$. An easy calculation shows that $\mathfrak{f}$ is a well-defined morphism of φ-modules and $\mathfrak{f}\otimes_{\mathfrak{S}}\mathcal{O}_{\mathcal{E}}$ is an isomorphism. Then $T_{\mathfrak{S}}(\mathfrak{f})$ is an isomorphism but $\mathfrak{f}$ is not. Also, $\mathrm{Cok}(\mathfrak{f})$ is not an object in $\mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$.

The following lemma is an analogy of "scheme-theoretic closure" in the theory of finite flat group schemes over $\mathcal{O}_K$.

LEMMA 2.3.6 (Scheme-theoretic closure). – *Let $f:\mathfrak{M}\to L$ be a morphism of φ-modules over $\mathfrak{S}$. Suppose that $\mathfrak{M}$ and L are p'-torsion free and $\mathfrak{M}\in\mathrm{Mod}^r_{/\mathfrak{S}}$. Put $\mathfrak{M}'=\mathrm{Ker}(f)$ and $\mathfrak{M}''=f(\mathfrak{M})$. Then $\mathfrak{M}'$ and $\mathfrak{M}''$ are étale and belong to $\mathrm{Mod}^r_{/\mathfrak{S}}$. In particular, if $\mathfrak{M}\in\mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$, then $\mathfrak{M}'$ and $\mathfrak{M}''\in\mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$.*

Proof. – By the construction, it is obvious that $\mathfrak{M}'$ and $\mathfrak{M}''$ are p'-torsion free. By Lemma 2.3.1, $\mathfrak{M}'$ and $\mathfrak{M}''$ are étale and belong to $\mathrm{Mod}^r_{/\mathfrak{S}}$. If $\mathfrak{M}\in\mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$, then $\mathfrak{M}'$ and $\mathfrak{M}'$ are u-torsion free. By Proposition 2.3.2 (2), we see that $\mathfrak{M}'$ and $\mathfrak{M}''$ belong to $\mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$. □

LEMMA 2.3.7. – *Let $\mathfrak{M}\in\mathrm{Mod}^r_{/\mathfrak{S}}$ be torsion free, $M=\mathfrak{M}\otimes_{\mathfrak{S}}\mathcal{O}_{\mathcal{E}}$. Then there exists a finite free $\mathfrak{S}$-module $\mathfrak{M}'\in\mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$ such that $\mathfrak{M}\subset\mathfrak{M}'\subset M$.*

Proof. – Let $\mathfrak{M}'=M\cap\mathfrak{M}[1/p]$. By Proposition B 1.2.4 of [8], we have $\mathfrak{M}\subset\mathfrak{M}'\subset M$ with $\mathfrak{M}'$ a finite free $\mathfrak{S}$-module. It is obvious that $\mathfrak{M}'$ is φ-stable, so it remains to check that $\mathrm{Cok}(\varphi^*_{\mathfrak{M}'})$ is killed by $E(u)^r$. Note that there exists an integer s such that $p^s\mathfrak{M}'\subset\mathfrak{M}$. Since $E(u)^r$ kills $\mathrm{Cok}(\varphi^*_{\mathfrak{M}})$, we have that $p^sE(u)^r$ kills $\mathrm{Cok}(\varphi^*_{\mathfrak{M}'})$. Let $\alpha_1,\dots,\alpha_d$ be a basis of $\mathfrak{M}'$ and $\varphi_{\mathfrak{M}'}(\alpha_1,\dots,\alpha_d)=(\alpha_1,\dots,\alpha_d)A$ where A is a $d\times d$ matrix with coefficients in $\mathfrak{S}$. Since $\mathfrak{M}'$ is étale, A^{-1} exists with coefficients in $\mathcal{O}_{\mathcal{E}}$. It suffices to prove that $E(u)^rA^{-1}$ has coefficients in $\mathfrak{S}$, but this follows easily from the fact that $p^sE(u)^rA^{-1}$ has coefficients in $\mathfrak{S}$. □

COROLLARY 2.3.8. – *Let $\mathfrak{f}:\mathfrak{M}\to\mathfrak{N}$ be a surjective morphism in $\mathrm{Mod}^r_{/\mathfrak{S}}$ with $\mathfrak{M}\in\mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$ a finite free $\mathfrak{S}$-module and $\mathfrak{N}\in\mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$ killed by some power of p. Then $\mathfrak{L}:=\mathrm{Ker}(\mathfrak{f})\in\mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$ is $\mathfrak{S}$-finite free.*

Proof. – By Lemma 2.3.1, $\mathfrak{L}\in\mathrm{Mod}^r_{/\mathfrak{S}}$. $\mathfrak{L}$ is obviously torsion free and of $\mathfrak{S}$-finite type. Let $L:=\mathfrak{L}\otimes_{\mathfrak{S}}\mathcal{O}_{\mathcal{E}}$ and $M:=\mathfrak{M}\otimes_{\mathfrak{S}}\mathcal{O}_{\mathcal{E}}$. Since $\mathfrak{N}$ is u-torsion free, we have $\mathfrak{M}\cap L=\mathfrak{L}$. By the proof of Lemma 2.3.7, we see that $L\cap\mathfrak{L}[\frac{1}{p}]$ is $\mathfrak{S}$-finite free. But $\mathfrak{L}[\frac{1}{p}]=\mathfrak{M}[\frac{1}{p}]$, so $L\cap\mathfrak{L}[\frac{1}{p}]=L\cap(M\cap\mathfrak{M}[\frac{1}{p}])=L\cap\mathfrak{M}=\mathfrak{L}$. Thus $\mathfrak{L}$ is $\mathfrak{S}$-finite free. □

COROLLARY 2.3.9. – *Let* $\mathfrak{M} \in \mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$ (*resp.* $\mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$), *let* $\mathfrak{N}$ *be a* φ*-stable* $\mathfrak{S}$*-submodule of* $\mathfrak{M}$ *and* $N := \mathfrak{N} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$. *Then there exists an* $\mathfrak{N}' \in \mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$ (*resp.* $\mathfrak{N}' \in \mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$) *such that* $\mathfrak{N} \subset \mathfrak{N}' \subset N \cap \mathfrak{M}$.

Proof. – Let $M = \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$ and $L = M/N$. By Lemma 2.3.6, in this case, there exists $\mathfrak{N}'' \in \mathrm{Mod}^{r}_{/\mathfrak{S}}$ such that $\mathfrak{N} \subset \mathfrak{N}'' \subset \mathfrak{M} \cap N$ and $\mathfrak{N}''$ is étale. If $\mathfrak{M}$ is in $\mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$, then put $\mathfrak{N}' = \mathfrak{N}''$, so $\mathfrak{N}'$ belongs to $\mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$ because $\mathfrak{N}'$ is obviously u-torsion free. If $\mathfrak{M}$ is a finite free $\mathfrak{S}$-module, then $\mathfrak{N}'' \in \mathrm{Mod}^{r}_{/\mathfrak{S}}$ is torsion free. Therefore, by Lemma 2.3.7, there exists $\mathfrak{N}' \in \mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$ such that $\mathfrak{N} \subset \mathfrak{N}'' \subset \mathfrak{N}' \subset \mathfrak{M} \cap N$. □

2.4. Main results of the first part

Now we can state the main theorems to be proved in the first part of this paper. The first theorem is an analog of Raynaud's theorem (Proposition 2.3.1 in [22]) which states that a Barsotti–Tate group H over K can be extended to a Barsotti–Tate group over $\mathcal{O}_K$ if and only if, for each n, $H[p^n]$ can be extended to a finite flat group scheme $\mathcal{H}_n$ over $\mathcal{O}_K$.

THEOREM 2.4.1. – *Let* T *be a finite free* $\mathbb{Z}_p$*-representation of* G_∞. *If for each* n, *there exists an* $\mathfrak{M}_{(n)} \in \mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$ *such that* $T_{\mathfrak{S}}(\mathfrak{M}_{(n)}) \simeq T/p^nT$, *then there exists a finite free* $\mathfrak{S}$*-module* $\mathfrak{M} \in \mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$ *such that* $T_{\mathfrak{S}}(\mathfrak{M}) \simeq T$.

Though the functor $T_{\mathfrak{S}}$ on $\mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$ is not a fully faithful functor if $er \geqslant p-1$ as explained in Example 2.3.5, we will prove that the functor $T_{\mathfrak{S}}$ enjoys "weak" full faithfulness.

THEOREM 2.4.2. – *Let* $\mathfrak{M}, \mathfrak{M}' \in \mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$, *let* $f : T_{\mathfrak{S}}(\mathfrak{M}') \to T_{\mathfrak{S}}(\mathfrak{M})$ *be a morphism of finite* $\mathbb{Z}_p[G_\infty]$*-modules. Then there exists a morphism* $\mathfrak{f} : \mathfrak{M} \to \mathfrak{M}'$ *such that* $T_{\mathfrak{S}}(\mathfrak{f}) = p^{\mathfrak{c}} f$, *where* $\mathfrak{c}$ *is a constant depending only on the absolute ramification index* $e = e(K/K_0)$ *and the height* r. *In particular,* $\mathfrak{c} = 0$ *if* $er < p-1$.

Remark 2.4.3. – The constant $\mathfrak{c}$ has an explicit (but complicated) formula. We do not optimize it, so there should still be room to improve. We have proved a similar, though weaker, result in [19] for truncated Barsotti–Tate groups (see also [3]). The constant obtained here is independent of the height of the truncated Barsotti–Tate group, though we do use many of the techniques found in [19].

To prove these theorems, we need to construct the Cartier dual on $\mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$ and a theorem (Theorem 3.2.2) to compare $\mathfrak{M}$ with $T_{\mathfrak{S}}(\mathfrak{M})$. These preparations will be discussed in §3.

2.5. Construction of $\mathfrak{S}^{\mathrm{f}(r)}$

For a fixed height r, $\mathfrak{S}^{\mathrm{ur}}$ is too big to work with. In this subsection, we cut out a $\mathfrak{S}$-submodule $\mathfrak{S}^{\mathrm{f}(r)}$ inside $\mathfrak{S}^{\mathrm{ur}}$ which is big enough for representations arising from $\mathrm{Mod}^{r}_{/\mathfrak{S}}$. Let Λ be a p'-torsion free φ-module over $\mathfrak{S}$. We denote by $F^{\mathrm{fr}}_{\mathfrak{S}}(\Lambda)$ the set of $\mathfrak{S}$-submodules $\mathfrak{M}$ of Λ such that $\mathfrak{M} \in \mathrm{Mod}^{r}_{/\mathfrak{S}}$. Since Λ is p'-torsion free, $\mathfrak{M}$ is étale, so $F^{\mathrm{fr}}_{\mathfrak{S}}(\Lambda) \subset F_{\mathfrak{S}}(\Lambda)$. (Recall that $F_{\mathfrak{S}}(\Lambda)$ is the set of $\mathfrak{S}$-submodules $\mathfrak{M}$ such that $\mathfrak{M}$ is of $\mathfrak{S}$-finite type and stable under φ.) Define

$$\mathfrak{S}^{\mathrm{f}(r)}_{(n)} = \bigcup_{\mathfrak{M} \in F^{\mathrm{fr}}_{\mathfrak{S}}(\mathfrak{S}^{\mathrm{ur}}_n)} \mathfrak{M} \quad \text{for each fixed } n \geqslant 1,$$

and

$$\mathfrak{S}^{\mathrm{f}(r)} = \bigcup_{\mathfrak{M}\in F^{\mathrm{fr}}_{\mathfrak{S}}(\mathfrak{S}^{\mathrm{ur}})} \mathfrak{M}.$$

Obviously, $\mathfrak{S}^{\mathrm{f}(r)}_{(n)}$ (resp. $\mathfrak{S}^{\mathrm{f}(r)}$) is a subset of $\mathfrak{S}^{\mathrm{ur}}_n$ (resp. $\mathfrak{S}^{\mathrm{ur}}$).

PROPOSITION 2.5.1. – *For each $n \geqslant 1$,*
(1) $\mathfrak{S}^{\mathrm{f}(r)}_{(n)}$ *is a G_∞-stable and φ-stable $\mathfrak{S}$-submodule of* $\mathfrak{S}^{\mathrm{ur}}_n$,
(2) $\mathfrak{S}^{\mathrm{f}(r)}$ *is a G_∞-stable and φ-stable $\mathfrak{S}$-submodule of* $\mathfrak{S}^{\mathrm{ur}}$,
(3) $\mathfrak{S}^{\mathrm{f}(r)}_{(n)} = \mathfrak{S}^{\mathrm{f}(r)}/p^n\mathfrak{S}^{\mathrm{f}(r)}$, *i.e.,* $\mathfrak{S}^{\mathrm{f}(r)}_n = \mathfrak{S}^{\mathrm{f}(r)}_{(n)}$.

Proof. – For each fixed $n \geqslant 1$, let $\mathfrak{M}'$ and $\mathfrak{M}'' \in F^{\mathrm{fr}}_{\mathfrak{S}}(\mathfrak{S}^{\mathrm{ur}}_n)$. To prove (1), it suffices to check that $\mathfrak{M} := \mathfrak{M}' + \mathfrak{M}'' \in F^{\mathrm{fr}}_{\mathfrak{S}}(\mathfrak{S}^{\mathrm{ur}}_n)$. It is obvious that $\mathfrak{S}^{\mathrm{f}(r)}_{(n)}$ is G_∞-stable and φ-stable. Since $\mathfrak{S}^{\mathrm{ur}}_n$ is p'-torsion free, $\mathfrak{M}$ is p'-torsion free. It therefore suffices to check that the cokernel of $\varphi^*_{\mathfrak{M}}: \mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \to \mathfrak{M}$ is killed by $E(u)^r$. This follows from the fact that the cokernels of $\varphi^*_{\mathfrak{M}'}$ and $\varphi^*_{\mathfrak{M}''}$ are killed by $E(u)^r$. The above argument also works for proving (2). For (3), we need to show that the natural map $\iota: \mathfrak{S}^{\mathrm{f}(r)}_n \to \mathfrak{S}^{\mathrm{f}(r)}_{(n)}$ induced by $\mathrm{pr}: \mathfrak{S}^{\mathrm{ur}} \to \mathfrak{S}^{\mathrm{ur}}_n$ is an isomorphism. We first prove the surjectivity by claiming that for any $\mathfrak{M} \in F^{\mathrm{fr}}_{\mathfrak{S}}(\mathfrak{S}^{\mathrm{ur}}_n)$ there exists an $\mathfrak{N} \in F^{\mathrm{fr}}_{\mathfrak{S}}(\mathfrak{S}^{\mathrm{ur}})$ such that $\mathrm{pr}(\mathfrak{N}) = \mathfrak{M}$. In fact, by Proposition 2.3.2, (3), there exists a finite free φ-module $\mathfrak{N}' \in \mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$ with a surjection $f: \mathfrak{N}' \twoheadrightarrow \mathfrak{M}$. Recall that the functor $T_{\mathfrak{S}}: \mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}} \to \mathrm{Rep}_{\mathbb{Z}_p}(G_\infty)$ is exact (Corollary 2.3.4). Thus $T_{\mathfrak{S}}(f): T_{\mathfrak{S}}(\mathfrak{N}') \to T_{\mathfrak{S}}(\mathfrak{M})$ is surjective, so by Lemma 2.2.2, there exists a morphism of φ-modules $h: \mathfrak{N}' \to \mathfrak{S}^{\mathrm{ur}}$ which lifts the identity embedding $\mathfrak{M} \hookrightarrow \mathfrak{S}^{\mathrm{ur}}_n$. Therefore $\mathfrak{N} = h(\mathfrak{N}') \in F^{\mathrm{fr}}_{\mathfrak{S}}(\mathfrak{S}^{\mathrm{ur}})$ and $\mathrm{pr}(\mathfrak{N}) = \mathfrak{M}$, as required. For the injectivity, it suffices to prove that for any $\mathfrak{M} \in F^{\mathrm{fr}}_{\mathfrak{S}}(\mathfrak{S}^{\mathrm{ur}})$ and $x \in \mathfrak{S}^{\mathrm{ur}}$, if $px \in \mathfrak{M}$, then there exists $\mathfrak{L} \in F^{\mathrm{fr}}_{\mathfrak{S}}(\mathfrak{S}^{\mathrm{ur}})$ such that $x \in \mathfrak{L}$. Let $\mathfrak{N}$ be the $\mathfrak{S}$-submodule in $\mathfrak{S}^{\mathrm{ur}}$ generated by $\{\varphi^m(x)\}_{m\geqslant 0}$ and $\tilde{\mathfrak{N}}$ the $\mathfrak{S}$-submodule of $\mathfrak{M}$ generated by $\varphi^m(px)$. Let $\alpha: \tilde{\mathfrak{N}} \to \mathfrak{N}$ be the morphism defined by

$$\alpha: \sum s_i\varphi^{m_i}(px) \mapsto \sum s_i\varphi^{m_i}(x).$$

Since $\mathfrak{S}^{\mathrm{ur}}$ is torsion free, α is an isomorphism and α extends to an isomorphism $\tilde{\mathfrak{N}} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} \xrightarrow{\sim} \mathfrak{N} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$ inside $\widehat{\mathcal{O}^{\mathrm{ur}}}$. By Corollary 2.3.9, we have $\tilde{\mathfrak{N}}' \in \mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$ such that $\tilde{\mathfrak{N}} \subset \tilde{\mathfrak{N}}' \subset \tilde{\mathfrak{N}} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$. Let $\mathfrak{L} = \alpha(\tilde{\mathfrak{N}}')$. We see that $x \in \mathfrak{N} \subset \mathfrak{L} \subset \widehat{\mathcal{O}^{\mathrm{ur}}}$ with $\mathfrak{L} \in \mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$, so by Proposition 2.2.1 (3), $x \in \mathfrak{N} \subset \mathfrak{L} \subset \mathfrak{S}^{\mathrm{ur}}$. $\square$

COROLLARY 2.5.2. – *For each $n \geqslant 1$, $\mathfrak{S}^{\mathrm{f}(r)}_n$ is flat over $\mathfrak{S}_n$.*

Proof. – By Proposition 2.5.1 (3), it suffices to prove that, for any $\mathfrak{M} \in F^{\mathrm{fr}}_{\mathfrak{S}}(\mathfrak{S}^{\mathrm{ur}})$, there exists a finite free $\mathfrak{S}$-modules $\mathfrak{M}' \in F^{\mathrm{fr}}_{\mathfrak{S}}(\mathfrak{S}^{\mathrm{ur}})$ such that $\mathfrak{M} \subset \mathfrak{M}'$. By Lemma 2.3.7, there exists such a module $\mathfrak{M}' \subset \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} \subset \widehat{\mathcal{O}^{\mathrm{ur}}}$. By Proposition 2.2.1 (3), we see that $\mathfrak{M}' \subset \mathfrak{S}^{\mathrm{ur}}$. $\square$

COROLLARY 2.5.3. – *For any $\mathfrak{M} \in \mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$,*

$$\mathrm{Hom}_{\mathfrak{S},\varphi}\big(\mathfrak{M}, \mathfrak{S}^{\mathrm{f}(r)} \otimes_{\mathbb{Z}} (\mathbb{Q}_p/\mathbb{Z}_p)\big) \simeq \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathcal{E}/\mathcal{O}^{\mathrm{ur}}_{\mathcal{E}}) = T_{\mathfrak{S}}(\mathfrak{M}).$$

3. A theorem to compare $\mathfrak{M}$ with $T_{\mathfrak{S}}(\mathfrak{M})$

In this section, we prove a "comparison" theorem (Theorem 3.2.2) to compare $\mathfrak{M}$ with $T_{\mathfrak{S}}(\mathfrak{M})$. This theorem will be the technical hearts in many of our proofs. In the following two sections, we will focus on torsion objects $\mathrm{Mod}_{/\mathfrak{S}}^{r,\mathrm{tor}}$. For $\mathfrak{M} \in \mathrm{Mod}_{/\mathfrak{S}}^{r,\mathrm{tor}}$, n will always denote an integer such that p^n kills $\mathfrak{M}$.

3.1. Cartier dual

We need to generalize to $\mathrm{Mod}_{/\mathfrak{S}}^{r,\mathrm{tor}}$ the concept of *Cartier dual* on finite flat group schemes over $\mathcal{O}_K$. Example 2.3.5 shows that if $r=1$, then $\mathfrak{S}^{\star}$ is the correct Cartier dual of $\mathfrak{S}$. Motivated by this example, we have:

Convention 3.1.1. – Define a φ-semi-linear morphism $\varphi^{\vee}: \mathfrak{S} \to \mathfrak{S}$ by $1 \mapsto c_0^{-r} E(u)^r$. We denote by $\mathfrak{S}^{\vee}$ the ring $\mathfrak{S}$ with φ-semi-linear morphism $\varphi^{\vee}$. The same notations apply for $\mathfrak{S}_n$ and $\mathfrak{S}_n^{\mathrm{f}(r)}$, etc. By Example 2.3.5, we have $T_{\mathfrak{S}}(\mathfrak{S}_n^{\vee}) \simeq \mathbb{Z}_p/p^n\mathbb{Z}_p(r)$.

Obviously, such "Cartier dual" (if it exists) must be compatible with the associated Galois representations, so we first analyze the dual on $\mathbf{\Phi M}_{\mathcal{O}_{\mathcal{E}}}^{\mathrm{tor}}$. Let $M \in \mathbf{\Phi M}_{\mathcal{O}_{\mathcal{E}}}^{\mathrm{tor}}$ and $M^{\vee} = \mathrm{Hom}_{\mathcal{O}_{\mathcal{E}}}(M, \mathcal{E}/\mathcal{O}_{\mathcal{E}})$. As an $\mathcal{O}_{\mathcal{E}}$-module, we have $M \simeq \bigoplus_{i=1}^{d} \mathcal{O}_{\mathcal{E},n_i}$, so there exists a canonical perfect pairing of $\mathcal{O}_{\mathcal{E}}$-modules

$$\langle\, ,\rangle : M \times M^{\vee} \to \mathcal{E}/\mathcal{O}_{\mathcal{E}}. \tag{3.1.1}$$

We equip $\mathcal{E}/\mathcal{O}_{\mathcal{E}}$ with a φ-structure by $1 \mapsto c_0^{-r}E(u)^r$. We will construct a φ-structure on $M^{\vee}$ such that (3.1.1) is also compatible with φ-structures. A $\mathfrak{S}$-linear map $f: \mathfrak{M} \to \mathfrak{N}$ is also called *φ-equivariant* if f is a morphism of φ-modules.

LEMMA 3.1.2. – *There exists a unique φ-semi-linear morphism $\varphi_{M^{\vee}}: M^{\vee} \to M^{\vee}$ such that*
(1) $(M^{\vee}, \varphi^{\vee}) \in \mathbf{\Phi M}_{\mathcal{O}_{\mathcal{E}}}^{\mathrm{tor}}$.
(2) *For any $x \in M$, $y \in M^{\vee}$, $\langle \varphi_M(x), \varphi_{M^{\vee}}(y)\rangle = \varphi(\langle x, y\rangle)$.*
(3) $T(M^{\vee}) \simeq T^{\vee}(M)(r)$ *as $\mathbb{Z}_p[G_{\infty}]$-modules.*

Proof. – We first construct a $\varphi_{M^{\vee}}$ satisfying (2). Let $M \simeq \bigoplus_{i=1}^{d} \mathcal{O}_{\mathcal{E},n_i}\alpha_i$ and let $\beta_1, \dots, \beta_d$ be the dual basis of $M^{\vee}$. Write $\varphi_M(\alpha_1, \dots, \alpha_d) = (\alpha_1, \dots, \alpha_d)A$, where A is a $d \times d$ matrix with coefficients in $\mathcal{O}_{\mathcal{E}}$. Define

$$\varphi_{M^{\vee}}(\beta_1, \dots, \beta_d) = (\beta_1, \dots, \beta_d)\big(c_0^{-r}E(u)^r\big)(A^{-1})^t.$$

Note that A is invertible in $\mathcal{O}_{\mathcal{E}}$ because M is étale. It is easy to check that $(M^{\vee}, \varphi_{M^{\vee}})$ satisfies (1), (2) and uniqueness, so it remains to check (3). We can extend the φ-equivariant perfect pairing $\langle\, ,\, \rangle$ to

$$\langle\, ,\rangle : (M \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{O}^{\mathrm{ur}}) \times (M^{\vee} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{O}^{\mathrm{ur}}) \to \mathcal{O}_n^{\mathrm{ur},\vee}, \tag{3.1.2}$$

where $n = \mathrm{Max}(n_1, \dots, n_d)$. Since the above pairing is φ-equivariant, we have a pairing

$$(M \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{O}^{\mathrm{ur}})^{\varphi=1} \times (M^{\vee} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{O}^{\mathrm{ur}})^{\varphi=1} \to (\mathcal{O}_n^{\mathrm{ur},\vee})^{\varphi=1}.$$

Thus, we have a pairing

$$T^{\vee}(M) \times T^{\vee}(M^{\vee}) \to \mathbb{Z}/p^n\mathbb{Z}(-r) \tag{3.1.3}$$

compatible with G_∞-action. It suffices to check the above pairing is perfect. By (2.2.2), we see that a $\mathbb{Z}_p$-basis of $T^\vee(M)$ is also a $\mathcal{O}^{\mathrm{ur}}$-basis of $M\otimes_{\mathcal{O}_\mathcal{E}}\mathcal{O}^{\mathrm{ur}}$. Then the fact that (3.1.3) is perfect follows from the fact that (3.1.2) is perfect. $\square$

Since the functor $T^\vee$ is an equivalence between $\mathbf{\Phi M}^{\mathrm{tor}}_{\mathcal{O}_\mathcal{E}}$ and $\mathrm{Rep}^{\mathrm{tor}}_{\mathbb{Z}_p}(G_\infty)$, we have

COROLLARY 3.1.3. – *The functor $M\to M^\vee$ is an anti-equivalence on $\mathbf{\Phi M}^{\mathrm{tor}}_{\mathcal{O}_\mathcal{E}}$ and $(M^\vee)^\vee=M$.*

Now let us extend Lemma 3.1.2 to $\mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$. Let $\mathfrak{M}\in\mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$ and $M=\mathfrak{M}[\frac{1}{u}]:=\mathfrak{M}\otimes_\mathfrak{S}\mathcal{O}_\mathcal{E}$. Define $\mathfrak{M}^\vee=\mathrm{Hom}_\mathfrak{S}(\mathfrak{M},\mathfrak{S}[1/p]/\mathfrak{S})$. Before we equip $\mathfrak{M}^\vee$ with a suitable φ-structure, a lemma is needed to compare the underlying space of $\mathfrak{M}^\vee$ with that of $M^\vee$.

LEMMA 3.1.4. – *Let $\mathfrak{M}\in\mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$ and $\Lambda=\mathfrak{S}[1/p]/\mathfrak{S}$. Then $\mathrm{Ext}^1_\mathfrak{S}(\mathfrak{M},\Lambda)=0$, where Ext is taken in the category of $\mathfrak{S}$-modules.*

Proof. – By Proposition 2.3.2 (4) and taking *dévissage*, we can reduce the problem to the case that $\mathfrak{M}$ is killed by p, where $\mathfrak{M}$ is a finite free $k[\![u]\!]$-module. So it suffices to show that $\mathrm{Ext}^1_\mathfrak{S}(k[\![u]\!],\Lambda)=0$. The short exact sequence

$$0\to\mathfrak{S}\xrightarrow{p}\mathfrak{S}\to k[\![u]\!]\to 0$$

yields a long exact sequence

$$0\to k[\![u]\!]\to\Lambda\xrightarrow{p}\Lambda\to\mathrm{Ext}^1_\mathfrak{S}(k[\![u]\!],\Lambda)\to 0,$$

so $\mathrm{Ext}^1_\mathfrak{S}(\mathfrak{M},\Lambda)=0$. $\square$

COROLLARY 3.1.5. – *Let $0\to\mathfrak{M}\to\mathfrak{N}\to\mathfrak{L}\to 0$ be an exact sequence in $\mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$. Then $0\to\mathfrak{L}^\vee\to\mathfrak{N}^\vee\to\mathfrak{M}^\vee\to 0$ is exact as $\mathfrak{S}$-modules.*

COROLLARY 3.1.6. – *With notations as above, $\mathfrak{M}^\vee$ is u-torsion free and $(\mathfrak{M}^\vee)[\frac{1}{u}]=(\mathfrak{M}[\frac{1}{u}])^\vee$.*

Proof. – The u-torsion freeness of $\mathfrak{M}^\vee$ is obvious by definition and u-torsion freeness of $\mathfrak{M}$ (Proposition 2.3.2 (2)). To see the natural map $\mathfrak{M}^\vee[1/u]\to(\mathfrak{M}[1/u])^\vee$ is bijective, we reduce the proof by Lemma 3.1.4 and *dévissage* to the case that $\mathfrak{M}$ is killed by p, where $\mathfrak{M}$ is a finite free $k[\![u]\!]$-module. Then the statement that $(\mathfrak{M}^\vee)[\frac{1}{u}]=(\mathfrak{M}[\frac{1}{u}])^\vee$ is obvious. $\square$

PROPOSITION 3.1.7. – *Keeping the above notations, there exists a unique φ-semi-linear endomorphism $\varphi_{\mathfrak{M}^\vee}$ on $\mathfrak{M}^\vee$ such that*
(1) $(\mathfrak{M}^\vee,\varphi_{\mathfrak{M}^\vee})\in\mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$,
(2) *the following diagram commutes*

(3.1.4)
$$\begin{tikzcd}\mathfrak{M}^\vee \arrow[r, "\varphi_{\mathfrak{M}^\vee}"] \arrow[d, hook] & \mathfrak{M}^\vee \arrow[d, hook] \\ M^\vee \arrow[r, "\varphi_{M^\vee}"] & M^\vee\end{tikzcd}$$

In particular, $\varphi_{\mathfrak{M}^\vee}\otimes_\mathfrak{S}\mathcal{O}_\mathcal{E}=\varphi_{M^\vee}$.
The assignment $\mathfrak{M}\mapsto\mathfrak{M}^\vee$ is an anti-equivalence on $\mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$ and $(\mathfrak{M}^\vee)^\vee=\mathfrak{M}$ for all $\mathfrak{M}\in\mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$.

Proof. – Of course, (2) implies that we need to define that $\varphi_{\mathfrak{M}^\vee} := \varphi_{M^\vee}|_{\mathfrak{M}^\vee}$. We claim that $\varphi_{\mathfrak{M}^\vee}$ is well defined in this way; that is, $\varphi_{M^\vee}(\mathfrak{M}^\vee) \subset \mathfrak{M}^\vee$ and the cokernel of $\varphi^*_{\mathfrak{M}^\vee} : \mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}^\vee \to \mathfrak{M}^\vee$ is killed by $E(u)^r$. To prove the claim, we first consider the case that $\mathfrak{M}$ is a finite free $\mathfrak{S}_n$-module. Let $(\alpha_1, \dots, \alpha_d)$ be a basis of $\mathfrak{M}$ and $\beta_1, \dots, \beta_d$ the dual basis of $\mathfrak{M}^\vee$. Write $\varphi_{\mathfrak{M}}(\alpha_1, \dots, \alpha_d) = (\alpha_1, \dots, \alpha_d)A$, where A is a $d \times d$ matrix with coefficients in $\mathfrak{S}_n$. Recall from the proof of Lemma 3.1.2 that we have defined

$$\varphi_{M^\vee}(\beta_1, \dots, \beta_d) = (\beta_1, \dots, \beta_d)\big(c_0^{-r} E(u)^r\big)(A^{-1})^t.$$

Since $\mathfrak{M} \in \mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$, we see that $E(u)^r (A^{-1})^t$ is a matrix with coefficients in $\mathfrak{S}_n$. Thus $\varphi_{M^\vee}(\mathfrak{M}^\vee) \subset \mathfrak{M}^\vee$ and $E(u)^r$ kills the cokernel of $\varphi^*_{\mathfrak{M}^\vee}$, i.e., $\mathfrak{M}^\vee \in \mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$. For a general $\mathfrak{M} \in \mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$, there exists by Proposition 2.3.2 (5) a right exact sequence

$$\mathfrak{L} \xrightarrow{f} \mathfrak{N} \to \mathfrak{M} \to 0 \tag{3.1.5}$$

in $\mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$, where $\mathfrak{N}, \mathfrak{L} \in \mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$ are finite free $\mathfrak{S}_n$-modules. By taking duals, we have a left exact sequence

$$0 \to \mathfrak{M}^\vee \to \mathfrak{N}^\vee \xrightarrow{f^\vee} \mathfrak{L}^\vee. \tag{3.1.6}$$

Since $\mathcal{O}_{\mathcal{E}}$ is flat over $\mathfrak{S}$, by tensoring $\mathcal{O}_{\mathcal{E}}$ and using Lemma 3.1.6, we have the following commutative diagram of φ-modules:

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{M}^\vee \arrow[r] \arrow[d, hook] & \mathfrak{N}^\vee \arrow[r, "f^\vee"] \arrow[d, hook] & \mathfrak{L}^\vee \arrow[d, hook] \\ \arrow[r] & M^\vee \arrow[r] & N^\vee \arrow[r, "f^\vee[\frac{1}{u}]"] & L^\vee, \end{tikzcd}$$

where $M = \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$, $N = \mathfrak{N} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$ and $L = \mathfrak{L} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$. Note that $f^\vee$ is a morphism in $\mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$. By Lemma 2.3.6, we have that $\varphi_{\mathfrak{M}^\vee} = \varphi_{M^\vee}|_{\mathfrak{M}^\vee}$ is well defined and $(\mathfrak{M}^\vee, \varphi_{\mathfrak{M}^\vee}) = \mathrm{Ker}(f^\vee) \in \mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$. This completes the proof of (1) and (2). Since $M \to M^\vee$ is an anti-equivalence on $\mathbf{\Phi M}^{\mathrm{tor}}_{\mathcal{O}_{\mathcal{E}}}$ by the characterizing properties of $\varphi_{\mathfrak{M}^\vee}$, we see that the assignment $(-)^\vee : \mathfrak{M} \to \mathfrak{M}^\vee$ is a functor from $\mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$ to itself which is exact by Corollary 3.1.5. It remains to check that the natural map $\mathfrak{M} \to (\mathfrak{M}^\vee)^\vee$ is an isomorphism. If $\mathfrak{M}$ is a finite free $\mathfrak{S}_n$-module, this is obvious. For a general $\mathfrak{M} \in \mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$, we use Proposition 2.3.2 (4) and *dévissage* to reduce the proof to the case that $\mathfrak{M}$ is killed by p, where $\mathfrak{M}$ is finite $k[\![u]\!]$-free, in which case that $\mathfrak{M} = (\mathfrak{M}^\vee)^\vee$ is obvious. □

3.2. Comparing $\mathfrak{M}$ with $T_{\mathfrak{S}}(\mathfrak{M})$

Let $\mathfrak{M}$, $\mathfrak{N}$ be φ-modules over $\mathfrak{S}$; note that $\varphi_{\mathfrak{M}} \otimes_{\mathfrak{S}} \varphi_{\mathfrak{N}}$ is a φ-semi-linear map on $\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{N}$. If $\mathfrak{L}$ is any finite $\mathbb{Z}_p$-module, we define a φ-semi-linear map on $\mathfrak{L} \otimes_{\mathbb{Z}_p} \mathfrak{M}$ by $1 \otimes \varphi_{\mathfrak{M}}$.

PROPOSITION 3.2.1. – *Let $\mathfrak{M} \in \mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$. There is a natural $\mathfrak{S}^{\mathrm{ur}}$-linear morphism*

$$\hat{\iota} : \mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}^{\mathrm{ur}} \to T^\vee_{\mathfrak{S}}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} \mathfrak{S}^{\mathrm{ur}}$$

such that

(1) *$\hat{\iota}$ is G_∞-equivariant and φ-equivariant,*
(2) *$\hat{\iota}\otimes_{\mathfrak{S}^{\mathrm{ur}}}\mathcal{O}^{\mathrm{ur}}=\tilde{\iota}$, where $\tilde{\iota}$ is defined in* (2.2.2).

Proof. – This is a tautological proof, given Proposition 2.5.3. We may assume that p^n kills $\mathfrak{M}$. First, observe that

$$T_{\mathfrak{S}}(\mathfrak{M})=\mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M},\mathfrak{S}_n^{\mathrm{ur}})=\mathrm{Hom}_{\mathfrak{S}^{\mathrm{ur}},\varphi}(\mathfrak{M}\otimes_{\mathfrak{S}}\mathfrak{S}^{\mathrm{ur}},\mathfrak{S}_n^{\mathrm{ur}}).$$

Note that for each $f\in\mathrm{Hom}_{\mathfrak{S}^{\mathrm{ur}},\varphi}(\mathfrak{M}\otimes_{\mathfrak{S}}\mathfrak{S}^{\mathrm{ur}},\mathfrak{S}_n^{\mathrm{ur}})$, the G_∞-action on f is defined as $f^\sigma(m)=\sigma(f(\sigma^{-1}(m\otimes s)))$ for any $\sigma\in G_\infty$ and $m\otimes s\in\mathfrak{M}\otimes_{\mathfrak{S}}\mathfrak{S}^{\mathrm{ur}}$. We can define a natural morphism $\hat{\iota}':\mathfrak{M}\otimes_{\mathfrak{S}}\mathfrak{S}^{\mathrm{ur}}\to\mathrm{Hom}_{\mathbb{Z}_p}(T_{\mathfrak{S}}(\mathfrak{M}),\mathfrak{S}_n^{\mathrm{ur}})$ by:

$$m\otimes s\mapsto\big(f\mapsto f(m\otimes s),\quad\forall f\in T_{\mathfrak{S}}(\mathfrak{M})\big).$$

On the other hand, since $T_{\mathfrak{S}}(\mathfrak{M})\simeq\bigoplus_{i\in I}\mathbb{Z}_p/p^i\mathbb{Z}_p$ as finite $\mathbb{Z}_p$-modules, we have a natural isomorphism $\mathrm{Hom}_{\mathbb{Z}_p}(T_{\mathfrak{S}}(\mathfrak{M}),\mathfrak{S}_n^{\mathrm{ur}})\simeq T_{\mathfrak{S}}^\vee(\mathfrak{M})\otimes_{\mathbb{Z}_p}\mathfrak{S}^{\mathrm{ur}}$. Combining this with $\hat{\iota}'$, we have a natural morphism $\hat{\iota}:\mathfrak{M}\otimes_{\mathfrak{S}}\mathfrak{S}^{\mathrm{ur}}\to T_{\mathfrak{S}}^\vee(\mathfrak{M})\otimes_{\mathbb{Z}_p}\mathfrak{S}^{\mathrm{ur}}$. It is easy to check that $\hat{\iota}'$ is G_∞- and φ-equivariant. This settles (1). To prove (2), let $M=\mathfrak{M}\otimes_{\mathfrak{S}}\mathcal{O}_{\mathcal{E}}\in\mathbf{\Phi M}_{\mathcal{O}_{\mathcal{E}}}$. By Lemma 2.2.2, we have

$$T_{\mathfrak{S}}(\mathfrak{M})=\mathrm{Hom}_{\mathfrak{S},\varphi}\big(\mathfrak{M},\mathfrak{S}^{\mathrm{ur}}[1/p]/\mathfrak{S}^{\mathrm{ur}}\big)\xrightarrow{\sim}\mathrm{Hom}_{\mathcal{O}_{\mathcal{E}},\varphi}(M,\mathcal{E}^{\mathrm{ur}}/\mathcal{O}^{\mathrm{ur}})=T(M).$$

Repeating the argument in (1), we get a natural map

$$\hat{\iota}\otimes_{\mathfrak{S}^{\mathrm{ur}}}\mathcal{O}^{\mathrm{ur}}:M\otimes_{\mathcal{O}_{\mathcal{E}}}\mathcal{O}^{\mathrm{ur}}\to T_{\mathfrak{S}}^\vee(M)\otimes_{\mathbb{Z}_p}\mathcal{O}^{\mathrm{ur}}.$$

By §1.2 of [8], $\hat{\iota}\otimes_{\mathfrak{S}^{\mathrm{ur}}}\mathcal{O}^{\mathrm{ur}}=\tilde{\iota}$ (with $\tilde{\iota}$ as defined in (2.2.2)) is an isomorphism. □

Combining Proposition 3.2.1 with Example 2.3.5, we have the following $\mathfrak{S}_n^{\mathrm{ur}}$-linear morphism:

$$\hat{\iota}:\mathfrak{S}_n^{\mathrm{ur},\vee}\to\mathfrak{S}_n^{\mathrm{ur}}(-r),$$

with $\hat{\iota}(1)=\mathfrak{t}^r$ and $\mathfrak{t}\in\mathfrak{S}^{\mathrm{ur}}$ satisfying $\varphi(\mathfrak{t})=c_0^{-1}E(u)\mathfrak{t}$. Such choice of $\mathfrak{t}$ is unique up to multiplication by a unit in $\mathbb{Z}_p$, so we also denote the above morphism by $\mathfrak{t}^r$.

THEOREM 3.2.2. – *Let $\mathfrak{M}\in\mathrm{Mod}_{/\mathfrak{S}}^{r,\mathrm{tor}}$ or $\mathrm{Mod}_{/\mathfrak{S}}^{r,\mathrm{fr}}$. There exist natural $\mathfrak{S}^{\mathrm{ur}}$-linear morphisms*

$$\hat{\iota}:\mathfrak{M}\otimes_{\mathfrak{S}}\mathfrak{S}^{\mathrm{ur}}\to T_{\mathfrak{S}}^\vee(\mathfrak{M})\otimes_{\mathbb{Z}_p}\mathfrak{S}^{\mathrm{ur}}\tag{3.2.1}$$

and

$$\hat{\iota}^\vee:T_{\mathfrak{S}}^\vee(\mathfrak{M})\otimes_{\mathbb{Z}_p}\mathfrak{S}^{\mathrm{ur},\vee}\to\mathfrak{M}\otimes_{\mathfrak{S}}\mathfrak{S}^{\mathrm{ur}}(-r)\tag{3.2.2}$$

such that
(1) *$\hat{\iota}$, $\hat{\iota}^\vee$ are compatible with G_∞-actions and φ-structures on both sides,*
(2) *if we identify $\mathfrak{S}^{\mathrm{ur}}$ with $\mathfrak{S}^{\mathrm{ur},\vee}$ by ignoring the φ-structures, then*

$$\hat{\iota}^\vee\circ\hat{\iota}=\mathrm{Id}\otimes_{\mathfrak{S}}\mathfrak{t}^r.$$

In the following, we only consider the case that $\mathfrak{M}$ is of p-power torsion. The case that $\mathfrak{M}$ is $\mathfrak{S}$-finite free is an easy consequence by taking inverse limits of torsion objects. The construction of $\hat{\iota}$ is completed in Proposition 3.2.1, but the construction of $\hat{\iota}^\vee$ requires the following lemma.

LEMMA 3.2.3. – *For any $\mathfrak{M} \in \mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$, there is a natural isomorphism*

$$\mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{S}^\vee, \mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}^{\mathrm{f}(r)}) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}^\vee, \mathfrak{S}_n^{\mathrm{f}(r)}) = T^\vee_{\mathfrak{S}}(\mathfrak{M})(r). \tag{3.2.3}$$

Proof. – By Cartier duality, § 3.1, it suffices to construct a natural isomorphism

$$\mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{S}^\vee, \mathfrak{M}^\vee \otimes_{\mathfrak{S}} \mathfrak{S}^{\mathrm{f}(r)}) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}_n^{\mathrm{f}(r)}).$$

We first claim that by ignoring φ-structures, we have natural isomorphisms

$$\mathrm{Hom}_{\mathfrak{S}}(\mathfrak{S}, \mathfrak{M}^\vee \otimes_{\mathfrak{S}} \mathfrak{S}^{\mathrm{f}(r)}) \xrightarrow{\sim} \mathfrak{M}^\vee \otimes_{\mathfrak{S}} \mathfrak{S}^{\mathrm{f}(r)} \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{S}}(\mathfrak{M}, \mathfrak{S}_n^{\mathrm{f}(r)}). \tag{3.2.4}$$

In fact, it suffices to check that the natural morphism

$$\mathfrak{M}^\vee \otimes_{\mathfrak{S}} \mathfrak{S}^{\mathrm{f}(r)} = \mathrm{Hom}_{\mathfrak{S}}(\mathfrak{M}, \mathfrak{S}_n) \otimes_{\mathfrak{S}} \mathfrak{S}^{\mathrm{f}(r)} \to \mathrm{Hom}_{\mathfrak{S}}(\mathfrak{M}, \mathfrak{S}_n^{\mathrm{f}(r)}) \tag{3.2.5}$$

is an isomorphism. (3.2.5) is certainly an isomorphism if $\mathfrak{M}$ is a finite free $\mathfrak{S}_n$-module. For general $\mathfrak{M}$, there exists by Proposition 2.3.2 (5) a morphism of φ-modules $f : \mathfrak{N}' \to \mathfrak{N}$ with $\mathfrak{N}$ and $\mathfrak{N}'$ finite free over $\mathfrak{S}_n$ such that $\mathfrak{M} = \mathrm{Cok}(f)$. Let $f' : \mathrm{Hom}_{\mathfrak{S}}(\mathfrak{N}, \mathfrak{S}_n^{\mathrm{f}(r)}) \to \mathrm{Hom}_{\mathfrak{S}}(\mathfrak{N}', \mathfrak{S}_n^{\mathrm{f}(r)})$ be the natural map induced by f. Then $\mathrm{Hom}_{\mathfrak{S}}(\mathfrak{M}, \mathfrak{S}_n^{\mathrm{f}(r)}) = \mathrm{Ker}(f')$. Similarly, we have $\mathfrak{M}^\vee \otimes_{\mathfrak{S}} \mathfrak{S}_n^{\mathrm{f}(r)} = \mathrm{Ker}(\tilde{f}^\vee)$ where $\tilde{f}^\vee : \mathfrak{N}^\vee \otimes_{\mathfrak{S}} \mathfrak{S}_n^{\mathrm{f}(r)} \to \mathfrak{N}'^\vee \otimes_{\mathfrak{S}} \mathfrak{S}_n^{\mathrm{f}(r)}$ is induced by $f^\vee$. Since (3.2.5) is an isomorphism for $\mathfrak{N}'$ and $\mathfrak{N}$, $\mathrm{Ker}(f') = \mathrm{Hom}_{\mathfrak{S}}(\mathfrak{M}, \mathfrak{S}_n^{\mathrm{f}(r)}) \simeq \mathfrak{M}^\vee \otimes_{\mathfrak{S}} \mathfrak{S}_n^{\mathrm{f}(r)} = \mathrm{Ker}(\tilde{f}^\vee)$. It remains to check that φ-structures on both sides of (3.2.4) cut out the same elements under the given isomorphism. Let $f \in \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{S}^\vee, \mathfrak{M}^\vee \otimes_{\mathfrak{S}} \mathfrak{S}^{\mathrm{f}(r)})$ and $f(1) = \sum_i f_i \otimes a_i$ with $f_i \in \mathfrak{M}^\vee$ and $a_i \in \mathfrak{S}^{\mathrm{f}(r)}$. Then we have

$$\varphi_{\mathfrak{M}^\vee \otimes \mathfrak{S}^{\mathrm{f}(r)}}\big(f(1)\big) = f\big(\varphi_{\mathfrak{S}^\vee}(1)\big) = f\big(c_0^{-r} E(u)^r\big) = c_0^{-r} E(u)^r f(1).$$

Setting $h = \sum_i a_i f_i \in \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{f}(r)})$, we have

$$c_0^{-r} E(u)^r \sum_i a_i f_i = \sum_i \varphi(a_i) \varphi_{\mathfrak{M}^\vee}(f_i). \tag{3.2.6}$$

It now suffices to check that for any $m \in \mathfrak{M}$, $\varphi(h(m)) = h(\varphi_{\mathfrak{M}}(m))$. Using (3.2.6), we have

$$c_0^{-r} E(u)^r h\big(\varphi_{\mathfrak{M}}(m)\big) = \sum_i \varphi(a_i) \varphi_{\mathfrak{M}^\vee}(f_i)\big(\varphi_{\mathfrak{M}}(m)\big).$$

By Lemma 3.1.2, $\varphi_{\mathfrak{M}^\vee}(f_i)(\varphi_{\mathfrak{M}}(m)) = c_0^{-r} E(u)^r \varphi(f_i(m))$. Then the above formula implies that $c_0^{-r} E(u)^r h(\varphi_{\mathfrak{M}}(m)) = c_0^{-r} E(u)^r \varphi(h(m))$, and we thus have that $\varphi(h(m)) = h(\varphi_{\mathfrak{M}}(m))$, as $c_0^{-1} E(u)$ is not a zero divisor in $\mathfrak{S}_n$. □

COROLLARY 3.2.4. – *Keep notations as above and let $M = \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$.*

(1) $T^\vee(M)(r) = \mathrm{Hom}_{\mathcal{O}^{\mathrm{ur}},\varphi}(\mathcal{O}^{\mathrm{ur},\vee}, M \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{O}^{\mathrm{ur}})$,

(2) *the natural map*

$$i : \mathrm{Hom}_{\mathfrak{S}^{\mathrm{ur}},\varphi}(\mathfrak{S}^{\mathrm{ur},\vee}, \mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}^{\mathrm{ur}}) \to \mathrm{Hom}_{\mathcal{O}^{\mathrm{ur}},\varphi}(\mathcal{O}^{\mathrm{ur},\vee}, M \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{O}^{\mathrm{ur}})$$

is an isomorphism of $\mathbb{Z}_p[G_\infty]$-modules.

Proof. – By Proposition 3.2.1 (2), we have an $\mathcal{O}^{\mathrm{ur}}$-linear isomorphism

$$\mathfrak{t}^r \otimes_{\mathfrak{S}^{\mathrm{ur}}} \mathcal{O}^{\mathrm{ur}} : \mathcal{O}_n^{\mathrm{ur},\vee} \simeq \mathcal{O}_n^{\mathrm{ur}}(-r). \tag{3.2.7}$$

Thus,

$$\begin{aligned}\mathrm{Hom}_{\mathcal{O}^{\mathrm{ur}},\varphi}(\mathcal{O}^{\mathrm{ur},\vee}, M \otimes \mathcal{O}^{\mathrm{ur}}) &= \mathrm{Hom}_{\mathcal{O}^{\mathrm{ur}},\varphi}(\mathcal{O}^{\mathrm{ur}}(-r), M \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{O}^{\mathrm{ur}})\\ &= (M \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{O}^{\mathrm{ur}})^{\varphi=1}(r)\\ &= T^{\vee}(M)(r),\end{aligned}$$

which settles (1). Consider the natural map

$$\begin{aligned}\mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{S}^{\vee}, \mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}^{\mathrm{f}(r)}) &\to \mathrm{Hom}_{\mathfrak{S}^{\mathrm{ur}},\varphi}(\mathfrak{S}^{\mathrm{ur},\vee}, \mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}^{\mathrm{ur}})\\ &\to \mathrm{Hom}_{\mathcal{O}^{\mathrm{ur}},\varphi}(\mathcal{O}^{\mathrm{ur},\vee}, M \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{O}^{\mathrm{ur}}).\end{aligned}$$

Since the first term and the last term have been proved to be isomorphic to $T^{\vee}(M)$, which is a finite set, it suffices to check the above natural maps are injections. Therefore, it is enough to check that the maps

$$\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}^{\mathrm{f}(r)} \to \mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}^{\mathrm{ur}} \to M \otimes_{\mathcal{O}} \mathcal{O}^{\mathrm{ur}}$$

are injections. By Proposition 2.3.2 (4), noting that $\mathfrak{S}_n^{\mathrm{f}(r)}$, $\mathfrak{S}_n^{\mathrm{ur}}$ and $\mathcal{O}_n^{\mathrm{ur}}$ are flat over $\mathfrak{S}_n$, we can reduce the problem to the case that $\mathfrak{M}$ is a finite free $k[\![u]\!]$-module, where the injectivity is obvious. □

Proof of Theorem 3.2.2. – By Corollary 3.2.4, we have

$$T^{\vee}_{\mathfrak{S}}(\mathfrak{M})(r) = \mathrm{Hom}_{\mathfrak{S}^{\mathrm{ur}},\varphi}(\mathfrak{S}^{\mathrm{ur},\vee}, \mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}^{\mathrm{ur}}).$$

Using the same idea as in the proof for Proposition 3.2.1, we see that there exists a natural φ-equivariant, G_∞-equivariant and $\mathfrak{S}^{\mathrm{ur}}$-linear morphism

$$\hat{\iota}^{\vee} : T^{\vee}_{\mathfrak{S}}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} \mathfrak{S}^{\mathrm{ur},\vee} \to \mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}^{\mathrm{ur}}(-r).$$

It now suffices to check that $\hat{\iota}^{\vee} \circ \hat{\iota} = \mathrm{Id} \otimes_{\mathfrak{S}} \mathfrak{t}^r$. Let $M = \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$. It suffices to check that

$$(\hat{\iota}^{\vee} \otimes_{\mathfrak{S}^{\mathrm{ur}}} \mathcal{O}^{\mathrm{ur}}) \circ (\hat{\iota} \otimes_{\mathfrak{S}^{\mathrm{ur}}} \mathcal{O}^{\mathrm{ur}}) = \mathrm{Id}_M \otimes_{\mathcal{O}_{\mathcal{E}}} (\mathfrak{t}^r \otimes_{\mathfrak{S}^{\mathrm{ur}}} \mathcal{O}^{\mathrm{ur}}). \tag{3.2.8}$$

Note that $M \simeq \bigoplus_{i=1}^d \mathcal{O}_{\mathcal{E},n_i}$ as $\mathcal{O}_{\mathcal{E}}$-modules, $T^{\vee}(M) \simeq \bigoplus_{i=1}^d \mathbb{Z}/p^{n_i}\mathbb{Z}$ as $\mathbb{Z}_p$-modules and $\hat{\iota} \otimes_{\mathfrak{S}^{\mathrm{ur}}} \mathcal{O}^{\mathrm{ur}} = \tilde{\iota}$ by Proposition 3.2.1 (2), so it suffices to show that

$$(\hat{\iota} \otimes_{\mathfrak{S}^{\mathrm{ur}}} \mathcal{O}^{\mathrm{ur}}) \circ (\hat{\iota}^{\vee} \otimes_{\mathfrak{S}^{\mathrm{ur}}} \mathcal{O}^{\mathrm{ur}}) = \tilde{\iota} \circ (\hat{\iota}^{\vee} \otimes_{\mathfrak{S}^{\mathrm{ur}}} \mathcal{O}^{\mathrm{ur}}) = \mathrm{Id}_{T^{\vee}(M)} \otimes_{\mathbb{Z}_p} (\mathfrak{t}^r \otimes_{\mathfrak{S}^{\mathrm{ur}}} \mathcal{O}^{\mathrm{ur}}).$$

Note that we have used the isomorphism (3.2.7) to establish

$$\mathrm{Hom}_{\mathcal{O}^{\mathrm{ur}},\varphi}(\mathcal{O}_n^{\mathrm{ur},\vee}, M \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{O}^{\mathrm{ur}}) = T^{\vee}(M)(r),$$

so $\hat{\iota}^{\vee} \otimes_{\mathfrak{S}^{\mathrm{ur}}} \mathcal{O}^{\mathrm{ur}}$ is a composition of the two maps

$$\mathrm{Id} \otimes_{\mathbb{Z}_p} (\mathfrak{t}^r \otimes_{\mathfrak{S}^{\mathrm{ur}}} \mathcal{O}^{\mathrm{ur}}) : T^{\vee}(M) \otimes_{\mathbb{Z}_p} \mathcal{O}^{\mathrm{ur},\vee}(r) \to T^{\vee}(M) \otimes_{\mathbb{Z}_p} \mathcal{O}^{\mathrm{ur}}$$

and

$$\tilde{\iota}^{-1}: T^{\vee}(M)\otimes_{\mathbb{Z}_p}\mathcal{O}^{\mathrm{ur}}\to M\otimes_{\mathcal{O}_{\mathcal{E}}}\mathcal{O}^{\mathrm{ur}}.$$

Therefore,

$$\tilde{\iota}\circ(\hat{\iota}^{\vee}\otimes_{\mathfrak{S}}\mathcal{O}_{\mathcal{E}})=\tilde{\iota}\circ\left(\mathrm{Id}\otimes_{\mathbb{Z}_p}(\mathfrak{t}^r\otimes_{\mathfrak{S}^{\mathrm{ur}}}\mathcal{O}^{\mathrm{ur}})\right)\circ\tilde{\iota}^{-1}=\mathrm{Id}\otimes_{\mathbb{Z}_p}(\mathfrak{t}^r\otimes_{\mathfrak{S}^{\mathrm{ur}}}\mathcal{O}^{\mathrm{ur}}),$$

as required. $\square$

COROLLARY 3.2.5. – *Restricting $\hat{\iota}^{\vee}$ to $T^{\vee}_{\mathfrak{S}}(\mathfrak{M})\otimes_{\mathbb{Z}_p}\mathfrak{S}^{\mathrm{f}(r)}$ gives a natural injection*

$$\hat{\iota}^{\vee}: T^{\vee}_{\mathfrak{S}}(\mathfrak{M})\otimes_{\mathbb{Z}_p}\mathfrak{S}^{\mathrm{f}(r)}\to\mathfrak{M}\otimes_{\mathfrak{S}}\mathfrak{S}^{\mathrm{f}(2r)}(-r).$$

Proof. – By Lemma 3.2.3, we see that $\hat{\iota}^{\vee}(T^{\vee}(\mathfrak{M})\otimes_{\mathbb{Z}_p}\mathfrak{S}^{\vee})\subset\mathfrak{M}\otimes_{\mathfrak{S}}\mathfrak{S}^{\mathrm{f}r}(-r)$. Since $\hat{\iota}^{\vee}$ is $\mathfrak{S}^{\mathrm{ur}}$-linear, it suffices to check that $\mathfrak{S}^{\mathrm{f}(r)}\cdot\mathfrak{S}^{\mathrm{f}(r)}\subset\mathfrak{S}^{\mathrm{f}(2r)}$. Recall from § 2.5 that $F^{\mathrm{f}r}_{\mathfrak{S}}(\mathfrak{S}^{\mathrm{ur}}_n)$ is the set consisting of finite $\mathfrak{S}$-submodules inside $\mathfrak{S}^{\mathrm{ur}}_n$ for which the cokernel of φ^* is killed by $E(u)^r$. Let $\mathfrak{M},\mathfrak{N}\in F^{\mathrm{f}r}_{\mathfrak{S}}(\mathfrak{S}^{\mathrm{ur}}_n)$ and let $\mathfrak{L}$ be the $\mathfrak{S}$-submodule generated by $\mathfrak{M}\cdot\mathfrak{N}$. We see that $\mathfrak{L}$ is a $\mathfrak{S}$-submodule inside $\mathfrak{S}^{\mathrm{ur}}_n$ and is obviously φ-stable. For any $x\in\mathfrak{M}$ and $y\in\mathfrak{N}$, since $\mathfrak{M},\mathfrak{N}\in F^{\mathrm{f}r}_{\mathfrak{S}}(\mathfrak{S}^{\mathrm{ur}}_n)$, there exist $x_i\in\mathfrak{M}$ and $y_j\in\mathfrak{N}$ such that $E(u)^r x=\sum_i a_i\varphi(x_i)$ and $E(u)^r y=\sum_j b_j\varphi(y_j)$ with $a_i,b_j\in\mathfrak{S}$. Thus, we have $E(u)^{2r}xy=\sum_{i,j}a_ib_j\varphi(x_iy_j)$. Therefore, $\mathfrak{L}=\mathfrak{M}\cdot\mathfrak{N}\in F^{\mathrm{f}(2r)}_{\mathfrak{S}}(\mathfrak{S}^{\mathrm{ur}}_n)$. $\square$

4. Proof of the main theorems in part I

4.1. Reducing the proof to the rank-1 case

We will use the Theorem 3.2.2 to reduce Theorem 2.4.2 to the case that $\mathfrak{M}$ is a finite free rank-1 $\mathfrak{S}_n$-module. As in the beginning of § 3, we assume that $\mathfrak{M}$ and $\mathfrak{M}'$ in Theorem 2.4.2 are killed by p^n. First of all,

LEMMA 4.1.1. – *To prove Theorem* 2.4.2 *it suffices to consider the case that*

$$f: T_{\mathfrak{S}}(\mathfrak{M}')\to T_{\mathfrak{S}}(\mathfrak{M})$$

is an isomorphism, with $\mathfrak{M}'$ a finite free $\mathfrak{S}_n$-module and there exists a morphism of φ-modules $\mathfrak{g}:\mathfrak{M}'\to\mathfrak{M}$ such that $T_{\mathfrak{S}}(\mathfrak{g})=f^{-1}$.

Proof. – We reduce the proof of Theorem 2.4.2 to the above case in three steps. Let $M=\mathfrak{M}\otimes_{\mathfrak{S}}\mathcal{O}_{\mathcal{E}}$, $M'=\mathfrak{M}'\otimes_{\mathfrak{S}}\mathcal{O}_{\mathcal{E}}$ and $\tilde{f}: M\to M'$ the morphism in $\mathbf{\Phi M}^{\mathrm{tor}}_{\mathcal{O}_{\mathcal{E}}}$ induced by f. Note that the statement of Theorem 2.4.2 is equivalent to the existence of a constant $\mathfrak{c}$ such that $p^{\mathfrak{c}}\tilde{f}(\mathfrak{M})\subset\mathfrak{M}'$. First, we reduce to the case that $\mathfrak{M}$ is a finite free $\mathfrak{S}_n$-module. By Proposition 2.3.2 (5), we have a surjection $q:\mathfrak{N}\to\mathfrak{M}$ in $\mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$ with $\mathfrak{N}$ a finite free $\mathfrak{S}_n$-module. Let $N=\mathfrak{N}\otimes_{\mathfrak{S}}\mathcal{O}_{\mathcal{E}}$ and $\tilde{q}=q\otimes_{\mathfrak{S}}\mathcal{O}_{\mathcal{E}}$. We see that $p^{\mathfrak{c}}\tilde{f}(\mathfrak{M})\subset\mathfrak{M}'$ if and only if $p^{\mathfrak{c}}\tilde{f}\circ\tilde{q}(\mathfrak{N})\subset\mathfrak{M}'$. Thus it suffices to prove the theorem when $\mathfrak{M}$ is a finite free module over $\mathfrak{S}_n$. Second, by taking the Cartier dual constructed in §3.1, we reduce the proof to the case that $\mathfrak{M}'$ is a finite free $\mathfrak{S}_n$-module. Finally, let Γ be the image of $1\times\tilde{f}$ in $M\times M'$. We have an exact sequence in $\mathbf{\Phi M}^{\mathrm{tor}}_{\mathcal{O}_{\mathcal{E}}}$:

$$0\to\Gamma\to M\times M'\xrightarrow{\mathrm{pr}}M'\to 0.$$

Let $\mathfrak{N}=\mathrm{pr}(\mathfrak{M}\times\mathfrak{M}')$, and let $i_1:\mathfrak{M}\hookrightarrow\mathfrak{M}\times\mathfrak{M}'$ and $i_2:\mathfrak{M}'\hookrightarrow\mathfrak{M}\times\mathfrak{M}'$ be the natural injections; we have

(1) $\mathfrak{N}\in\mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$,

(2) $(\mathrm{pr}\circ i_2)\otimes_{\mathfrak{S}}\mathcal{O}_{\mathcal{E}}:M'\to N$ is an isomorphism, where $N=\mathfrak{N}\otimes_{\mathfrak{S}}\mathcal{O}_{\mathcal{E}}$,

(3) $((\mathrm{pr}\circ i_2)\otimes_{\mathfrak{S}}\mathcal{O}_{\mathcal{E}})^{-1}\circ((\mathrm{pr}\circ i_1)\otimes_{\mathfrak{S}}\mathcal{O}_{\mathcal{E}})=\tilde f$.

Thus we get $\mathrm{pr}\circ i_2:\mathfrak{M}'\to\mathfrak{N}$ with $\mathfrak{M}'$ a finite free $\mathfrak{S}_n$-module and $(\mathrm{pr}\circ i_2)\otimes_{\mathfrak{S}}\mathcal{O}_{\mathcal{E}}$ is an isomorphism. Thus, if we can prove Theorem 2.4.2 for this case, i.e., assuming that there exists $\mathfrak{g}':\mathfrak{N}\to\mathfrak{M}'$ such that $\mathfrak{g}'\otimes_{\mathfrak{S}}\mathcal{O}_{\mathcal{E}}=p^{\mathfrak{c}}((\mathrm{pr}\circ i_2)\otimes_{\mathfrak{S}}\mathcal{O}_{\mathcal{E}})^{-1}$, then let $\mathfrak{g}:=\mathfrak{g}'\circ(\mathrm{pr}\circ i_1)$, and we see that $\mathfrak{g}\otimes_{\mathfrak{S}}\mathcal{O}_{\mathcal{E}}=p^{\mathfrak{c}}\tilde f$ as required. □

Since $\mathfrak{g}\otimes_{\mathfrak{S}}\mathcal{O}_{\mathcal{E}}=\tilde f^{-1}$ is an isomorphism, $\mathfrak{g}:\mathfrak{M}'\to\mathfrak{M}$ is an injection, so we may regard $\mathfrak{M}'$ as a submodule of $\mathfrak{M}$. It thus suffices to prove the following:

LEMMA 4.1.2. – *Let $\mathfrak{M}$, $\mathfrak{M}'\in\mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$ with $\mathfrak{M}'$ finite $\mathfrak{S}_n$-free such that $\mathfrak{M}'\subset\mathfrak{M}$ and $\mathfrak{M}\otimes_{\mathfrak{S}}\mathcal{O}_{\mathcal{E}}=\mathfrak{M}'\otimes_{\mathfrak{S}}\mathcal{O}_{\mathcal{E}}$. There exists a constant $\mathfrak{c}$ only depending on e and r such that $p^{\mathfrak{c}}\mathfrak{M}\subset\mathfrak{M}'$.*

By Corollary 3.2.5, we have the following commutative diagram:

$$\begin{tikzcd}
\mathfrak{M}' \arrow[r, "\hat\iota"] \arrow[d, hook] & (T^\vee_{\mathfrak{S}}(\mathfrak{M}')\otimes_{\mathbb{Z}_p}\mathfrak{S}^{\mathrm{f}(r)})^{G_\infty} \arrow[r, "\hat\iota^\vee"] \arrow[d, equal] & (\mathfrak{M}'\otimes_{\mathfrak{S}}\mathfrak{S}^{\mathrm{f}(2r)}(-r))^{G_\infty} \\
\mathfrak{M} \arrow[r, "\hat\iota_{\mathfrak{M}}"] & (T^\vee_{\mathfrak{S}}(\mathfrak{M})\otimes_{\mathbb{Z}_p}\mathfrak{S}^{\mathrm{f}(r)})^{G_\infty} &
\end{tikzcd}$$

Since $\mathfrak{M}'$ is a finite free $\mathfrak{S}_n$-module, we have

$$\left(\mathfrak{M}'\otimes_{\mathfrak{S}}\mathfrak{S}^{\mathrm{f}(2r)}(-r)\right)^{G_\infty}=\mathfrak{M}'\otimes_{\mathfrak{S}_n}\left(\mathfrak{S}_n^{\mathrm{f}(2r)}(-r)\right)^{G_\infty}.$$

By Theorem 3.2.2, we have

$$\hat\iota^\vee\circ\hat\iota(\mathfrak{M}')=\mathfrak{M}'\otimes_{\mathfrak{S}}\mathfrak{S}_n\cdot\mathfrak{t}^r\subset\hat\iota^\vee\circ\hat\iota_{\mathfrak{M}}(\mathfrak{M})\subset\mathfrak{M}'\otimes_{\mathfrak{S}_n}\left(\mathfrak{S}_n^{\mathrm{f}(2r)}(-r)\right)^{G_\infty},$$

so it suffices to prove that

$$p^{\mathfrak{c}}\left(\mathfrak{S}_n^{\mathrm{f}(2r)}(-r)\right)^{G_\infty}\subset\mathfrak{S}_n\cdot\mathfrak{t}^r. \tag{4.1.1}$$

Let us further shrink $(\mathfrak{S}_n^{\mathrm{f}(2r)}(-r))^{G_\infty}$ by claiming that $(\mathfrak{S}_n^{\mathrm{f}(2r)}(-r))^{G_\infty}\subset\mathcal{O}_{\mathcal{E},n}\cdot\mathfrak{t}^r$. In fact, recall that we have an isomorphism $\mathfrak{t}^r\otimes_{\mathfrak{S}^{\mathrm{ur}}}\mathcal{O}^{\mathrm{ur}}:\mathcal{O}_n^{\mathrm{ur},\vee}\xrightarrow{\sim}\mathcal{O}_n^{\mathrm{ur}}(-r)$. Taking G_∞-invariants of both sides, we have $(\mathcal{O}_n^{\mathrm{ur}}(-r))^{G_\infty}=\mathcal{O}_{\mathcal{E},n}\cdot\mathfrak{t}^r$. Thus,

$$\left(\mathfrak{S}_n^{\mathrm{f}(2r)}(-r)\right)^{G_\infty}\subset\left(\mathcal{O}_n^{\mathrm{ur}}(-r)\right)^{G_\infty}=\mathcal{O}_{\mathcal{E},n}\cdot\mathfrak{t}^r.$$

Now we have reduced the proof of Lemma 4.1.2 (hence the proof of Theorem 2.4.2) to proving that there exists a constant $\mathfrak{c}$ only depending on e and r such that

$$p^{\mathfrak{c}}\left(\mathfrak{S}_n^{\mathrm{f}(2r)}\cap\mathcal{O}_{\mathcal{E},n}\cdot\mathfrak{t}^r\right)\subset\mathfrak{S}_n\cdot\mathfrak{t}^r.$$

For any $x\in\mathfrak{S}_n^{\mathrm{f}(2r)}\cap\mathcal{O}_{\mathcal{E},n}\cdot\mathfrak{t}^r$, let $\mathfrak{N}$ be the $\mathfrak{S}$-submodule generated by $\varphi^n(x)$ for all n. Using Corollary 2.3.9, we can reduce the proof to the following:

LEMMA 4.1.3. – *There exists a constant $\mathfrak{c}$ only depending on e and r such that for any $\mathfrak{M} \in \mathrm{Mod}_{/\mathfrak{S}}^{2r,\mathrm{tor}}$, if $\mathfrak{S}_n \cdot \mathfrak{t}^r \subset \mathfrak{M} \subset \mathcal{O}_{\mathcal{E},n} \cdot \mathfrak{t}^r$, then $p^{\mathfrak{c}}\mathfrak{M} \subset \mathfrak{S}_n \cdot \mathfrak{t}^r$.*

4.2. Proof of Lemma 4.1.3

We first need a Weierstrass Preparation Theorem to proceed with our calculation. There are several versions of such a theorem available; the version we use is from [23]. For any $f \in \mathfrak{S}_n$, let $\bar{f} = f \mod p$, the *order* of f is defined to be the order of $\bar{f}$, i.e., $\mathrm{ord}(f) = \min\{i \mid a_i \mod p \text{ is a unit}\}$ where $f = \sum_{i=0}^{\infty} a_i u^i$.

THEOREM 4.2.1 (Venjakob). – *Let $f \in \mathfrak{S}_n$ have order d. Then there exist a unit $\epsilon \in \mathfrak{S}_n$ and a polynomial $F \in W_n(k)[u]$ of degree d such that $F = u^d \mod p$ and $f = \epsilon F$.*

Proof. – Corollary 3.2 in [23]. □

The above corollary allows us to study division by an irreducible polynomial in $\mathfrak{S}_n$. For $f \in \mathfrak{S}_n$ and a positive integer $m \leqslant n$, we write $E(u) \mid f \mod p^m$ if there exists $h \in \mathfrak{S}_n$ such that $f = E(u)h \mod p^m$. For a real number x, recall that $[x] = \max\{m \mid m \text{ is an integer such that } m \leqslant x\}$.

LEMMA 4.2.2. – *Let $f, g \in \mathfrak{S}_n$ and $n \geqslant 2$. Suppose that $E(u) \mid gf \mod p^n$. Then either $E(u) \mid g \mod p^{[n/2]}$ or $E(u) \mid h \mod p^{[n/2]}$.*

Proof. – By Theorem 4.2.1, we may assume that g and h are polynomials of degrees $\tilde{d}$ and $\tilde{d}'$ such that $g \equiv u^{\tilde{d}} \mod p$ and $h \equiv u^{\tilde{d}'} \mod p$, respectively. Since $E(u)$ is a monomial, we can write $g = E(u)q_1 + \tilde{g}_1$ and $h = E(u)q_1' + \tilde{h}_1$ with $\deg(\tilde{g}_1), \deg(\tilde{h}_1) < \deg(E(u))$. If either $\tilde{g}_1$ or $\tilde{h}_1$ is zero then the proof is complete. Suppose that neither of them is zero. We may write $\tilde{g}_1 = p^{\alpha} g_1$ and $\tilde{h}_1 = p^{\alpha'} h_1$ with $g_1, h_1 \not\equiv 0 \mod p$. It suffices to prove that

$$\alpha + \alpha' \geqslant n - 1.$$

Suppose that the above inequality is not true. Then there exists $\delta \in \mathfrak{S}_2$ such that $g_1 h_1 \equiv E(u)\delta \mod p^2$. By Theorem 4.2.1, we may assume that g_1 (resp. h_1) has degree d (resp. d') and $g_1 \equiv u^d \mod p$ (resp. $h_1 \equiv u^{d'} \mod p$). Since $g_1 h_1 \equiv E(u)\delta \mod p$, we have $d + d' = e + e'$ where e' is the degree of $\delta \mod p$, so we get $0 < d, d' < e$ and $e' < \min(d, d')$. Write $g_1 = \sum_{i=0}^{d} a_i u^i$, $h_1 = \sum_{j=0}^{d'} b_j u^j$, $E(u) = u^e + \sum_{i=0}^{e-1} c_i u^i$, $\delta = \sum_{j=0}^{\infty} f_j u^j$. Comparing the e'-degree terms on both sides of the equation $g_1 h_1 = E(u)\delta \mod p^2$, we have

$$\sum_{i+j=e'} a_i b_j = \sum_{i+j=e'} c_j f_j = c_0 f_{e'} + \sum_{i=1}^{e'} c_i f_{e'-i}.$$

Since $d, d' > e'$, we have $p \mid a_i, p \mid b_j$ for any i, j satisfying $i + j = e'$, so the left hand side is $0 \mod p^2$. On the other hand, since $e' < e$, we have $p \mid e_i, p \mid f_{e'-i}$ for all $i = 1, \ldots, e'$ and $f_{e'} \neq 0 \mod p$, so we have that the right-hand side is $p\mu \mod p^2$ with μ a unit in $\mathbb{Z}_p$, a contradiction. □

Let $\mathfrak{c}_2 = r([\frac{er}{p-1}]) + 1$. Put $\mathfrak{c}_1 = 0$ if $er < p - 1$ and $\mathfrak{c}_1 = 2^{2r}\mathfrak{c}_2$ if $er \geqslant p - 1$.

LEMMA 4.2.3. – *With hypotheses as in Lemma 4.1.3, suppose that $\mathfrak{M}$ is a finite free rank-1 $\mathfrak{S}_n$-module. Then if $n \geqslant \mathfrak{c}_1$ we have $\mathfrak{M} = \mathfrak{S}_n \cdot \mathfrak{t}^r$.*

Proof. – Since $\mathfrak{M} \subset \mathcal{O}_{\mathcal{E},n} \cdot \mathfrak{t}^r$ and $\mathfrak{M}$ is $\mathfrak{S}_n$-free of rank 1, there exists $f' \in \mathcal{O}_{\mathcal{E},n}$ such that $\mathfrak{M} = \mathfrak{S}_n \cdot f'\mathfrak{t}^r$. Note that $\mathfrak{S}_n \cdot \mathfrak{t}^r \subset \mathfrak{M}$, so there exists $f \in \mathfrak{S}_n$ such that $f'f = 1$. Thus, we can write $\mathfrak{M} = \mathfrak{S}_n \cdot \frac{\mathfrak{t}^r}{f}$. By Theorem 4.2.1, we may assume that f is a polynomial with $f = u^d \mod p$. It suffices to prove that f is a unit in $\mathfrak{S}_n$, or equivalently, $d = 0$ if $n \geqslant \mathfrak{c}_1$. We have

$$\varphi\left(\frac{1}{f}\mathfrak{t}^r\right) = \left(c_0^{-1}E(u)\right)^r f/\varphi(f) \cdot \frac{1}{f}\mathfrak{t}^r.$$

Since the cokernel of $\varphi^*_{\mathfrak{M}}$ is killed by $E(u)^{2r}$, if we let $g := E(u)^r f/\varphi(f) \in \mathfrak{S}_n$, then there exists $h \in \mathfrak{S}_n$ such that $gh = E(u)^{2r}$. Put $\bar{f} := f \mod p$ and $\bar{g} := g \mod p$. Then

$$\deg(u^{re}\bar{f}/\bar{f}^p) = re - (p-1)d = \deg(\bar{g}) \geqslant 0.$$

Therefore, $d \leqslant \frac{er}{p-1}$ and $\deg(\bar{g}) \leqslant er$. In particular, if $er < p-1$, then $d = 0$, i.e., f is a unit. Now suppose that $d > 0$, so $\deg(\bar{g}) < er$. Since $E(u)^{2r} = gh \mod p^n$, by Lemma 4.2.2, we see that either $E(u) \mid g \mod p^{[n/2]}$ or $E(u) \mid h \mod p^{[n/2]}$. Suppose that $E(u) \mid g \mod p^{[n/2]}$ and write $g = E(u)g_1 \mod p^{[n/2]}$. Then we have $E(u)^{2r-1} = g_1 h \mod p^{[n/2]}$. Similarly, we have $E(u)^{2r-1} = gh_1 \mod p^{[n/2]}$ if $E(u) \mid h \mod p^{[n/2]}$. Induction on $2r$ shows that

$$g = \epsilon E(u)^{r_1} \mod p^{\mathfrak{c}_2} \tag{4.2.1}$$

with $r_1 < r$ and $\epsilon \in \mathfrak{S}_n$ a unit, so $E(u)^r f/\varphi(f) = E(u)^{r_1}\epsilon \mod p^{\mathfrak{c}_2}$; that is,

$$E(u)^{r-r_1} f = \varphi(f)\epsilon \mod p^{\mathfrak{c}_2}. \tag{4.2.2}$$

Write $f = \sum_{i=0}^d a_i u^i$ and let b_0 be the coefficient of the constant term of ϵ. Comparing the constant terms of both sides of (4.2.2), we get $(c_0 p)^{r-r_1} a_0 = \varphi(a_0) b_0 \mod p^{\mathfrak{c}_2}$. Since b_0 is a unit of $\mathbb{Z}_p$, $a_0 = 0 \mod p^{\mathfrak{c}_2}$. Therefore, $E(u)^{r-r_1} f^1 = \varphi(f^1)\epsilon \mod p^{\mathfrak{c}_2}$ with $f^1 = \sum_{i=1}^d a_i u^i$. Comparing the coefficients of u-terms both sides, we have $(c_0 p)^{r-r_1} a_1 = 0 \mod p^{\mathfrak{c}_2}$. Hence $a_1 = 0 \mod p^{\mathfrak{c}_2 - r}$. Since $\mathfrak{c}_2 = r([\frac{er}{p-1}]) + 1 \geqslant rd + 1$, an easy induction shows that $a_d = 0 \mod p$. This contradicts the fact that $f = u^d \mod p$, which we assumed at the beginning of the proof. Thus, $\deg(\bar{g}) = er$, $d = 0$ and therefore f is a unit. $\square$

Let $\mathfrak{M} \in \mathrm{Mod}^{2r,\mathrm{tor}}_{/\mathfrak{S}}$ with $M := \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$ a finite free $\mathcal{O}_{\mathcal{E},n}$-module. In general, $\mathfrak{M}$ may not be a finite free $\mathfrak{S}_n$-module. However, we will prove that $\mathfrak{M}$ "contains" finite free pieces by employing the following trick. For $0 \leqslant i \leqslant j \leqslant n$, let

$$\mathfrak{M}^{i,j} := \mathrm{Ker}(p^i\mathfrak{M} \xrightarrow{p^{j-i}} p^j\mathfrak{M}). \tag{4.2.3}$$

Since $p^i\mathfrak{M}$ and $p^j\mathfrak{M}$ are in $\mathrm{Mod}^{2r,\mathrm{tor}}_{/\mathfrak{S}}$, $\mathfrak{M}^{i,j} \in \mathrm{Mod}^{2r,\mathrm{tor}}_{/\mathfrak{S}}$ by Lemma 2.3.3. Easy computations show that $p^s\mathfrak{M}^{i,j} = \mathfrak{M}^{s+i,j}$ for any $s \leqslant j-i$ and $\mathfrak{M}^{i,j} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} \simeq M/p^{j-i}M$. For any $l \geqslant 0$ such that $l + j \leqslant n$, the natural injections $p^{i+l}\mathfrak{M} \hookrightarrow p^i\mathfrak{M}$ and $p^{j+l}\mathfrak{M} \hookrightarrow p^j\mathfrak{M}$ induce a map

$$\alpha^{i,j,l} : \mathfrak{M}^{i+l,j+l} \to \mathfrak{M}^{i,j}.$$

It is easy to check that $\alpha^{i,j,l} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$ is an isomorphism. In particular, for $l = 1$ and $i = j$ we get the following decreasing chain

$$\mathfrak{M}^{n-1,n} \cdots \subset \mathfrak{M}^{1,2} \subset \mathfrak{M}^{0,1} \subset M_1 \tag{4.2.4}$$

such that $\mathfrak{M}^{i,i+1} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} = M_1$ for $0 \leqslant i \leqslant n-1$.

LEMMA 4.2.4. – *Notations as above. In the decreasing chain* (4.2.4)*, if there exist i_0 and s such that*

$$\mathfrak{M}^{i_0+s-1,i_0+s} = \cdots = \mathfrak{M}^{i_0+1,i_0+2} = \mathfrak{M}^{i_0,i_0+1}, \tag{4.2.5}$$

then $\mathfrak{M}^{i_0,i_0+s}$ is $\mathfrak{S}_s$-finite free.

Proof. – For any $0 \leqslant m \leqslant s$, let $\Gamma_m = \mathfrak{M}^{i_0+(s-m),i_0+s}$ and $\Gamma = \Gamma_s$. Obviously, we have $\Gamma_m = p^{s-m}\Gamma$. We claim that $\Gamma_{m+1}/p^m\Gamma_{m+1} = \Gamma_m$. To see the claim, considering the following commutative diagram:

$$\begin{tikzcd} \Gamma_{m+1} \arrow[r, equal] & \mathfrak{M}^{i_0+(s-m-1),i_0+s} \arrow[r, "p^m"] \arrow[d, "\beta"] & \mathfrak{M}^{i_0+(s-m-1),i_0+s} \arrow[r, equal] & \Gamma_{m+1} \\ & \mathfrak{M}^{i_0+(s-1),i_0+s} \arrow[r, "\alpha"] & \mathfrak{M}^{i_0+(s-m-1),i_0+(s-m)} \arrow[u, "\gamma"'] & \end{tikzcd} \tag{4.2.6}$$

where $\alpha := \alpha^{i_0+(s-m-1),i_0+(s-m),m}$ is an isomorphism by (4.2.5). The map β is induced by $p^m : p^{i_0+(s-m-1)}\mathfrak{M} \to p^{i_0+(s-1)}\mathfrak{M}$ and it is a surjection. The map γ is induced by $p^m : p^{i_0+(s-m)}\mathfrak{M} \to p^{i_0+s}\mathfrak{M}$ and it is an injection. Tensoring (4.2.6) by $\mathcal{O}_{\mathcal{E}}$, it is easy to check that diagram (4.2.6) is commutative. Since α is an isomorphism, we see that $\Gamma_{m+1}/p^m\Gamma_{m+1} = \mathrm{Cok}(\gamma)$. By the Snake lemma and chasing the diagram, we have

$$\mathrm{Cok}(\gamma) = \mathrm{Ker}(p^{i_0+(s-m)}\mathfrak{M} \xrightarrow{p^m} p^{i_0+s}\mathfrak{M}) = \Gamma_m.$$

Therefore, we have $\Gamma_{m+1}/p^m\Gamma_{m+1} = \Gamma_m$. Now we prove that Γ_m is a finite free $\mathfrak{S}_m$-module by induction on m. The case $m=1$ is obvious. Now assume that Γ_m is a finite free $\mathfrak{S}_m$-module with rank d. Select $x_1, \dots, x_d \in \Gamma_{m+1}$ such that $px_1, \dots, px_d$ is a basis of Γ_m. Since $\Gamma_{m+1}/p^m\Gamma_{m+1} = \Gamma_m$, by Nakayama's lemma, $x_1, \dots, x_d$ generates Γ_{m+1}. Therefore, we have a natural surjection $f : \bigoplus_{i=1}^d \mathfrak{S}_{m+1} \to \Gamma_{m+1}$. Since $M_{m+1} = M/p^{m+1}M$ is a finite free $\mathcal{O}_{\mathcal{E},m+1}$-module with rank d, we see that $f \otimes \mathcal{O}_{\mathcal{E}}$ is a bijection. Note that Γ is u-torsion free. So f is an injection. Thus Γ_{m+1} is $\mathfrak{S}_{m+1}$-finite free. $\square$

Let $\mathfrak{c} = 0$ if $er < p-1$ and $\mathfrak{c} = [\frac{er}{p-1}](\mathfrak{c}_1 - 1) + 1$ if $er \geqslant p-1$.

Proof of Lemma 4.1.3. – We will follow the idea of the proof for Proposition 1.0.6 in [19]. Keep notations as in Lemma 4.1.3. Define

$$\mathfrak{M}^{i,j} := \mathrm{Ker}(p^i\mathfrak{M} \xrightarrow{p^{j-i}} p^j\mathfrak{M}).$$

Then, as in the argument above Lemma 4.2.4, we have

$$\mathfrak{S}_1 \cdot \mathfrak{t}^r \subset \mathfrak{M}^{n-1,n} \cdots \subset \mathfrak{M}^{1,2} \subset \mathfrak{M}^{0,1} \subset \mathcal{O}_{\mathcal{E},1} \cdot \mathfrak{t}^r. \tag{4.2.7}$$

Suppose that $\mathfrak{M}^{i,i+1} = \mathfrak{S}_1 \cdot \frac{\mathfrak{t}^r}{f_i}$ with $f_i \in \mathfrak{S}_1$. By Theorem 4.2.1, we may assume that $f_i = u^{\lambda_i}$ since the cokernel of $\varphi^*_{\mathfrak{M}^{i,i+1}}$ has to be killed by u^{2er}. As in the beginning of the proof of Lemma 4.2.3, we have that $0 \leqslant \lambda_i \leqslant \frac{er}{p-1}$, so if $er < p-1$, then $\mathfrak{M}^{i,i+1} = \mathfrak{S}_1 \cdot \mathfrak{t}^r$ for all i. Thus $\mathfrak{M}$ is a finite free $\mathfrak{S}_n$-module and $\mathfrak{M} = \mathfrak{S}_n \cdot \mathfrak{t}^r$ by Lemma 4.2.3. If $er \geqslant p-1$, then there are at most $[\frac{er}{p-1}]+1$ distinct terms in (4.2.7). Thus, if $n > [\frac{er}{p-1}]+1$, then there must be repeated terms in (4.2.7). If $n \geqslant \mathfrak{c} = [\frac{er}{p-1}](\mathfrak{c}_1 - 1) + 1$, then there exist at least $\mathfrak{c}_1$ terms which coincide in (4.2.7). By Lemma 4.2.4, there exists $i_0 \leqslant \mathfrak{c}$ such that $\mathfrak{M}^{i_0, i_0+\mathfrak{c}_1}$ is a finite free rank $\mathfrak{S}_{\mathfrak{c}_1}$-module of rank 1. By Lemma 4.2.3, we see that $\mathfrak{M}^{i_0, i_0+\mathfrak{c}_1} = \mathfrak{S}_{\mathfrak{c}_1} \cdot \mathfrak{t}^r$. In particular, $\mathfrak{M}^{i_0, i_0+1} = \mathfrak{S}_1 \cdot \mathfrak{t}^r$ and so $\mathfrak{M}^{i,i+1} = \mathfrak{S}_1 \cdot \mathfrak{t}^r$ for all $i \geqslant i_0$. Therefore, $\mathfrak{M}^{i_0,n} \subset \mathfrak{S}_n \cdot \mathfrak{t}^r$, so $p^{\mathfrak{c}}\mathfrak{M} \subset \mathfrak{S}_n \cdot \mathfrak{t}^r$. □

Now we complete the proof of Theorem 2.4.2. As a consequence, we have

COROLLARY 4.2.5. – *Suppose that $\tilde{\mathfrak{M}}$, $\mathfrak{M} \in \mathrm{Mod}_{/\mathfrak{S}}^{r,\mathrm{tor}}$ are such that $T_{\mathfrak{S}}(\mathfrak{M}) \simeq T_{\mathfrak{S}}(\tilde{\mathfrak{M}})$. If we identify $\tilde{\mathfrak{M}} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$ with $\mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$, then*

(1) *$p^{\mathfrak{c}}\tilde{\mathfrak{M}} \subset \mathfrak{M}$ and $p^{\mathfrak{c}}\mathfrak{M} \subset \tilde{\mathfrak{M}}$,*

(2) *if $\mathfrak{M}$ and $\tilde{\mathfrak{M}}$ are finite free $\mathfrak{S}_n$-modules with $n \geqslant \mathfrak{c}$ then $\mathfrak{M}_{n-\mathfrak{c}} = \tilde{\mathfrak{M}}_{n-\mathfrak{c}}$.*

Proof. – We only need to prove (2). $p^{\mathfrak{c}}\mathfrak{M} \subset \tilde{\mathfrak{M}}$ implies that $p^{\mathfrak{c}}\mathfrak{M} \subset \tilde{\mathfrak{M}}^{0,n-\mathfrak{c}}$, where $\tilde{\mathfrak{M}}^{0,n-\mathfrak{c}} = \mathrm{Ker}(p^{n-\mathfrak{c}} : \tilde{\mathfrak{M}} \to p^{n-\mathfrak{c}}\tilde{\mathfrak{M}})$. Since $\tilde{\mathfrak{M}}$ is finite $\mathfrak{S}_n$-free, $\tilde{\mathfrak{M}}^{0,n-\mathfrak{c}} = p^{\mathfrak{c}}\tilde{\mathfrak{M}}$. Therefore, $p^{\mathfrak{c}}\mathfrak{M} \subset p^{\mathfrak{c}}\tilde{\mathfrak{M}}$, and for the same reason we have $p^{\mathfrak{c}}\tilde{\mathfrak{M}} \subset p^{\mathfrak{c}}\mathfrak{M}$. □

As the consequence of Theorem 2.4.2, we also get another proof of Proposition 2.1.12 in [14]. Let $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{fr}}(G_\infty)$ denote the category of continuous finite free $\mathbb{Z}_p$-representations of G_∞.

COROLLARY 4.2.6. – *The functor $T_{\mathfrak{S}} : \mathrm{Mod}_{/\mathfrak{S}}^{r,\mathrm{fr}} \to \mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{fr}}(G_\infty)$ is fully faithful.*

Proof. – Let $\mathfrak{M}, \mathfrak{N} \in \mathrm{Mod}_{/\mathfrak{S}}^{r,\mathrm{fr}}$, $M := \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$, $N := \mathfrak{N} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$ and $\tilde{f} : T_{\mathfrak{S}}(\mathfrak{N}) \to T_{\mathfrak{S}}(\mathfrak{M})$ a morphism of $\mathbb{Z}_p[G_\infty]$-modules. Then we get a morphism $f : M \to N$ such that $T(f) = \tilde{f}$. It suffices to show that $f(\mathfrak{M}) \subset \mathfrak{N}$. By Theorem 2.4.2, we see that $p^{\mathfrak{c}} f(\mathfrak{M}_n) \subset \mathfrak{N}_n$ for any $n \geqslant \mathfrak{c}$. Since $\mathfrak{M}$, $\mathfrak{N}$ are finite free, we see that $f(\mathfrak{M}_{n-\mathfrak{c}}) \subset \mathfrak{N}_{n-c}$. Thus $f(\mathfrak{M}) \subset \mathfrak{N}$. □

4.3. Proof of Theorem 2.4.1

LEMMA 4.3.1. – *Let $\mathfrak{M} \in \mathrm{Mod}_{/\mathfrak{S}}^{r,\mathrm{tor}}$ such that $M := \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$ is a finite free $\mathcal{O}_{\mathcal{E},n}$-module. Suppose that $n \geqslant 2\mathfrak{c} + 1$; then there exists a finite free $\mathfrak{S}_{n-2\mathfrak{c}}$-module $\tilde{\mathfrak{M}} \in \mathrm{Mod}_{/\mathfrak{S}}^{r,\mathrm{tor}}$ such that $\tilde{\mathfrak{M}} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} \simeq M/p^{2\mathfrak{c}}M$.*

Proof. – Use the same notations in Lemma 4.2.4 and set $\mathfrak{M}^{i,j} := \mathrm{Ker}(p^i\mathfrak{M} \xrightarrow{p^{j-i}} p^j\mathfrak{M})$, and $\tilde{\mathfrak{M}} := \mathfrak{M}^{\mathfrak{c},n-\mathfrak{c}}$. We claim that $\tilde{\mathfrak{M}}$ is a finite free $\mathfrak{S}_{n-2\mathfrak{c}}$-module. By Lemma 4.2.4, it suffices to prove

$$\mathfrak{M}^{n-\mathfrak{c}-1,n-\mathfrak{c}} = \cdots = \mathfrak{M}^{\mathfrak{c},\mathfrak{c}+1}.$$

For any $0 \leqslant i \leqslant n - \mathfrak{c} - 1$, we have a natural injection $\alpha : p^{n-\mathfrak{c}-1}\mathfrak{M} \hookrightarrow \mathfrak{M}^{i,i+\mathfrak{c}+1}$. It is easy to see that $\alpha \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$ is an isomorphism. Thus, by Corollary 4.2.5,

$$p^{\mathfrak{c}}\mathfrak{M}^{i,i+\mathfrak{c}+1} \subset p^{n-\mathfrak{c}-1}\mathfrak{M}.$$

Since p kills $p^{\mathfrak{c}}\mathfrak{M}^{i,i+\mathfrak{c}+1}$, we have $p^{\mathfrak{c}}\mathfrak{M}^{i,i+\mathfrak{c}+1} \subset \mathfrak{M}^{n-\mathfrak{c}-1,n-\mathfrak{c}}$. On the other hand, we have $p^{\mathfrak{c}}\mathfrak{M}^{i,i+\mathfrak{c}+1} = \mathfrak{M}^{i+\mathfrak{c},i+\mathfrak{c}+1}$. Therefore, $\mathfrak{M}^{i+\mathfrak{c},i+\mathfrak{c}+1} \subset \mathfrak{M}^{n-\mathfrak{c}-1,n-\mathfrak{c}}$. But by (4.2.7), we always have a decreasing chain $\mathfrak{M}^{i,i+1} \subset \mathfrak{M}^{i+1,i+2} \subset M_1$ for $0 \leqslant i \leqslant n-2$, so we get $\mathfrak{M}^{n-\mathfrak{c}-1,n-\mathfrak{c}} = \cdots = \mathfrak{M}^{\mathfrak{c},\mathfrak{c}+1}$, as required. □

Proof of Theorem 2.4.1. – Suppose that for each n, there exists $\mathfrak{M}_{(n)} \in \mathrm{Mod}_{/\mathfrak{S}}^{r,\mathrm{tor}}$ such that $T_{\mathfrak{S}}(\mathfrak{M}_{(n)}) \simeq T_n = T/p^nT$. For n a fixed integer, let

$$(4.3.1) \quad \mathfrak{M}'_{(n)} = \mathfrak{M}_{(n+3\mathfrak{c})}^{2\mathfrak{c},n+2\mathfrak{c}} = p^{\mathfrak{c}}\mathfrak{M}_{(n+3\mathfrak{c})}^{\mathfrak{c},n+2\mathfrak{c}} = p^{\mathfrak{c}}\,\mathrm{Ker}(p^{\mathfrak{c}}\mathfrak{M}_{(n+3\mathfrak{c})} \xrightarrow{p^{n+\mathfrak{c}}} p^{n+2\mathfrak{c}}\mathfrak{M}_{(n+3\mathfrak{c})}).$$

We claim that $\mathfrak{M}'_{(n)} \in \mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$ and is finite free over $\mathfrak{S}_n$ and that we have $\mathfrak{M}'_{(n+1)}/p^n\mathfrak{M}'_{(n+1)} \simeq \mathfrak{M}'_{(n)}$. If this is the case, letting $\mathfrak{M} = \varprojlim_n \mathfrak{M}'_{(n)}$, we see that $\mathfrak{M} \in \mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$ and $T_{\mathfrak{S}}(\mathfrak{M}) = T$, as required. Hence, it suffices to prove the claim. By the proof of Lemma 4.3.1, we see that $\mathfrak{M}^{\mathfrak{c},n+2\mathfrak{c}}_{(n+3\mathfrak{c})}$ is a finite free $\mathfrak{S}_{n+\mathfrak{c}}$-module, so $\mathfrak{M}'_{(n)} = p^{\mathfrak{c}}\mathfrak{M}^{\mathfrak{c},n+2\mathfrak{c}}_{(n+3\mathfrak{c})}$ is $\mathfrak{S}_n$-free. To see $\mathfrak{M}'_{(n+1)}/p^n\mathfrak{M}'_{(n+1)} \simeq \mathfrak{M}'_{(n)}$, it suffices to show that $p\mathfrak{M}'_{(n+1)} \simeq \mathfrak{M}'_{(n)}$. Note that $p\mathfrak{M}^{\mathfrak{c},n+2\mathfrak{c}}_{(n+1+3\mathfrak{c})}$ and $\mathfrak{M}^{\mathfrak{c},n+2\mathfrak{c}}_{(n+3\mathfrak{c})}$ are both finite free $\mathfrak{S}_{n+\mathfrak{c}}$-modules and give the same finite free $\mathbb{Z}_{n+\mathfrak{c}}$-representation $T_{n+\mathfrak{c}}$ of G_∞. Thus, by Corollary 4.2.5,

$$p\mathfrak{M}'_{(n+1)} = p^{\mathfrak{c}}p\mathfrak{M}^{\mathfrak{c},n+2\mathfrak{c}}_{(n+1+3\mathfrak{c})} \simeq p^{\mathfrak{c}}\mathfrak{M}^{\mathfrak{c},n+2\mathfrak{c}}_{(n+3\mathfrak{c})} = \mathfrak{M}'_{(n)}. \qquad \square$$

4.4. A refinement of Theorem 2.4.1

In order to prove Conjecture 1.0.1, we need a slight variant of Theorem 2.4.1. Recall that $G := \mathrm{Gal}(\bar{K}/K)$. Let T be a finite free $\mathbb{Z}_p$-representation of G. Suppose that, for each n, there exist G-stable $\mathbb{Z}_p$-lattices $L'_{(n)} \subset L_{(n)}$ in a $\mathbb{Q}_p$-representation $V_{(n)}$ of G such that

(1) $L_{(n)}/L'_{(n)} \simeq T_n = T/p^nT$ as $\mathbb{Z}_p[G]$-modules,

(2) there exist finite free $\mathfrak{S}$-modules $\mathfrak{L}_{(n)}, \mathfrak{L}'_{(n)} \in \mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$ such that

$$T_{\mathfrak{S}}(\mathfrak{L}_{(n)}) = L_{(n)}|_{G_\infty} \quad \text{and} \quad T_{\mathfrak{S}}(\mathfrak{L}'_{(n)}) = L'_{(n)}|_{G_\infty}.$$

Letting $\mathfrak{M}_{(n)} := \mathfrak{L}'_{(n)}/\mathfrak{L}_{(n)}$, we have $T_{\mathfrak{S}}(\mathfrak{M}_{(n)}) \simeq T_n|_{G_\infty}$. By Theorem 2.4.1, there exists an $\mathfrak{M} \in \mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$ such that $T_{\mathfrak{S}}(\mathfrak{M}) \simeq T|_{G_\infty}$. In general, it is not necessarily true that $\mathfrak{M}_n \simeq \mathfrak{M}_{(n)}$. To remedy this, we have the following:

LEMMA 4.4.1. – *We can always choose G-stable lattices $L'_{(n)} \subset L_{(n)}$ in $V_{(n)}$ such that $\mathfrak{M}_n = \mathfrak{L}'_{(n)}/\mathfrak{L}_{(n)}$.*

Proof. – Using the covariant functor will be more convenient here. For $\mathfrak{M} \in \mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$ or $\mathfrak{M} \in \mathrm{Mod}^{r,\mathrm{tor}}_{/\mathfrak{S}}$, recall that

$$T^\vee(\mathfrak{M}) := T^\vee(\mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}) = (\mathfrak{M} \otimes_{\mathfrak{S}} \widehat{\mathcal{O}^{\mathrm{ur}}})^{\varphi=1} = \big(T_{\mathfrak{S}}(\mathfrak{M})\big)^\vee.$$

Applying the functor $T^\vee$ to the exact sequence $0 \to \mathfrak{L}_{(n)} \to \mathfrak{L}'_{(n)} \to \mathfrak{M}_{(n)} \to 0$, we get an exact sequence of $\mathbb{Z}_p[G]$-modules $0 \to L^\vee_{(n)} \to L'^\vee_{(n)} \to T^\vee_n \to 0$. By (4.3.1) in the proof of Theorem 2.4.1, we see that

$$\mathfrak{M}_n := \mathfrak{M}^{2\mathfrak{c},n-\mathfrak{c}}_{(n+3\mathfrak{c})} = \mathrm{Ker}(p^{2\mathfrak{c}}\mathfrak{M}_{(n+3\mathfrak{c})} \to p^{n+2\mathfrak{c}}\mathfrak{M}_{(n+3\mathfrak{c})}).$$

Let $\mathfrak{f} : \mathfrak{L}'_{(n+3\mathfrak{c})} \to \mathfrak{M}_{(n+3\mathfrak{c})}$ be the surjection such that $T^\vee(\mathfrak{f})$ is the surjection $L'^\vee_{(n+3\mathfrak{c})} \to T^\vee_{n+3\mathfrak{c}}$. Then $p^{2\mathfrak{c}}\mathfrak{f} : p^{2\mathfrak{c}}\mathfrak{L}'_{(n+3\mathfrak{c})} \to p^{2\mathfrak{c}}\mathfrak{M}_{(n+3\mathfrak{c})}$ is a surjection and $T^\vee(p^{2\mathfrak{c}}\mathfrak{f})$ is a surjection of $\mathbb{Z}_p[G]$-modules $p^{2\mathfrak{c}}L'^\vee_{(n+3\mathfrak{c})} \to p^{2\mathfrak{c}}T^\vee_{n+3\mathfrak{c}}$. For the same reason, $T^\vee(p^{n+2\mathfrak{c}}\mathfrak{f})$ is a surjection of $\mathbb{Z}_p[G]$-modules $p^{n+2\mathfrak{c}}L'^\vee_{(n+3\mathfrak{c})} \to p^{n+2\mathfrak{c}}T^\vee_{n+3\mathfrak{c}}$. Let $\mathfrak{N} := \mathrm{Ker}(p^{2\mathfrak{c}}\mathfrak{f})$, $\mathfrak{N}' := \mathrm{Ker}(p^{n+2\mathfrak{c}}\mathfrak{f})$, $N := \mathrm{Ker}(T^\vee(p^{2\mathfrak{c}}\mathfrak{f}))$ and $N' := \mathrm{Ker}(T^\vee(p^{n+2\mathfrak{c}}\mathfrak{f}))$. By Lemma 2.3.8, $\mathfrak{N}$ and $\mathfrak{N}'$ are $\mathfrak{S}$-finite free. Therefore, we get an exact sequence $0 \to \mathfrak{N}' \to \mathfrak{N} \to \mathfrak{M}_n \to 0$ in $\mathrm{Mod}^r_{/\mathfrak{S}}$; applying the functor $T^\vee$ to this sequence, we get an exact sequence of $\mathbb{Z}_p[G]$-modules $0 \to N' \to N \to T^\vee_n \to 0$. $\square$

5. Preliminaries on semi-stable Galois representations

We begin the second part with this section. In this section we first briefly review several theories for constructions of semi-stable p-adic Galois representations from Fontaine, Breuil and Kisin and then set up several variations of Theorem 3.2.2 to connect Galois representations and their various associated p-adic Hodge structures. These comparisons will play central technical roles in the later calculations.

5.1. Semi-stable Galois representations and (φ, N)-modules

Recall that a p-adic representation is a continuous linear representation of $G := \mathrm{Gal}(\bar{K}/K)$ on a finite dimensional $\mathbb{Q}_p$-vector space V.

Definition 5.1.1 [10]. – A p-adic representation V of G is called *semi-stable* if

$$\dim_{K_0}(B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^G = \dim_{\mathbb{Q}_p} V, \tag{5.1.1}$$

where B_{st} is the period ring constructed by Fontaine, see for example [9] or § 5.2 for the construction.

If V is *any* p-adic representation of G, then one always has $\dim_{K_0}(B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^G \leqslant \dim_{\mathbb{Q}_p} V$ ([11]). To prove that $T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ in Conjecture 1.0.1 is semi-stable, it therefore suffices to prove that $\dim_{K_0}(B_{\mathrm{st}} \otimes_{\mathbb{Z}_p} T)^G \geqslant \mathrm{Rank}_{\mathbb{Z}_p} T$.

Recall that a *filtered* (φ, N)*-module* is a finite dimensional K_0-vector space D endowed with:
(1) a Frobenius semi-linear injection: $\varphi : D \to D$,
(2) a linear map $N : D \to D$ such that $N\varphi = p\varphi N$,
(3) a decreasing filtration $(\mathrm{Fil}^i D_K)_{i \in \mathbb{Z}}$ on $D_K := K \otimes_{K_0} D$ by K-vector spaces such that $\mathrm{Fil}^i D_K = D_K$ for $i \ll 0$ and $\mathrm{Fil}^i D_K = 0$ for $i \gg 0$.

If D is a 1-dimensional (φ, N)-module and $v \in D$ is a basis vector, then $\varphi(v) = \alpha v$ for some $\alpha \in K_0$. We write $t_N(D)$ for the p-adic valuation of α and $t_H(D)$ the unique integer i such that $\mathrm{gr}^i D_K$ is nonzero. If D has dimension $d \in \mathbb{N}^+$, then we write $t_N(D) = t_N(\bigwedge^d D)$ and $t_H(D) = t_H(\bigwedge^d D)$. A filtered (φ, N)-module is called *weakly admissible* if $t_H(D) = t_N(D)$ and for any (φ, N)-submodule $D' \subset D$, $t_H(D') \leqslant t_N(D')$, where $D'_K \subset D_K$ is equipped with the induced filtration. A (φ, N)-module is called *positive* if $\mathrm{Fil}^0 D_K = D_K$. We denote by $\mathrm{MF}(\varphi, N)$ the category of positive filtered (φ, N)-modules, and by $\mathrm{MF}^{\mathrm{w}}(\varphi, N)$ the subcategory consisting of weakly admissible (φ, N)-modules. In [7], Fontaine and Colmez proved that the functor $D^*_{\mathrm{st}} : V \to (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^G$ establishes an equivalence of categories between the category of semi-stable p-adic representations of G and the category of weakly admissible filtered (φ, N)-modules. Therefore, we can always use weakly admissible filtered (φ, N)-modules to describe semi-stable Galois representations. In the sequel, we will instead use the contravariant functor $D_{\mathrm{st}}(V) := D^*_{\mathrm{st}}(V^\vee)$, where $V^\vee$ is the dual representation of V. The advantage of this is that the Hodge–Tate weights of V are exactly the $i \in \mathbb{Z}$ such that $\mathrm{gr}^i D_{\mathrm{st}}(V)_K \neq 0$. A quasi-inverse to D_{st} is then given by

$$V_{\mathrm{st}}(D) := \mathrm{Hom}_{\varphi, N}(D, B_{\mathrm{st}}) \cap \mathrm{Hom}_{\mathrm{Fil}^\cdot}(D_K, K \otimes_{K_0} B_{\mathrm{st}}). \tag{5.1.2}$$

Convention 5.1.2. – From now on, we always assume that the filtration on the weakly admissible filtered (φ, N)-module D under consideration is such that $\mathrm{Fil}^0 D_K = D_K$ and $\mathrm{Fil}^{r+1} D_K = 0$. Equivalently, the Hodge–Tate weights of the semi-stable p-adic Galois representation $V_{\mathrm{st}}(D)$ are always contained in $\{0, \ldots, r\}$.

5.2. Theory of Breuil modules

We denote by S the p-adic completion of the divided power envelope of $W(k)[u]$ with respect to $\mathrm{Ker}(s)$ where $s: W(k)[u] \to \mathcal{O}_K$ is the canonical surjection sending u to π. For any positive integer i, let $\mathrm{Fil}^i S \subset S$ be the p-adic closure of the ideal generated by the divided powers $\gamma_j(u) = \frac{E(u)^j}{j!}$ for all $j \geqslant i$. There is a unique map (Frobenius) $\varphi: S \to S$ which extends the Frobenius on $W(k)$ and satisfies $\varphi(u) = u^p$. Define a continuous K_0-linear derivation $N: S \to S$ such that $N(u) = -u$. Finally, we denote $S[1/p]$ by S_{K_0}. Following [5], a *filtered φ-module over S_{K_0}* is a finite free S_{K_0}-module $\mathcal{D}$ with

(1) a $\varphi_{S_{K_0}}$-semi-linear morphism $\varphi_{\mathcal{D}}: \mathcal{D} \to \mathcal{D}$ such that the determinant of $\varphi_{\mathcal{D}}$ is invertible in S_{K_0},
(2) a decreasing filtration over $\mathcal{D}$ of S_{K_0}-modules $(\mathrm{Fil}^i(\mathcal{D}))_{i\in\mathbb{Z}}$ with $\mathrm{Fil}^0(\mathcal{D}) = \mathcal{D}$ and $\mathrm{Fil}^i S_{K_0} \cdot \mathrm{Fil}^j(\mathcal{D}) \subset \mathrm{Fil}^{i+j}(\mathcal{D})$.

Similarly, a *filtered φ-module over S* is a finite free S-module $\mathcal{M}$ with

(1) a φ_S-semi-linear morphism $\varphi_{\mathcal{M}}: \mathcal{M} \to \mathcal{M}$ such that the determinant of $\varphi_{\mathcal{M}}$ is invertible in S_{K_0},
(2) a decreasing filtration over $\mathcal{M}$ of S-modules $(\mathrm{Fil}^i(\mathcal{M}))_{i\in\mathbb{Z}}$ with $\mathrm{Fil}^0(\mathcal{M}) = \mathcal{M}$ and $\mathrm{Fil}^i S \cdot \mathrm{Fil}^j(\mathcal{M}) \subset \mathrm{Fil}^{i+j}(\mathcal{M})$.

Clearly, if $\mathcal{M}$ is a filtered φ-module over S, then $\mathcal{M} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a filtered φ-module over S_{K_0}. A *filtered (φ, N)-module over S_{K_0}* or a *Breuil module* is a filtered φ-module $\mathcal{D}$ over S_{K_0} with following extra monodromy structure:

(1) a K_0-linear (monodromy) map $N: \mathcal{D} \to \mathcal{D}$ such that
 (a) for all $f \in S_{K_0}$ and $m \in \mathcal{D}$, $N(fm) = N(f)m + fN(m)$,
 (b) $N\varphi = p\varphi N$,
 (c) $N(\mathrm{Fil}^i \mathcal{D}) \subset \mathrm{Fil}^{i-1}(\mathcal{D})$.

We denote the category of filtered φ-modules over $S_{K_0}$by $\mathrm{Mod}^{\varphi}_{/S_{K_0}}$, the category of filtered φ-modules over S by $\mathrm{Mod}^{\varphi}_{/S}$ and the category of Breuil modules by $\mathrm{Mod}^{\varphi,N}_{/S_{K_0}}$. It turns out that the categories $\mathrm{MF}(\varphi, N)$ and $\mathrm{Mod}^{\varphi,N}_{/S_{K_0}}$ are equivalent. More precisely, for any filtered (φ, N)-module $D \in \mathrm{MF}(\varphi, N)$, we can associate an object $\mathcal{D} \in \mathrm{Mod}^{\varphi,N}_{/S_{K_0}}$ by defining $\mathcal{D} = S \otimes_{W(k)} D$; $\varphi_{\mathcal{D}} := \varphi_S \otimes \varphi_D$; $N_{\mathcal{D}} := N \otimes \mathrm{Id} + \mathrm{Id} \otimes N$; $\mathrm{Fil}^i(\mathcal{D}) := \mathcal{D}$ if $\mathrm{Fil}^i D_K = D_K$ and by induction

$$\mathrm{Fil}^{i+1} \mathcal{D} := \{x \in \mathcal{D} \mid N(x) \in \mathrm{Fil}^i \mathcal{D} \text{ and } f_\pi(x) \in \mathrm{Fil}^{i+1} D_K\},$$

where $f_\pi: \mathcal{D} \twoheadrightarrow D_K$ is defined by $s(u) \otimes x \mapsto s(\pi)x$. In §6 of [5], Breuil proved the above functor $\mathcal{D}: D \to D \otimes_{W(k)} S$ is an equivalence of categories. Furthermore, D and $\mathcal{D}(D)$ give rise to the same Galois representations. Several periods rings have to be constructed to make the statement more precise. Recall $R = \varprojlim \mathcal{O}_{\bar{K}}/p$ and the unique surjective map $\theta: W(R) \to \widehat{\mathcal{O}_{\bar{K}}}$ which lifts the projection $R \to \mathcal{O}_{\bar{K}}/p$ onto the first factor in the inverse limit. We denote by A_{cris} the p-adic completion of the divided power envelope of $W(R)$ with respect to $\mathrm{Ker}(\theta)$. Recall that $[\underline{\pi}] \in W(R)$ is the Teichmüller representative of $\underline{\pi} = (\pi_n)_{n\geqslant 0} \in R$. We embed the $W(k)$-algebra $W(k)[u]$ into $W(R)$ via $u \mapsto [\underline{\pi}]$. Since $\theta(\underline{\pi}) = \pi$, this embedding extends to an embedding $\mathfrak{S} \hookrightarrow S \hookrightarrow A_{\mathrm{cris}}$, and $\theta|_S$ is the K_0-linear map $s: S \to \mathcal{O}_K$ defined by sending u to π. The embedding is compatible with Frobenius endomorphisms. As usual, we write $B^+_{\mathrm{cris}} = A_{\mathrm{cris}}[1/p]$, and denote by B^+_{dR} the $\mathrm{Ker}(\theta)$-adic completion of $W(R)[1/p]$.

For any field extension $F/\mathbb{Q}_p$, set $F_{p^\infty} = \bigcup_{n=1}^\infty F(\zeta_{p^n})$ with ζ_{p^n} a primitive p^n-th root of unity. Note that $K_{\infty,p^\infty} = \bigcup_{n=1}^\infty K(\sqrt[p^n]{\pi}, \zeta_{p^n})$ is Galois over K. Let $G_0 := \mathrm{Gal}(K_{\infty,p^\infty}, K_{p^\infty})$, $H_K = \mathrm{Gal}(K_{\infty,p^\infty}, K_\infty)$ and $\hat{G} := \mathrm{Gal}(K_{\infty,p^\infty}/K)$. If $K_{p^\infty} \cap K_\infty = K$ then we easily see that

$\hat{G} = G_0 \rtimes H_K$ and $G_0 \simeq \mathbb{Z}_p(1)$. In fact, Lemma 5.1.2 in [18] shows that $K_{p^\infty} \cap K_\infty = K$ always holds unless $p = 2$. Therefore,

Assumption 5.2.1. – From now to §7, we always assume that $p \geqslant 3$ or $K_{p^\infty} \cap K_\infty = K$ if $p = 2$.

§8 will deal with the case when the above assumption breaks.

For any $g \in G$, let $\underline{\epsilon}(g) = g([\underline{\pi}])/[\underline{\pi}]$. Then $\underline{\epsilon}(g)$ is a cocycle from G to the group of units of A_{cris}. In particular, fixing a topological generator τ of G_0, Assumption 5.2.1 implies that $\underline{\epsilon}(\tau) = [(\epsilon_i)_{i \geqslant 0}] \in W(R)$ with ϵ_i a primitive p^i-th root of unity. Therefore, $t := -\log(\underline{\epsilon}(\tau)) \in A_{\mathrm{cris}}$ is well defined and for any $g \in G$, $g(t) = \chi(g)t$ where χ is the cyclotomic character. Let $B_{\mathrm{dR}} := B^+_{\mathrm{dR}}[\frac{1}{t}]$. $\mathfrak{u} := \log([\underline{\pi}]) \in B_{\mathrm{dR}}$ is well defined. We define $B^+_{\mathrm{st}} := B^+_{\mathrm{cris}}[\mathfrak{u}]$ and $B_{\mathrm{st}} := B^+_{\mathrm{st}}[\frac{1}{t}]$. Let $\mathcal{D} \in \mathrm{Mod}^{\varphi,N}_{/S_{K_0}}$ be a Breuil module. Using the monodromy N on $\mathcal{D}$, we can define a semi-linear G-action on $\mathcal{D} \otimes_S A_{\mathrm{cris}}$ by

$$\sigma(x \otimes a) = \sum_{i=0}^{\infty} N^i(x) \otimes \sigma(a)\gamma_i\big(-\log\big(\underline{\epsilon}(\sigma)\big)\big) \tag{5.2.1}$$

for $\sigma \in G$, $x \in \mathcal{D}$ and $a \in A_{\mathrm{cris}}$. In particular, the G-action preserves the Frobenius and filtration on $\mathcal{D} \otimes_S A_{\mathrm{cris}}$ and for any $g \in G_\infty$ and $x \otimes a \in \mathcal{D} \otimes_S A_{\mathrm{cris}}$, we have $g(x \otimes a) = x \otimes g(a)$ (see Lemma 5.1.1 in [18]). Define

$$V_{\mathrm{st}}(\mathcal{D}) := \mathrm{Hom}_{A_{\mathrm{cris}},\mathrm{Fil},\varphi}(\mathcal{D} \otimes_S A_{\mathrm{cris}}, B^+_{\mathrm{cris}}).$$

Since $\mathcal{D} \otimes_S A_{\mathrm{cris}}$ has a natural G-action defined by (5.2.1), we can define a G-action on $V_{\mathrm{st}}(\mathcal{D})$ by $(g \circ f)(x) = g(f(g^{-1}(x)))$ for any $f \in V_{\mathrm{st}}(\mathcal{D})$, $g \in G$ and $x \in \mathcal{D} \otimes_S A_{\mathrm{cris}}$. Thus, $V_{\mathrm{st}}(\mathcal{D})$ is a $\mathbb{Q}_p[G]$-module.

PROPOSITION 5.2.2 (Breuil). – *For any $D \in \mathrm{MF}(\varphi, N)$, let $\mathcal{D} := \mathcal{D}(D) = D \otimes_{W(k)} S$. Then there is a natural isomorphism $V_{\mathrm{st}}(\mathcal{D}) \simeq V_{\mathrm{st}}(D)$ of $\mathbb{Q}_p[G]$-modules.*

Proof. – This result has been explicitly or non-explicitly used in several papers (e.g., Proposition 2.1.5 in [14]). Lemma 5.2.1 in [18] gives a proof by using the main result of [5]. □

By the above proposition, we always identify $V_{\mathrm{st}}(\mathcal{D})$ with $V_{\mathrm{st}}(D)$ as the same Galois representations.

5.3. Comparisons

In this subsection, we set up a variant of Theorem 3.2.2 to compare filtered φ-modules over S with their associated G_∞-representations. Note that the natural embedding $\mathfrak{S} \hookrightarrow S$ is compatible with Frobenius structures. As in [4], for any finite free φ-module $\mathfrak{M} \in \mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$ of finite height, we can associate a filtered φ-module over S via $\mathcal{M}_{\mathfrak{S}}(\mathfrak{M}) := S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$, a φ_S-semi-linear endomorphism $\varphi_{\mathcal{M}_{\mathfrak{S}}(\mathfrak{M})} := \varphi_S \otimes \varphi_{\mathfrak{M}}$ (as usual, we drop the subscript from $\varphi_{\mathcal{M}_{\mathfrak{S}}(\mathfrak{M})}$ if no confusion will arise) and a decreasing filtration on $\mathcal{M}_{\mathfrak{S}}(\mathfrak{M})$ via

$$\mathrm{Fil}^i(\mathcal{M}_{\mathfrak{S}}(\mathfrak{M})) = \{m \in \mathcal{M}_{\mathfrak{S}}(\mathfrak{M}) \mid (1 \otimes \varphi)(m) \in \mathrm{Fil}^i S \otimes_{\mathfrak{S}} \mathfrak{M}\}.$$

To see that $\mathcal{M} := \mathcal{M}_{\mathfrak{S}}(\mathfrak{M})$ is a filtered φ-module over S, note that the cokernel of $\varphi^*_{\mathfrak{M}}$ is killed by $E(u)^r$, so the determinant of $\varphi_{\mathcal{M}}$ is a divisor of $\varphi(E(u)^r)$, which is a unit in S_{K_0}. Once can easily check that $\mathrm{Fil}^i S \cdot \mathrm{Fil}^j \mathcal{M} \subset \mathrm{Fil}^{i+j} \mathcal{M}$ from the definition. We set

$\mathcal{D}_{\mathfrak{S}}(\mathfrak{M}) = \mathcal{M}_{\mathfrak{S}}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, which is a filtered φ-module over S_{K_0}. To any $\mathfrak{M} \in \mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$ and $\mathcal{M} := \mathcal{M}_{\mathfrak{S}}(\mathfrak{M})$, we can associate a $\mathbb{Z}_p[G_\infty]$-module by:

$$T_{\mathrm{st}}(\mathcal{M}) := \mathrm{Hom}_{S,\varphi,\mathrm{Fil}^\cdot}(\mathcal{M}, A_{\mathrm{cris}}).$$

Since G_∞ acts trivially on S, $T_{\mathrm{st}}(\mathcal{M})$ is a $\mathbb{Z}_p[G_\infty]$-module. Note that $\mathfrak{S}^{\mathrm{ur}} \subset W(R) \subset A_{\mathrm{cris}}$. Given $\mathfrak{f} \in T_{\mathfrak{S}}(\mathfrak{M}) = \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}})$, we define an S-linear map $f : \mathcal{M}_{\mathfrak{S}}(\mathfrak{M}) \to A_{\mathrm{cris}}$ by

$$f(s \otimes m) = s\varphi\big(\mathfrak{f}(m)\big), \quad \text{for any } s \otimes m \in S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}. \tag{5.3.1}$$

It is easy to check that f is compatible with Frobenius and filtration. Thus, we have a map

$$T_{\mathfrak{S}}(\mathfrak{M}) \to T_{\mathrm{st}}\big(\mathcal{M}_{\mathfrak{S}}(\mathfrak{M})\big). \tag{5.3.2}$$

LEMMA 5.3.1. – *The map* (5.3.2) *is an injection, and is compatible with G_∞-actions.*

Proof. – It suffices to check that the map is an injection. By (5.3.1), if $f = 0$, then note that $\mathfrak{M}$ is a finite free $\mathfrak{S}$-module, so $\varphi(\mathfrak{f}) = 0$. But $\varphi : \mathfrak{S}^{\mathrm{ur}} \to A_{\mathrm{cris}}$ is easily checked to be an injection, so $\mathfrak{f} = 0$. □

Remark 5.3.2. – If $r < p-1$, then the map (5.3.2) is an isomorphism (cf. Lemma 3.3.4 of [18]). However, if $p \geqslant r-1$, we may only get an injection as in the following example.

Example 5.3.3. – Let $S^\vee := \mathcal{M}_{\mathfrak{S}}(\mathfrak{S}^\vee)$. Then $\varphi_{S^\vee}(1) = \varphi(c_0^{-r} E(u)^r)$, $\mathrm{Fil}^i S^\vee = S$ for $0 \leqslant i \leqslant r$ and $\mathrm{Fil}^i S^\vee = \mathrm{Fil}^{i-r} S$ for $i > r$. Let $c = \prod_{n=0}^\infty \varphi^n(\frac{\varphi(c_0^{-1}E(u))}{p})$ and S^* be the rank-1 φ-module over S with $\varphi_{S^*}(1) = p^r$, $\mathrm{Fil}^i S^* = S$ for $0 \leqslant i \leqslant r$ and $\mathrm{Fil}^i S^* = \mathrm{Fil}^{i-r} S$ for $i > r$. Note that c is a unit in S. Then the map $c^r : S^* \to S^\vee$ sending 1 to c^r is an isomorphism of filtered φ-modules over S, so $T_{\mathrm{st}}(S^*) \simeq T_{\mathrm{st}}(S^\vee) \simeq \mathbb{Z}_p(r)$. In particular, there exists a generator $f \in T_{\mathrm{st}}(S^*)$ such that $f(1) = t^{\{r\}}$, where $t^{\{n\}} = t^{r(n)}\gamma_{\tilde{q}(n)}(t^{p-1}/p)$ and $n = (p-1)\tilde{q}(n) + r(n)$ with $0 \leqslant r(n) < p-1$ (here we use the notations in §5.2 of [9]). Hence, if $r = p-1$, then we see that $t^{\{p-1\}} \notin W(R)$ and $T_{\mathfrak{S}}(\mathfrak{S}^\vee) \hookrightarrow T_{\mathrm{st}}(S^*)$ is not surjective. If $p \geqslant 3$ and $r = 1 < p-1$, by Remark 5.3.2, we have

$$T_{\mathfrak{S}}(\mathfrak{S}^\vee) \simeq \mathrm{Hom}_{S,\mathrm{Fil}^\cdot,\varphi}(S^*, A_{\mathrm{cris}}).$$

Therefore, $c\varphi(\mathfrak{t}) = u_0 t$ with u_0 a unit in $\mathbb{Z}_p$. If $p = 2$ and $r = 1$, then we only have an injection $T_{\mathfrak{S}}(\mathfrak{S}^\vee) \hookrightarrow T_{\mathrm{st}}(S^*)$. Therefore, $c\varphi(\mathfrak{t}) = \lambda t$ with a $\lambda \in \mathbb{Z}_p$. We claim that λ is a unit in $\mathbb{Z}_p$. In fact, using that $\varphi(\mathfrak{t}) = {c_0}^{-1}E(u)\mathfrak{t}$, one can easily compute that $\varphi(\mathfrak{t}) - c'E(u)^2 \in 2W(R)$ with c' a unit in $W(\bar{k})$. Therefore, $\varphi(\mathfrak{t}) \in 2A_{\mathrm{cris}}$ and $\varphi(\mathfrak{t}) \notin 4A_{\mathrm{cris}}$ and we still have that $c\varphi(\mathfrak{t}) = 2u_0\frac{t}{2} = u_0 t$ with a u_0 an unit in $\mathbb{Z}_2$.

Let $\mathfrak{M} \in \mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$ be a finite free φ-module over $\mathfrak{S}$ of finite $E(u)$-height. By Theorem 3.2.2, we have Frobenius equivariant $\mathfrak{S}^{\mathrm{ur}}$-linear morphisms

$$\hat{\imath} : \mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}^{\mathrm{ur}} \to T^\vee_{\mathfrak{S}}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} \mathfrak{S}^{\mathrm{ur}}$$

and

$$\hat{\imath}^\vee : T^\vee_{\mathfrak{S}}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} \mathfrak{S}^{\mathrm{ur},\vee} \to \mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}^{\mathrm{ur}}(-r)$$

such that $\hat{\iota}^\vee \circ \hat{\iota} = \mathrm{Id} \otimes \mathfrak{t}^r$. In order to extend the comparison of G-actions, we tensor $\hat{\iota}$ and $\hat{\iota}^\vee$ with A_{cris} via the map $\varphi : \mathfrak{S}^{\mathrm{ur}} \to A_{\mathrm{cris}}$. We have

$$\hat{\iota} \otimes_\varphi A_{\mathrm{cris}} : \mathfrak{M} \otimes_{\mathfrak{S},\varphi} A_{\mathrm{cris}} \to T^\vee_{\mathfrak{S}}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}}$$

and

$$\hat{\iota}^\vee \otimes_\varphi A_{\mathrm{cris}} : T^\vee_{\mathfrak{S}}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} A^\vee_{\mathrm{cris}} \to \mathfrak{M} \otimes_{\mathfrak{S},\varphi} A_{\mathrm{cris}}(-r),$$

where φ on $A^\vee_{\mathrm{cris}}$ sends $1 \mapsto \varphi(c_0^{-1} E(u)^r)$. Let $A^*_{\mathrm{cris}} = S^* \otimes_S A_{\mathrm{cris}}$ where S^* is constructed in Example 5.3.3. For the same reason as in Example 5.3.3, the A_{cris}-linear isomorphism $c^r : A^*_{\mathrm{cris}} \to A^\vee_{\mathrm{cris}}$ sending 1 to c^r is compatible with Frobenius and filtration on both sides. We summarize the above discussion in the following lemma:

LEMMA 5.3.4. – *Notations as above, there exist A_{cris}-linear injections*

$$\iota : \mathfrak{M} \otimes_{\mathfrak{S},\varphi} A_{\mathrm{cris}} \to T^\vee_{\mathfrak{S}}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}}$$

and

$$\iota^* : T^\vee_{\mathfrak{S}}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} A^*_{\mathrm{cris}} \to \mathfrak{M} \otimes_{\mathfrak{S},\varphi} A_{\mathrm{cris}}(-r)$$

such that ι and ι^* are compatible with Frobenius and G_∞-actions on both sides. Furthermore, $\iota^* \circ \iota = \mathrm{Id} \otimes t^r$ if we identify A^*_{cris} with A_{cris}.

Proof. – Let $\iota = \hat{\iota} \otimes_\varphi A_{\mathrm{cris}}$ and $\iota^* = c^r(\hat{\iota}^\vee \otimes_\varphi A_{\mathrm{cris}})$. Note that $\hat{\iota} \otimes_\varphi A_{\mathrm{cris}}$ and $\hat{\iota}^\vee \otimes_\varphi A_{\mathrm{cris}}$ are A_{cris}-linear. By Theorem 3.2.2, $\iota^* \circ \iota = \mathrm{Id} \otimes (\varphi(\mathfrak{t})c)^r$. In Example 5.3.3, we showed that $\varphi(\mathfrak{t})c = u_0 t$ for a unit $u_0 \in \mathbb{Z}_p$, and we can modify ι^* by multiplication by u_0^{-r} so that $\iota^* \circ \iota = \mathrm{Id} \otimes t^r$. Since t^r is not a zero divisor in A_{cris}, we see that ι and ι^* are injections. □

Remark 5.3.5. – In the applications that follow, we abuse the notation by identifying A^*_{cris} with A_{cris}. The map ι^* is no longer compatible with Frobenius with such identification. However, we do not use the Frobenius compatibility of ι^* in our applications (cf. Proposition 6.1.1).

5.4. Kisin's theory on (φ, N)-modules over $\mathfrak{S}$

In this subsection, we input Kisin's theory ([14]) on the classification of semi-stable Galois representations by (φ, N)-modules over $\mathfrak{S}$. A (φ, N)-*module over* $\mathfrak{S}$ is a finite free φ-module $\mathfrak{M} \in \mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$ equipped with a K_0-linear endomorphism $N : \mathfrak{M}/u\mathfrak{M} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \mathfrak{M}/u\mathfrak{M} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ such that $N\varphi = p\varphi N$. We denote by $\mathrm{Mod}^{\varphi,N}_{/\mathfrak{S}}$ the category of (φ, N)-modules over $\mathfrak{S}$, and by $\mathrm{Mod}^{\varphi,N}_{/\mathfrak{S}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ the associated isogeny category. The following theorem summarizes results we need from [14] (cf. Corollary 1.3.15, Proposition 2.1.5 and Lemma 2.1.15 there).

THEOREM 5.4.1 (Kisin). – *There exists a fully faithful $\otimes$-functor Θ from the category of positive weakly admissible filtered (φ, N)-modules $\mathrm{MF}^{\mathrm{w}}(\varphi, N)$ to $\mathrm{Mod}^{\varphi,N}_{/\mathfrak{S}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Let $D \in \mathrm{MF}^{\mathrm{w}}(\varphi, N)$ and $\mathfrak{M} := \Theta(D)$. Then there exists a canonical bijection*

$$\eta : T_{\mathfrak{S}}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\sim} V_{\mathrm{st}}(D) \tag{5.4.1}$$

compatible with the action of G_∞ on both sides. Let $V = T_{\mathfrak{S}}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. The map $\mathfrak{N} \to \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{N}, \mathfrak{S}^{\mathrm{ur}})$ is a bijection between the set of finite free φ-stable $\mathfrak{S}$-modules $\mathfrak{N} \subset \mathcal{E} \otimes_{\mathfrak{S}} \mathfrak{M}$ such that $\mathfrak{N}/\varphi^*\mathfrak{N}$ is killed by $E(u)^r$ and the set of G_∞-stable $\mathbb{Z}_p$-lattices $L \subset V$.

In fact, Kisin also gave a criterion to detect whether an $\mathfrak{M} \in \mathrm{Mod}^{\varphi,N}_{/\mathfrak{S}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is in the essential image of functor Θ. In particular, let $\mathfrak{M} = \Theta(D)$ for $D \in \mathrm{MF}^{\mathrm{w}}(\varphi, N)$ and $\mathcal{D} := \mathcal{D}_{\mathfrak{S}}(\mathfrak{M})$. §3.2 in [18] showed that this criterion implies that there exists a unique monodromy operator N defined over $\mathcal{D}$ such that the data $(\mathcal{D}, \mathrm{Fil}^i \mathcal{D}, \varphi, N)$ is a Breuil module and $\mathcal{D}(D) \simeq \mathcal{D}$ in $\mathrm{Mod}^{\varphi,N}_{/S_{K_0}}$ (where $\mathcal{D}(\cdot)$ is the functor constructed in §5.2). For our purposes, it will be convenient to reconstruct (5.4.1) somewhat differently from [14] following the idea in [18]. By Lemma 5.3.1, we have injections of $\mathbb{Z}_p[G_\infty]$-modules:

$$T_{\mathfrak{S}}(\mathfrak{M}) \hookrightarrow T_{\mathrm{st}}(\mathcal{M}) \hookrightarrow \mathrm{Hom}_{S,\mathrm{Fil}^{\cdot},\varphi}(\mathcal{D}, B^+_{\mathrm{cris}}). \tag{5.4.2}$$

Note that

$$\mathrm{Hom}_{S,\mathrm{Fil},\varphi}(\mathcal{D}, B^+_{\mathrm{cris}}) \simeq \mathrm{Hom}_{A_{\mathrm{cris}},\mathrm{Fil},\varphi}(\mathcal{D} \otimes_S A_{\mathrm{cris}}, B^+_{\mathrm{cris}}) = V_{\mathrm{st}}(\mathcal{D})$$

which is compatible with G_∞-actions on both sides. By Proposition 5.2.2, we have a natural injection

$$T_{\mathfrak{S}}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \hookrightarrow V_{\mathrm{st}}(\mathcal{D}) \simeq V_{\mathrm{st}}(D)$$

compatible with G_∞-actions on both sides. On the other hand, since D is weakly admissible, an argument in Proposition 4.5 of [7] shows that $\dim_{\mathbb{Q}_p} V_{\mathrm{st}}(D) \leqslant \dim_{K_0}(D) = d$. Since (5.4.2) is an injection and $\mathrm{rank}_{\mathbb{Z}_p} T_{\mathfrak{S}}(\mathfrak{M}) = d$, we must have $\dim_{\mathbb{Q}_p} V_{\mathrm{st}}(D) = \dim_{K_0}(D) = d$ and

$$T_{\mathfrak{S}}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\sim} \mathrm{Hom}_{A_{\mathrm{cris}},\mathrm{Fil},\varphi}(\mathcal{D} \otimes_S A_{\mathrm{cris}}, B^+_{\mathrm{cris}}) \simeq V_{\mathrm{st}}(D), \tag{5.4.3}$$

where the first isomorphism is compatible with G_∞-actions and the second is compatible with G-actions. Note that the second isomorphism allows us to construct a B^+_{cris}-linear map

$$\iota' : \mathcal{D} \otimes_S A_{\mathrm{cris}} \to V^\vee_{\mathrm{st}}(\mathcal{D}) \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}}$$

that is compatible with G-actions, Frobenius and filtration. On the other hand, by Lemma 5.3.4, we have

$$\iota : \mathfrak{M} \otimes_{\mathfrak{S},\varphi} A_{\mathrm{cris}} \to T^\vee_{\mathfrak{S}}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}}.$$

Note that $(\mathfrak{M} \otimes_{\mathfrak{S},\varphi} A_{\mathrm{cris}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \mathfrak{M} \otimes_{\mathfrak{S},\varphi} S_{K_0} \otimes_{S_{K_0}} B^+_{\mathrm{cris}} = \mathcal{D}_{\mathfrak{S}}(\mathfrak{M}) \otimes_S A_{\mathrm{cris}}$, we claim that $\iota \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \iota'$; that is, $\hat{\iota} \otimes_\varphi B^+_{\mathrm{cris}} = \iota'$. To prove the claim, note that $T_{\mathfrak{S}}(\mathfrak{M}) = \mathrm{Hom}_{\mathfrak{S}^{\mathrm{ur}},\varphi}(\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}^{\mathrm{ur}}, \mathfrak{S}^{\mathrm{ur}})$. The functor

$$\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}^{\mathrm{ur}} \mapsto (\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}^{\mathrm{ur}}) \otimes_{\mathfrak{S}^{\mathrm{ur}},\varphi} B^+_{\mathrm{cris}} = \mathcal{D}_{\mathfrak{S}}(\mathfrak{M}) \otimes_S B^+_{\mathrm{cris}}$$

induces a natural map

$$\mathrm{Hom}_{\mathfrak{S}^{\mathrm{ur}},\varphi}(\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}^{\mathrm{ur}}, \mathfrak{S}^{\mathrm{ur}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \mathrm{Hom}_{A_{\mathrm{cris}},\mathrm{Fil},\varphi}\big(\mathcal{D}_{\mathfrak{S}}(\mathfrak{M}) \otimes_S B^+_{\mathrm{cris}}, B^+_{\mathrm{cris}}\big).$$

Since the left-hand side is $T_{\mathfrak{S}}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and the right-hand side is isomorphic to $V_{\mathrm{st}}(D)$, (5.4.3) shows that the above map is an isomorphism. Therefore, by the construction of ι' and ι, we have $\iota \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \iota'$. In summary, we have proved the following:

THEOREM 5.4.2. – *Let T be a G_∞-stable $\mathbb{Z}_p$-lattice in a semi-stable Galois representation V, and let $\mathfrak{M} \in \mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$ be such that $\eta(T_{\mathfrak{S}}(\mathfrak{M})) = T$, as in Theorem* 5.4.1. *We have the* *following commutative diagram*:

$$
\begin{tikzcd}
\mathfrak{M}\otimes_{\mathfrak{S},\varphi} B^+_{\mathrm{cris}} \arrow[r, "\iota\otimes_{\mathbb{Z}_p}\mathbb{Q}_p"] & T^\vee\otimes_{\mathbb{Z}_p} B^+_{\mathrm{cris}} \\
\mathfrak{M}\otimes_{\mathfrak{S},\varphi} A_{\mathrm{cris}} \arrow[r, "\iota"] \arrow[u, hook] & T^\vee\otimes_{\mathbb{Z}_p} A_{\mathrm{cris}} \arrow[u, hook]
\end{tikzcd}
\tag{5.4.4}
$$

where $\iota\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ preserves G-actions, Frobenius and filtrations and ι preserves G_∞-actions and Frobenius.

Remark 5.4.3. – In the following applications of Theorem 5.4.2 and Lemma 5.3.4, we sometimes replace $\mathfrak{M}\otimes_{\mathfrak{S},\varphi} A_{\mathrm{cris}}$ by $\mathcal{M}\otimes_S A_{\mathrm{cris}}$ where $\mathcal{M}:=\mathcal{M}_{\mathfrak{S}}(\mathfrak{M})$. It is possible to define filtrations on both sides of ι such that ι is compatible with these filtrations, but we do not need filtrations in the applications below.

6. The G-action on $\mathfrak{M}\otimes_{\mathfrak{S},\varphi} A_{\mathrm{cris}}$

Let T be the finite free $\mathbb{Z}_p$-representation of G in Conjecture 1.0.1. By the hypotheses of Conjecture 1.0.1, we have $T_n = T/p^nT = L_{(n)}/L'_{(n)}$ for each $n\geqslant 0$, where $L'_{(n)}\subset L_{(n)}$ are G-stable $\mathbb{Z}_p$-lattices in a semi-stable Galois representation $V_{(n)}$ with Hodge–Tate weights in $\{0,\dots,r\}$. By Theorem 5.4.1, there exist finite free φ-modules $\mathfrak{L}_{(n)}$, $\mathfrak{L}'_{(n)}\in \mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$ and an injection $\mathfrak{i}_n:\mathfrak{L}_{(n)}\hookrightarrow\mathfrak{L}'_{(n)}$ in $\mathrm{Mod}^r_{/\mathfrak{S}}$ such that $T_{\mathfrak{S}}(\mathfrak{L}_{(n)})\simeq L_{(n)}|_{G_\infty}$, $T_{\mathfrak{S}}(\mathfrak{L}'_{(n)})\simeq L'_{(n)}|_{G_\infty}$ and $T_{\mathfrak{S}}(\mathfrak{i}_n)$ is the inclusion $L'_{(n)}\subset L_{(n)}$. Setting $\tilde{\mathfrak{M}}_{(n)}:=\mathfrak{L}'_{(n)}/\mathfrak{L}_{(n)}$, we have $T_{\mathfrak{S}}(\tilde{\mathfrak{M}}_{(n)})\simeq T_n|_{G_\infty}$. Thus, by Theorem 2.4.1, there exists a finite free $\mathfrak{S}$-module $\mathfrak{M}\in\mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$ such that $T_{\mathfrak{S}}(\mathfrak{M})\simeq T|_{G_\infty}$. A refinement of Theorem 2.4.1 in §4.4 shows that we can assume that $\mathfrak{M}_n\simeq\tilde{\mathfrak{M}}_{(n)}$. Let $\mathcal{M}=\mathcal{M}_{\mathfrak{S}}(\mathfrak{M})=S\otimes_{\varphi,\mathfrak{S}}\mathfrak{M}$. Note that $\mathcal{M}\otimes_S A_{\mathrm{cris}}=\mathfrak{M}\otimes_{\mathfrak{S},\varphi}A_{\mathrm{cris}}$. By Lemma 5.3.4, we have the following commutative diagram

$$
\begin{tikzcd}
\mathcal{M}\otimes_S B^+_{\mathrm{cris}} \arrow[r, "\iota\otimes_{\mathbb{Z}_p}\mathbb{Q}_p"] & T^\vee\otimes_{\mathbb{Z}_p} B^+_{\mathrm{cris}} & \\
\mathcal{M}\otimes_S A_{\mathrm{cris}} \arrow[r, "\iota"] \arrow[u, hook] & T^\vee\otimes_{\mathbb{Z}_p} A_{\mathrm{cris}} \arrow[u, hook] \arrow[r, "\iota^*"] & \mathcal{M}\otimes_S A_{\mathrm{cris}}(-r).
\end{tikzcd}
\tag{6.0.1}
$$

Since T is a $\mathbb{Z}_p$-representation of G, the second column has a natural G-action. Unlike diagram (5.4.4), we do not know whether $\mathcal{M}\otimes_S B^+_{\mathrm{cris}}$ is stable under the G-action on $T^\vee\otimes_{\mathbb{Z}_p} B^+_{\mathrm{cris}}$ because there is no monodromy on $\mathcal{D}_{\mathfrak{S}}(\mathfrak{M})=\mathcal{M}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ available, which is crucial in defining the G-action on $\mathcal{M}\otimes_S B^+_{\mathrm{cris}}$ via (5.2.1).

This section is devoted to proving that $\mathcal{M}\otimes_S B^+_{\mathrm{cris}}$ is indeed G-stable under the hypotheses of Conjecture 1.0.1. We also describe the precise image of the G-action of $\mathcal{M}$ in $\mathcal{M}\otimes_S B^+_{\mathrm{cris}}$. For any integer $n\geqslant 0$, recall that $t^{\{n\}}=t^{r(n)}\gamma_{\tilde{q}(n)}(t^{p-1}/p)$ and $n=(p-1)\tilde{q}(n)+r(n)$ with $0\leqslant r(n)<p-1$. Define a subring $\mathcal{R}_{K_0}$ of B^+_{cris} by

$$\mathcal{R}_{K_0}=\left\{x=\sum_{i=0}^{\infty} f_i t^{\{i\}},\ f_i\in S_{K_0}\text{ and } f_i\to 0\text{ as } i\to+\infty\right\}.$$

Put $\mathcal{R}:=\mathcal{R}_{K_0}\cap A_{\mathrm{cris}}$. The main goal of this section is to prove the following:

PROPOSITION 6.0.4. – *Under the hypotheses of Conjecture* 1.0.1*,* $\mathcal{M}\otimes_S B^+_{\mathrm{cris}}$ *is stable under the action of* G *and* $G(\mathcal{M})\subset\mathcal{M}\otimes_S\mathcal{R}_{K_0}$.

6.1. Action of G_0 on $\mathcal{M}\otimes_S A_{\mathrm{cris}}$

Suppose that T is a G-stable $\mathbb{Z}_p$-lattice in a *semi-stable* Galois representation V. Let $\mathfrak{M}\in\mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$ be such that $\eta(T_{\mathfrak{S}}(\mathfrak{M}))=T$ as in Theorem 5.4.2 and $\mathcal{M}:=\mathcal{M}_{\mathfrak{S}}(\mathfrak{M})$. We first analyze the action of G_0 on $\mathcal{M}\otimes_S B^+_{\mathrm{cris}}$. Recall that τ is a fixed topological generator of $G_0:=\mathrm{Gal}(K_{\infty,p^\infty},K_{p^\infty})$ (§5.2).

PROPOSITION 6.1.1. – *There exists a constant* $s_0\geqslant 0$ *only depending on the maximal Hodge–Tate weight* r *of* V *such that* $p^{s_0}\tau(\mathcal{M})\subset\mathcal{M}\otimes_S\mathcal{R}$.

Remark 6.1.2. – When $r<p-1$, we proved in §5.3 [18] that $\tau(\mathcal{M})\subset\mathcal{M}\otimes_S\mathcal{R}$. Thus, s_0 may be chosen to be 0. Little is known about the minimal bound for s_0 if $r\geqslant p-1$.

To prove Proposition 6.1.1, we need a fact about A_{cris}. Following the notations in §5.2 of [9], let

$$I^{[i]}=\{a\in A_{\mathrm{cris}}\mid \varphi^n(a)\in\mathrm{Fil}^i A_{\mathrm{cris}}\text{ for all }n\}.$$

By Proposition 5.3.1 in [9],

$$I^{[i]}=\left\{\sum_{j\geqslant i}^{\infty}a_jt^{\{j\}}\;\middle|\;a_j\in W(R),\ a_j\to 0\text{ as }j\to+\infty\right\}.$$

LEMMA 6.1.3. – *There exists a constant* $\lambda\geqslant 0$ *only depending on* r *such that for all* $m\geqslant\lambda$ *and all* $a\in A_{\mathrm{cris}}$*, if* $t^ra\in p^mA_{\mathrm{cris}}$ *then*

$$a\in\sum_{i+j=m-\lambda}p^iI^{[j]}.$$

Proof. – By Theorem 5.2.7 in [9], for any $a\in A_{\mathrm{cris}}$, we can write a in the following form: $a=\sum_{n=0}^{\infty}a_nt^{\{n\}}$, where $a_n\in W(R)$ and $a_n\to 0$ as $n\to+\infty$. Thus, $t^ra=\sum_{n=0}^{\infty}a_nc_nt^{\{n+r\}}$ with

$$c_n=\frac{p^{\tilde q(n+r)}\tilde q(n+r)!}{p^{\tilde q(n)}\tilde q(n)!}.$$

It is easy to check that $\tilde q(n+r)-\tilde q(n)$ is bounded and $n-v(c_n)\to+\infty$ as $n\to+\infty$, where $v(\cdot)$ is the standard valuation in $\mathbb{Z}_p$. Thus, $\lambda=-\min\{n-v(c_n)\}\geqslant 0$ is well defined. Now suppose that $m>\lambda$ and $t^ra\in p^mA_{\mathrm{cris}}$. Then there exists $b_n\in W(R)$ such that

$$\sum_{n=0}^{\infty}a_nc_nt^{\{n+r\}}=p^m\left(\sum_{n=0}^{\infty}b_nt^{\{n\}}\right).$$

Looking at this equation modulo $I^{[r]}$, we get $p^m(\sum_{n=0}^{r-1}b_nt^{\{n\}})=0\mod I^{[r]}$. By Proposition 5.3.5 in [9], $A_{\mathrm{cris}}/I^{[r]}$ has no p-torsion. Therefore, $\sum_{n=0}^{r-1}b_nt^{\{n\}}\in I^{[r]}$, so without loss of generality, we can assume that

$$\sum_{n=0}^{\infty}a_nc_nt^{\{n+r\}}=p^m\left(\sum_{n=0}^{\infty}b_nt^{\{n+r\}}\right).$$

Looking modulo $I^{[r+1]}$ gives that $(a_0c_0 - p^m b_0)t^{\{r\}} \in I^{[r+1]}$. Since

$$\varphi^n\big((a_0c_0 - p^m b_0)t^{\{r\}}\big) = \varphi^n(a_0c_0 - p^m b_0)p^n t^{\{r\}} \in \mathrm{Fil}^{r+1} A_{\mathrm{cris}}$$

and $t^r \in \mathrm{Fil}^r A_{\mathrm{cris}} - \mathrm{Fil}^{r+1} A_{\mathrm{cris}}$, we get $(a_0c_0 - p^m b_0) \in I^{[1]}$. Note that $m > \lambda \geqslant v(c_0)$, so we may write $a_0c_0 - p^m b_0 = p^{v(c_0)} d_0$, and, clearly, $d_0 \in I^{[1]}$. Therefore, $a_0 = (c_0^{-1} p^{v(c_0)})d_0 + p^m(c_0^{-1})b_0$ and we get

$$a = p^m(c_0^{-1})b_0 + \left(c_0^{-1}p^{v(c_0)}d_0 + \sum_{n=1}^{\infty} a_i t^{\{n\}}\right).$$

Hence, we can write $a = \sum_{n=0}^{\infty} a_n t^{\{n\}}$, where $a_0 \in p^{m-\lambda} A_{\mathrm{cris}}$ and $a_0c_0 \in p^m A_{\mathrm{cris}}$. It now suffices to prove that we can always write $a = \sum_{n=0}^{\infty} a_n t^{\{n\}}$ such that $a_n \in p^{m-\lambda-n} A_{\mathrm{cris}}$ and $p^m \mid a_n c_n$ for $0 \leqslant n \leqslant m - \lambda$. We prove this by induction on n. The above argument settles the case $n = 0$. Now suppose that we have $a_n \in p^{m-\lambda-n} A_{\mathrm{cris}}$ and $p^m \mid a_n c_n$ for $0 \leqslant n \leqslant l-1$. Consider the case that $n = l$. Since $a_n c_n \in p^m A_{\mathrm{cris}}$ for $0 \leqslant n \leqslant l-1$, we have

$$\sum_{n=l}^{\infty} a_n c_n t^{\{n+r\}} = p^m\left(\sum_{n=0}^{\infty} b_n t^{\{n+r\}} - \sum_{n=0}^{l-1} \frac{a_n c_n}{p^m} t^{\{n+r\}}\right). \tag{6.1.1}$$

As in the case $n = 0$, using the fact that $A_{\mathrm{cris}}/I^{[l+r]}$ has no p-torsion, we can rewrite (6.1.1) as the following:

$$\sum_{n=l}^{\infty} a_n c_n t^{\{n+r\}} = p^m\left(\sum_{n=l}^{\infty} b_n t^{\{n+r\}}\right).$$

Repeating the same argument as in the case $n = 0$, we have $a_l c_l - p^m b_l = d_l$ with $d_l \in I^{[1]}$. We claim that $v(c_l) \leqslant m$. In fact, if $v(c_l) \leqslant l$, then $v(c_l) \leqslant l \leqslant m - \lambda \leqslant m$; if $v(c_l) > l$, then $v(c_l) = (v(c_l) - l) + l \leqslant \lambda + l \leqslant \lambda + m - \lambda = m$. Therefore, $d_l \in p^{v(c_l)} A_{\mathrm{cris}}$ and we can write $a_l = (c_l^{-1}p^m)b_l + c_l^{-1}d_l$ with $c_l^{-1}d_l \in I^{[1]}$. Thus,

$$a = \sum_{n=1}^{l-1} a_n t^{\{n\}} + (c_l^{-1}p^m)b_l t^{\{l\}} + (c_l^{-1}d_l)t^{\{l\}} + \sum_{n=l+1}^{\infty} a_n t^{\{n\}}.$$

Rewrite $a_l = (c_l^{-1}p^m)b_l$. Note that $v(c_l) - l \leqslant \lambda$, so $m - v(c_l) \geqslant m - \lambda - l$, and obviously $a_l c_l \in p^m A_{\mathrm{cris}}$. Thus we have proved the case $n = l$, which proves the lemma. □

Proof of Proposition 6.1.1. – Let $s_0 = \lambda + r$. We choose m big enough so that $p^m\tau(\mathcal{M} \otimes_S A_{\mathrm{cris}}) \subset (\mathcal{M} \otimes_S A_{\mathrm{cris}})$. Put $\tilde{\tau} := p^m\tau$. By Theorem 5.4.2 and Lemma 5.3.4, we have the following commutative diagram:

$$\begin{tikzcd} \mathcal{M} \otimes_S A_{\mathrm{cris}} \arrow[r, "\iota"] \arrow[d, "\tilde{\tau}"] & T^\vee \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}} \arrow[d, "p^m\tau"] & \\ \mathcal{M} \otimes_S A_{\mathrm{cris}} \arrow[r, "\iota"] & T^\vee \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}} \arrow[r, "\iota^*"] & \mathcal{M} \otimes_S A_{\mathrm{cris}}. \end{tikzcd} \tag{6.1.2}$$

It suffices to show that if $m > s_0$, then $p \mid \tilde{\tau}$. Note that $\mathcal{M} = \mathfrak{M} \otimes_{\mathfrak{S},\varphi} S$ with $\mathfrak{M} \in \mathrm{Mod}_{/\mathfrak{S}}^{r,\mathrm{fr}}$. Let $(e_1, \ldots, e_d)$ be a basis of $\mathfrak{M}$ and $\varphi_{\mathfrak{M}}(e_1, \ldots, e_d) = (e_1, \ldots, e_d)A$, where A is a $d \times d$ matrix with coefficients in $\mathfrak{S}$. Then there exists a matrix B such that $BA = AB = E(u)^r I$ (I is the identity matrix). Clearly, we can regard $e_1, \dots, e_d$ as a basis of $\mathcal{M}$. Let $(y_1, \dots, y_d) := (e_1, \dots, e_d)B$ in $\mathcal{M}$. Then

$$\begin{aligned}\varphi_{\mathcal{M}}(y_1, \dots, y_d) &= \varphi_{\mathcal{M}}(e_1, \dots, e_d)\varphi(B)\\ &= (e_1, \dots, e_d)\varphi(A)\varphi(B) = p^r (c_1)^r (e_1, \dots, e_d),\end{aligned}$$

where $c_1 = \varphi(E(u))/p$ is a unit in S. By (6.1.2), we have

$$\iota^* \circ \iota \circ \tilde{\tau}(y_1, \dots, y_d) = \iota^* \circ p^m \tau \circ \iota(y_1, \dots, y_d).$$

Write $\tilde{\tau}(y_1, \dots, y_d) = (e_1, \dots, e_d)C$ where $C = (c_{ij})_{d\times d}$ is a $d \times d$ matrix with coefficients in A_{cris}. Since $\iota^* \circ \iota = \mathrm{Id} \otimes t^r$ by Lemma 5.3.4, $t^r c_{ij} \in p^m A_{\mathrm{cris}}$ for all $i, j = 1, \dots, d$. Thus, by Lemma 6.1.3, we have

$$c_{ij} \in \sum_{i+j=m-\lambda} p^i I^{[j]}, \quad i, j = 1, \dots, d.$$

In particular, $\varphi(c_{ij}) \in p^{m-\lambda} A_{\mathrm{cris}}$ for all $i, j = 1, \dots, d$. On the other hand,

$$\begin{aligned}\big(\varphi(e_1), \dots, \varphi(e_d)\big)\varphi(C) = \varphi\big(\tilde{\tau}(y_1, \dots, y_d)\big) &= \tilde{\tau}\big(\varphi(y_1, \dots, y_d)\big)\\ &= \tilde{\tau}\big(p^r (c_1)^r (e_1, \dots, e_d)\big),\end{aligned}$$

so we have $\tilde{\tau}(e_1, \dots, e_d) \in p^{m-s_0} A_{\mathrm{cris}}$; that is, $p \mid \tilde{\tau}$. $\square$

6.2. Proof of Proposition 6.0.4

Since G_∞ acts on $\mathcal{M}$ trivially, it suffices to prove that there must exist a constant s_1 only depending on e and r such that $p^{s_1}\tau(\mathcal{M}) \subset \mathcal{M} \otimes_S \mathcal{R}$. Since T/p^nT is torsion semi-stable, there exist G-stable lattices $L'_{(n)} \subset L_{(n)}$ in semi-stable Galois representations $V_{(n)}$ with Hodge–Tate weights in $\{0, \dots, r\}$ such that $L_{(n)}/L'_{(n)} \simeq T_n$. Let $\mathfrak{L}_{(n)} \hookrightarrow \mathfrak{L}'_{(n)}$ be the injection in $\mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$ corresponding to $L'_{(n)} \subset L_{(n)}$ as $\mathbb{Z}_p[G_\infty]$-modules. We may assume that $\mathfrak{L}'_{(n)}/\mathfrak{L}_{(n)} = \mathfrak{M}_n$ as explained in § 4.4. By Theorem 3.2.2, we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L^\vee_{(n)} \otimes_{\mathbb{Z}_p} \mathfrak{S}^{\mathrm{ur}} & \longrightarrow & L'^\vee_{(n)} \otimes_{\mathbb{Z}_p} \mathfrak{S}^{\mathrm{ur}} & \longrightarrow & (T/p^nT)^\vee \otimes_{\mathbb{Z}_p} \mathfrak{S}^{\mathrm{ur}} & \longrightarrow & 0\\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathfrak{L}_{(n)} \otimes_{\mathfrak{S}} \mathfrak{S}^{\mathrm{ur}} & \longrightarrow & \mathfrak{L}'_{(n)} \otimes_{\mathfrak{S}} \mathfrak{S}^{\mathrm{ur}} & \longrightarrow & \mathfrak{M}_n \otimes_{\mathfrak{S}} \mathfrak{S}^{\mathrm{ur}} & \longrightarrow & 0 \end{array}$$

where the two rows are short exact. Tensoring the above diagram with A_{cris} via $\varphi : \mathfrak{S}^{\mathrm{ur}} \to A_{\mathrm{cris}}$, we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L^\vee_{(n)} \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}} & \longrightarrow & L'^\vee_{(n)} \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}} & \longrightarrow & (T/p^nT)^\vee \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}} & \longrightarrow & 0\\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathfrak{L}_{(n)} \otimes_{\mathfrak{S},\varphi} A_{\mathrm{cris}} & \longrightarrow & \mathfrak{L}'_{(n)} \otimes_{\mathfrak{S},\varphi} A_{\mathrm{cris}} & \longrightarrow & \mathfrak{M}_n \otimes_{\mathfrak{S},\varphi} A_{\mathrm{cris}} & \longrightarrow & 0. \end{array}$$

The injectivity of the first two columns is guaranteed by Theorem 5.4.2. Since A_{cris} is flat over $\mathbb{Z}_p$, the top row is exact, then the second row is also exact by the injectivity of the first column.

For the same reason, we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & L^{\vee}_{(n)} \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}} \arrow[r] & L'^{\vee}_{(n)} \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}} \arrow[r] & (T/p^nT)^{\vee} \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}} \arrow[r] & 0 \\
0 \arrow[r] & \mathfrak{L}_{(n)} \otimes_{\mathfrak{S},\varphi} A_{\mathrm{cris}} \arrow[r] \arrow[u, hook] & \mathfrak{L}'_{(n)} \otimes_{\mathfrak{S},\varphi} A_{\mathrm{cris}} \arrow[r] \arrow[u, hook] & \mathfrak{M}_n \otimes_{\mathfrak{S},\varphi} A_{\mathrm{cris}} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathfrak{L}_{(n)} \otimes_{\mathfrak{S},\varphi} \mathcal{R} \arrow[r] \arrow[u, hook] & \mathfrak{L}'_{(n)} \otimes_{\mathfrak{S},\varphi} \mathcal{R} \arrow[r] \arrow[u, hook] & \mathfrak{M}_n \otimes_{\mathfrak{S},\varphi} \mathcal{R} \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

and the third row is exact. By Proposition 6.1.1, $p^{s_0}\tau(\mathfrak{L}_{(n)}) \subset \mathfrak{L}_{(n)} \otimes_{\mathfrak{S},\varphi} \mathcal{R}$ and $p^{s_0}\tau(\mathfrak{L}'_{(n)}) \subset \mathfrak{L}'_{(n)} \otimes_{\mathfrak{S},\varphi} \mathcal{R}$. Then $\tilde{\tau}_n := p^{s_0}\tau$ and $\tau_n := p^{s_0}\tau$ are well defined on $\mathfrak{M}_n \otimes_{\mathfrak{S},\varphi} \mathcal{R}$ and $\mathfrak{M}_n \otimes_{\mathfrak{S},\varphi} A_{\mathrm{cris}}$, respectively. Let $\iota_n := \iota \mod p^n$, where ι is constructed in Lemma 5.3.4; we have the following commutative diagram:

$$\begin{tikzcd}
\mathfrak{M}_n \otimes_{\mathfrak{S},\varphi} \mathcal{R} \arrow[r] \arrow[d, "\tilde{\tau}_n"] & \mathfrak{M}_n \otimes_{\mathfrak{S},\varphi} A_{\mathrm{cris}} \arrow[r, "\iota_n"] \arrow[d, "\tau_n"] & (T/p^nT)^{\vee} \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}} \arrow[d, "p^{s_0}\tau"] \\
\mathfrak{M}_n \otimes_{\mathfrak{S},\varphi} \mathcal{R} \arrow[r] & \mathfrak{M}_n \otimes_{\mathfrak{S},\varphi} A_{\mathrm{cris}} \arrow[r, "\iota_n"] & (T/p^nT)^{\vee} \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}}.
\end{tikzcd} \tag{6.2.1}$$

The above diagram tells us that for all n,

$$p^{s_0}\tau\big(\iota(\mathfrak{M} \otimes_{\mathfrak{S},\varphi} \mathcal{R})\big) \subset \iota(\mathfrak{M} \otimes_{\mathfrak{S},\varphi} \mathcal{R}) + p^n(T^{\vee} \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}}).$$

Since $\mathfrak{M} \otimes_{\mathfrak{S},\varphi} \mathcal{R}$ is p-adically complete, we have

$$p^{s_0}\tau\big(\iota(\mathfrak{M} \otimes_{\mathfrak{S},\varphi} \mathcal{R})\big) \subset \iota(\mathfrak{M} \otimes_{\mathfrak{S},\varphi} \mathcal{R})$$

and therefore τ is stable on $\mathcal{M} \otimes_S B^+_{\mathrm{cris}}$ and $G(\mathcal{M}) \subset \mathcal{M} \otimes_S \mathcal{R}_{K_0}$. This proves Proposition 6.0.4.

7. G-invariants in $\mathcal{M} \otimes_S B^+_{\mathrm{st}}$

In this section, we will show $\dim_{K_0}(\mathcal{M} \otimes_S B^+_{\mathrm{st}})^G \geqslant d$, where $d = \mathrm{rank}_{\mathbb{Z}_p}(T)$, and then prove that $T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is semi-stable. Recall that under Assumption 5.2.1, we have $\hat{G} := \mathrm{Gal}(K_{\infty,p^\infty}/K) \simeq G_0 \rtimes H_K$ where $G_0 := \mathrm{Gal}(K_{\infty,p^\infty}/K_{p^\infty}) \simeq \mathbb{Z}_p(1)$ and $H_K := \mathrm{Gal}(K_{\infty,p^\infty}/K_\infty)$.

7.1. $\hat{G}$-action on $\mathcal{D} \otimes_S \mathcal{R}_{K_0}$

Recall that $\mathcal{D} := \mathcal{M} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \mathfrak{M} \otimes_{\mathfrak{S},\varphi} S_{K_0}$. By Proposition 6.0.4, we have a $\hat{G}$-action on $\mathcal{D} \otimes_S \mathcal{R}_{K_0}$ such that

(1) the action is $\mathcal{R}_{K_0}$-semi-linear, i.e., for any $x \otimes a \in \mathcal{D} \otimes_S \mathcal{R}_{K_0}$ and $g \in \hat{G}$, $g(x \otimes a) = g(x) \otimes g(a)$,
(2) the action is compatible with Frobenius, i.e., $\varphi(g(x \otimes a)) = g(\varphi(x \otimes a))$,
(3) H_K acts trivially on $\mathcal{D}$.

Let $D := \mathcal{D}/u\mathcal{D}$. Then there is a Frobenius $\varphi : D \to D$ induced by Frobenius on $\mathcal{D}$. Proposition 6.2.1.1 in [5] (also see Lemma 7.3.1) shows that there exists a unique section $s : D \to \mathcal{D}$ such that s is Frobenius equivariant, i.e., $s \circ \varphi_D = \varphi_{\mathcal{D}} \circ s$.

Thinking of D as a K_0-submodule of $\mathcal{D}$ via s, we have

PROPOSITION 7.1.1. – *Notations as above, there exists a K_0-morphism $N : D \to D$ such that*
(1) $p\varphi N = N\varphi$,
(2) *for any $g \in \hat{G}$ and $x \in D$,*

$$g(x) = \sum_{i=0}^{\infty} N^i(x) \otimes \gamma_i\bigl(-\log \underline{\epsilon}(g)\bigr), \tag{7.1.1}$$

where $\underline{\epsilon}(g) = g([\underline{\pi}])/[\underline{\pi}]$ and $\gamma_i(x) = x^i/i!$.

Note that $p\varphi N = N\varphi$ implies that N is nilpotent. Thus (7.1.1) is well defined. To prove the above proposition, we need to analyze the structure of $\mathcal{R}_{K_0}$ more carefully.

LEMMA 7.1.2. – *Let $x = \sum_{j=0}^{\infty} f_j t^{\{j\}} \in \mathcal{R}_{K_0}$ with $f_j \in S_{K_0}$ for all $j \geqslant 0$. If $x = 0$; then $f_j = 0$ for all $j \geqslant 0$.*

Proof. – Without loss of generality, we can assume that $x = \sum_{j=0}^{\infty} f_j t^{\{j\}} \in A_{\mathrm{cris}}$ with $f_j \in S$ for all j. Let f_{j_0} be the first nonzero term. For any $n \geqslant 0$,

$$0 = \varphi^n(x) = \sum_{j=j_0}^{\infty} \varphi^n(f_j)\varphi^n(t^{\{j\}}) = \sum_{j=j_0}^{\infty} \varphi^n(f_j) p^n t^{\{j\}}.$$

Note that $t \in \mathrm{Fil}^1 A_{\mathrm{cris}}$, so $\varphi^n(f_{j_0}) t^{j_0} \in \mathrm{Fil}^{j_0+1} A_{\mathrm{cris}}$ for all n. Since $t^{j_0} \in \mathrm{Fil}^{j_0} A_{\mathrm{cris}}$ and $t^{j_0} \notin \mathrm{Fil}^{j_0+1} A_{\mathrm{cris}}$, $\varphi^n(f_{j_0}) \in \mathrm{Fil}^1 S$ for all $n \geqslant 1$. We claim that this is impossible unless $f_{j_0} = 0$. In fact, write

$$f_{j_0}(u) = \sum_{i=i_0}^{\infty} w_i \frac{u^i}{e(i)!}, \quad w_i \in W(k),\ \lim_{i\to\infty} w_i = 0,\ w_{i_0} \neq 0,$$

where $i = e \cdot e(i) + r(i)$ with $0 \leqslant r(i) < e$ and w_{i_0} is the first nonzero term. $\varphi^n(f_{j_0}) \in \mathrm{Fil}^1 S$ implies that $\sigma^n(f_{j_0})(\pi^{p^n}) = 0$ where $\sigma^n(f) := \sum_{i=i_0}^{\infty} \varphi^n(w_i) \frac{u^i}{e(i)!}$. But it is easy to see that there exists n_0 such that for any $n \geqslant n_0$,

$$v\left(\frac{\pi^{ip^n}}{e(i)!}\right) > v\left(\frac{\varphi^n(w_{i_0})\pi^{i_0 p^n}}{e(i_0)!}\right) \quad \text{for all } i > i_0,$$

where $v(\cdot)$ is the valuation on $W(k)$. Thus, $v(\sigma^n(f_{j_0})(\pi^{p^n})) = v(\frac{\varphi^n(w_{i_0})\pi^{i_0 p^n}}{e(i_0)!})$, which contradicts the fact that $\sigma^n(f_{j_0})(\pi^{p^n}) = 0$. Therefore, $f_{j_0} = 0$, so $f_j = 0$ for all j. $\square$

By Lemma 7.1.2, we may regard $K_0[\![t]\!]$ and $\mathcal{R}_{K_0}$ as subrings of $K_0[\![x,y]\!]$ via $u \mapsto x$ and $t \mapsto y$. Define $\tilde{\mathcal{R}} := \mathcal{R}_{K_0} \cap K_0[\![t]\!]$. The element $x \in \tilde{\mathcal{R}}$ has the following shape:

$$x = \sum_{i=0}^{\infty} a_i t^{\{i\}}, \quad a_i \in K_0,\ a_i \to 0 \text{ as } n \to +\infty.$$

LEMMA 7.1.3. – *Notations as in Proposition* 7.1.1*,* $\hat{G}(D) \subset D \otimes_{K_0} \tilde{\mathcal{R}}$.

Proof. – Let $e_1, \ldots, e_d$ be a basis of D. Recall that τ is a topological generator of G_0. Write $\tau(e_1, \ldots, e_d) = (x_1, \ldots, x_d)A$ where A is a $d \times d$ matrix with coefficients in $\mathcal{R}_{K_0}$. For any $n \geqslant 0$, we have

$$\tau\big(\varphi^n(e_1, \ldots, e_d)\big) = \varphi^n\big(\tau(e_1, \ldots, e_d)\big) = \varphi^n(e_1, \ldots, e_d)\varphi^n(A). \tag{7.1.2}$$

Note that φ is a bijection on D, so there exists an invertible matrix B_n with coefficients in K_0 such that $\varphi^n(e_1, \ldots, e_d) = (e_1, \ldots, e_d)B_n$. Thus, comparing both sides of (7.1.2), we have $AB_n = B_n\varphi^n(A)$.

Write $A = \sum_{i=0}^{\infty} A_i t^{\{i\}}$ where A_i is a $d \times d$ matrix with coefficients in S_{K_0}. Then we have

$$\sum_{i=0}^{\infty} B_n A_i t^{\{i\}} = \sum_{i=0}^{\infty} \varphi^n(A_i) B_n p^n t^{\{i\}}.$$

By Lemma 7.1.2, $B_n A_i = p^n \varphi^n(A_i) B_n$ for all n and all i. Now we claim that all coefficients of A_i have to be in K_0. In fact, write $A_i = \sum_{j=0}^{\infty} C_j u^j$ with the C_j coefficients in K_0. Note that $\varphi^n(A_i) = \sum_{j=0}^{\infty} \varphi^n(C_j) u^{p^n j}$ and B_n is an invertible matrix with coefficients in K_0. Then we have $C_j = 0$ for all $j > 0$ by comparing the coefficients of u^j terms. □

Proof of Proposition 7.1.1. – Recall that (§5.2) $\hat{G} = G_0 \rtimes H_K$, $H_K \simeq \mathrm{Gal}(K_{p^\infty}/K) \subset \mathrm{Gal}(\mathbb{Q}_{p,p^\infty}/\mathbb{Q}_p) \simeq \mathbb{Z}_p^\times$ and $G_0 \simeq \mathbb{Z}_p(1)$. If we identify H_K with a closed subgroup of $\mathbb{Z}_p^\times$, H_K acts on G_0 via the p-adic cyclotomic character χ; that is, for any $g \in H_K$, we have $g\tau = \tau^{\chi(g)} g$. Let $(e_1, \ldots, e_d)$ be a basis of D. Write

$$\tau(e_1, \ldots, e_d) = (e_1, \ldots, e_d)A, \quad A = \sum_{i=0}^{\infty} A_i \gamma_i(t),$$

where A_i is a $d \times d$ matrix with coefficients in K_0. Then for any $g \in H_K$,

$$\begin{aligned} g\tau(e_1, \ldots, e_d) &= (e_1, \ldots, e_d) g\left(\sum_{i=0}^{\infty} A_i \gamma_i(t)\right) \\ &= (e_1, \ldots, e_d) \sum_{i=0}^{\infty} A_i \gamma_i\big(g(t)\big) \\ &= (e_1, \ldots, e_d) \sum_{i=0}^{\infty} A_i \gamma_i\big(\chi(g)t\big). \end{aligned}$$

On the other hand,

$$g\tau(e_1, \ldots, e_d) = \tau^{\chi(g)} g(e_1, \ldots, e_d) = \tau^{\chi(g)}(e_1, \ldots, e_d) = (e_1, \ldots, e_d)A^{\chi(g)}.$$

Writing $A := A(t)$, we have $A(\chi(g)t) = A(t)^{\chi(g)}$ and $\log(A(\chi(g)t)) = \chi(g)\log(A(t))$. Choosing $g \in H_K$ such that $\chi(g) \neq 1$, we have $\log(A(t)) = Nt$ for some matrix N; thus,

$$A(t) = \sum_{i=0}^{\infty} N^i \gamma_i(t) \quad \text{and} \quad \tau(e_1, \ldots, e_d) = (e_1, \ldots, e_d) \sum_{i=0}^{\infty} N^i \gamma_i(t).$$

We can then define a K_0-linear endomorphism on D by using the matrix N, which settles (7.1.1). To check that $p\varphi N = N\varphi$, note that $\varphi\tau(e_1, \ldots, e_d) = \tau\varphi(e_1, \ldots, e_d)$. We get

$$(e_1, \ldots, e_d)B\varphi\left(\sum_{i=0}^{d} N^i \gamma_i(t)\right) = (e_1, \ldots, e_d)\sum_{i=0}^{d} N^i \gamma_i(t) B,$$

where $\varphi(e_1, \ldots, e_d) = (e_1, \ldots, e_d)B$. Therefore, $pBN = NB$ and we have shown that $p\varphi N = N\varphi$. $\square$

7.2. The proof of Conjecture 1.0.1 for semi-stable representations

We now calculate $\dim_{K_0}(\mathcal{M} \otimes_S B_{\mathrm{st}}^+)^G$. Recall that $\mathfrak{u} = \log([\underline{\pi}]) \in B_{\mathrm{st}}^+$ and for any $g \in G$, $g(\mathfrak{u}) - \mathfrak{u} = \log(\underline{\epsilon}(g))$. Consider the K_0-vector space

$$\bar{D} := \left\{ \sum_{i=0}^{\infty} N^i(y) \otimes \gamma_i(\mathfrak{u}) \in \mathcal{M} \otimes_S B_{\mathrm{st}}^+ \mid y \in D \right\}.$$

It is easy to see that $\dim_{K_0}(\bar{D}) = \dim_{K_0}(D) = d$. We claim that $\bar{D} \subset (\mathcal{M} \otimes_S B_{\mathrm{st}}^+)^G$. In fact, since G_∞ acts on $\mathfrak{u}$ and D trivially, it suffices to check that $\tau(x) = x$ for any $x = \sum_{i \geqslant 0} N^i(y) \otimes \gamma_i(\mathfrak{u}) \in \bar{D}$ with $y \in D$. We have

$$\begin{aligned}
\tau(x) &= \sum_{i=0}^{\infty} N^i\big(\tau(y)\big) \otimes \gamma_i\big(\tau(\mathfrak{u})\big) \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} N^{i+j}(y) \otimes \gamma_j\big(-\log\big(\underline{\epsilon}(\tau)\big)\big) \cdot \gamma_i\big(\log\big(\underline{\epsilon}(\tau)\big) + \mathfrak{u}\big) \\
&= \sum_{i=0}^{\infty} N^i(y) \otimes \sum_{j+l=i} \gamma_j(t)\gamma_l(-t + \mathfrak{u}) \\
&= \sum_{i=0}^{\infty} N^i(y) \otimes \gamma_i(\mathfrak{u}) \\
&= x.
\end{aligned}$$

Therefore,

$$\dim_{K_0}(T^\vee \otimes_{\mathbb{Z}_p} B_{\mathrm{st}}^+)^G \geqslant \dim_{K_0}(\mathcal{M} \otimes_S B_{\mathrm{st}}^+)^G \geqslant \dim_{K_0}(\bar{D}) = d.$$

Thus, $T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is semi-stable.

7.3. The case of crystalline representations

In this subsection, we give the proof of Conjecture 1.0.1 for crystalline representations. Though the arguments above have already shown that $T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ has to be semi-stable provided that T/p^nT is torsion crystalline for all n, we need a further argument to prove that $T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is indeed crystalline. This is mainly due to the fact that we need more precise information from torsion representations. Use the notations of the previous subsection and further suppose that T/p^nT is torsion crystalline. Recall that there exists $\mathfrak{M} \in \mathrm{Mod}_{/\mathfrak{S}}^{r,\mathrm{fr}}$ corresponding to the representation $T|_{G_\infty}$. Let $\mathcal{M} := \mathcal{M}_{\mathfrak{S}}(\mathfrak{M})$, $\mathcal{D} = \mathcal{M} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and $M := \mathfrak{M}/u\mathfrak{M} = \mathcal{M}/u\mathcal{M}$.

LEMMA 7.3.1. – *There exists a unique section* $\eta' : D \to \mathcal{D}$ *such that* $\eta' \circ \varphi = \varphi \circ \eta'$. *Also there exists a constant* s_2 *only depending on the absolute ramification index* e *and the maximal Hodge–Tate weight* r *such that*

$$p^{s_2}\eta'(M) \subset \mathcal{M}.$$

Proof. – The case $r = 1$ was proved in an early version of [13], but is no longer included there. Here we include the details of the proof for any $r > 0$ by modifying Kisin's argument. Let $\eta_0 : M \to \mathcal{M}$ be any $W(k)$-linear section. Since $M = \mathcal{M}/u\mathcal{M} = \mathfrak{M}/u\mathfrak{M}$ and $E(u)^r$ kills $\mathfrak{M}/\varphi^*\mathfrak{M}$, we see that $p^r M \subset \varphi(M)$. Therefore,

$$(\varphi \circ \eta_0 \circ \varphi^{-1} - \eta_0)(M) \subset p^{-r}u\mathcal{M},$$

so for $i \geqslant 1$, $(\varphi^i \circ \eta_0 \circ \varphi^{-i} - \varphi^{i-1} \circ \eta_0 \circ \varphi^{1-i})(M) \subset p^{-ir}u^{p^{i-1}}\mathcal{M}$. Thus,

$$\eta' = \eta_0 + \sum_{i=1}^{\infty}(\varphi^i \circ \eta_0 \circ \varphi^{-i} - \varphi^{i-1} \circ \eta_0 \circ \varphi^{1-i}) : M \to \mathcal{M} \otimes_{W(k)} K_0$$

is a well defined map and satisfies $\eta' \circ \varphi = \varphi \circ \eta'$. Taking $s_2 = \mathrm{Max}\{ri - v(e(p^{i-1})!)\}$ where $p^{i-1} = e \cdot e(p^{i-1}) + r(p^{i-1})$ with $0 \leqslant r(p^{i-1}) < e$, we have $\eta'(M) \subset p^{-s_2}\mathcal{M}$. The uniqueness of η' will be a consequence of Lemma 7.3.3 below, which extends the uniqueness of such η' to the torsion level. $\square$

Let $\eta = p^{s_2}\eta'$. Then $\eta : M \to \mathcal{M}$ is well defined and $q \circ \eta = p^{s_2}\,\mathrm{Id}$ where $q : \mathfrak{M} \to M$ is the canonical projection. If $V = T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is crystalline, then N acts as 0 on $D := D_{\mathrm{st}}(V)$. Thus, the semi-linear G-action defined by (5.2.1) is trivial on D. Therefore, we have

LEMMA 7.3.2. – $T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ *is crystalline if and only if* $\tilde{\iota} \circ \eta(M) \subset (T^\vee \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}})^G$, *where* $\tilde{\iota} : \mathcal{M} \hookrightarrow T^\vee \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}}$ *is the composite of the embedding* $\mathcal{M} \to \mathcal{M} \otimes_S A_{\mathrm{cris}}$ *and* $\iota : \mathcal{M} \otimes_S A_{\mathrm{cris}} \to T^\vee \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}}$ *is constructed in Lemma* 5.3.4.

Since T/p^nT is torsion crystalline, as in the beginning of §6.2 we have a short exact sequence in $\mathrm{Mod}^r_{/\mathfrak{S}}$

$$0 \to \mathfrak{L}_{(n)} \to \mathfrak{L}'_{(n)} \to \mathfrak{M}_n \to 0$$

corresponding to the short exact sequence of $\mathbb{Z}_p[G]$-modules

$$0 \to L^\vee_{(n)} \to L'^\vee_{(n)} \to (T/p^nT)^\vee \to 0.$$

Let $\mathcal{L}_{(n)} := \mathcal{M}_{\mathfrak{S}}(\mathfrak{L}_{(n)})$, $\mathcal{L}'_{(n)} := \mathcal{M}_{\mathfrak{S}}(\mathfrak{L}'_{(n)})$, $\mathcal{M}_n := \mathcal{M}/p^n\mathcal{M}$ and $M_n := \mathcal{M}_n/u\mathcal{M}_n$. We then have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{L}_{(n)} \arrow[r] & \mathcal{L}'_{(n)} \arrow[r] & \mathcal{M}_n \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{L}_{(n)}/u\mathcal{L}_{(n)} \arrow[r] \arrow[u, hook, "\eta_{\mathcal{L}_{(n)}}"] & \mathcal{L}'_{(n)}/u\mathcal{L}'_{(n)} \arrow[r] \arrow[u, hook, "\eta_{\mathcal{L}'_{(n)}}"] & M_n \arrow[r] \arrow[u, "\tilde{\eta}_n"] & 0
\end{tikzcd}$$

where $\tilde{\eta}_n$ is induced by $\eta_{\mathcal{L}_{(n)}}$ and $\eta_{\mathcal{L}'_{(n)}}$. Note that the bottom row is short exact because $\mathcal{L}_{(n)}/u\mathcal{L}_{(n)}$ is finite $W(k)$-free. Therefore, $\tilde{\eta}_n$ is φ-equivariant and $q_n \circ \tilde{\eta}_n = p^{s_2}\,\mathrm{Id}$, where

$q_n : \mathcal{M}_n \to M_n$ is the canonical projection. Furthermore, since $L_{(n)}$ and $L'_{(n)}$ are lattices in a crystalline representation, Lemma 7.3.2 implies that

$$\tilde{\iota}_{\mathcal{L}_{(n)}} \circ \eta_{\mathcal{L}_{(n)}}(\mathcal{L}_{(n)}/u\mathcal{L}_{(n)}) \subset (L^\vee_{(n)} \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}})^G$$

and

$$\tilde{\iota}_{\mathcal{L}'_{(n)}} \circ \eta_{\mathcal{L}'_{(n)}}(\mathcal{L}'_{(n)}/u\mathcal{L}'_{(n)}) \subset (L'^\vee_{(n)} \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}})^G.$$

Hence, letting $\tilde{\iota}_n := \tilde{\iota}_{\mathcal{M}} \mod p^n$, we have

$$\tilde{\iota}_n \circ \tilde{\eta}_n(M_n) \subset \big((T/p^nT)^\vee \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}}\big)^G.$$

Now let $\eta_n := \eta \mod p^n$ where $\eta := p^{s_2}\eta' : M \to \mathcal{M}$ is constructed in Lemma 7.3.1. By Lemma 7.3.2, to prove that $T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is crystalline, it suffices to show that there exists a constant λ_2 only depending on r and e such that $p^{\lambda_2}\tilde{\eta}_n = p^{\lambda_2}\eta_n$. This is settled in the following lemma.

LEMMA 7.3.3. – *Let $\mathfrak{M}_n = \mathfrak{M}/p^n\mathfrak{M}$ with $\mathfrak{M} \in \mathrm{Mod}^{r,\mathrm{fr}}_{/\mathfrak{S}}$ finite $\mathfrak{S}$-free, $\mathcal{M} := \mathcal{M}_{\mathfrak{S}}(\mathfrak{M})$, $\mathcal{M}_n := \mathcal{M}/p^n\mathcal{M}$ and $M_n := \mathcal{M}_n/u\mathcal{M}_n$. Suppose that there exist two $W(k)$-linear morphisms $\eta_1, \eta_2 : M_n \to \mathcal{M}_n$ such that*

(1) *η_1 and η_2 are φ-equivariant, i.e., $\eta_i \circ \varphi_{M_n} = \varphi_{\mathcal{M}_n} \circ \eta_i$ for $i = 1, 2$,*

(2) *$q \circ \eta_1 = q \circ \eta_2$ where $q : \mathcal{M}_n \to M_n$ is the canonical projection.*

Then there exists a constant λ_2 depending only on e and r such that $p^{\lambda_2}(\eta_1 - \eta_2) = 0$.

Proof. – Select a basis $e_1, \dots, e_d$ of $\mathcal{M}_n$ such that $q(e_1), \dots, q(e_d)$ is a basis of M_n. Suppose that $(\eta_1 - \eta_2)(q(e_1), \dots, q(e_d)) = (e_1, \dots, e_d)A$ where A is a $d \times d$ matrix with coefficients in S. Let I be the ideal of S given by

$$I = \Big\{ \sum_{i \geqslant 1} w_i \frac{u^i}{e(i)!} \mid w_i \in W(k), w_i \to 0 \text{ as } i \to +\infty \Big\},$$

where $i = e \cdot e(i) + r(i)$ with $0 \leqslant r(i) < e$. Since $q \circ (\eta_1 - \eta_2) = 0$, all the coefficients of A belong to I. Note that $(\eta_1 - \eta_2)$ is Frobenius equivariant, so we have

$$\begin{aligned} (\eta_1 - \eta_2)\big(\varphi\big(q(e_1)\big), \dots, \varphi\big(q(e_d)\big)\big) &= \varphi\big((\eta_1 - \eta_2)\big(q(e_1), \dots, q(e_d)\big)\big) \\ &= \varphi(e_1, \dots, e_d)\varphi(A). \end{aligned} \tag{7.3.1}$$

Write $\varphi(e_1, \dots, e_d) = (e_1, \dots, e_d)X$ where X is a $d \times d$ matrix with coefficients in S. Then $\varphi(q(e_1), \dots, q(e_d)) = (q(e_1), \dots, q(e_d))X_0$, where $X_0 = X \mod I$. By (7.3.1), we get $X\varphi(A) = AX_0$. Since $\mathfrak{M}_n = \mathfrak{M}/p^n\mathfrak{M}$, by repeating the same argument as in the proof of Proposition 6.1.1, there exists a matrix Y such that $XY = p^r {c_1}^r$, where $c_1 = \varphi(E(u))/p$ a unit in S. Then we have $X_0Y_0 = (c_0p)^r \,\mathrm{Id}$ where $Y_0 := Y \mod I$ and pc_0 is the constant term of $E(u)$. Therefore,

$$X\varphi(A)Y_0 = (c_0p)^r A. \tag{7.3.2}$$

Write $A = \sum_{i=0}^{\infty} A_i \frac{u^i}{e(i)!}$ with the coefficients of the A_i in $W_n(k)$; then

$$\varphi(A) = \sum_{i=0}^{\infty} \varphi(A_i) \frac{e(pi)!}{e(i)!} \frac{u^{pi}}{e(pi)!}.$$

An easy calculation shows that $v_p(e(pi)!/e(i)!) \to +\infty$ as $i \to +\infty$ and there exists a constant i_0 depending only on e and r such that $v_p(e(pi)!/e(i)!) \geqslant r$ for all $i \geqslant i_0$. Now put $\lambda_2 = i_0 r$. To prove the lemma, it suffices to show that $p^{\lambda_2} A = 0$. We first prove that $p^{\lambda_2} A_i = 0$ for $i < i_0$. To see this, note that $A \in I$, so $A_0 = 0$. If $A_1 \neq 0$, then the lowest term in the right-hand side of (7.3.2) is $(c_0 p)^r A_1 u$, but then the lowest term in $\varphi(A)$ is $\varphi(A_1) u^p$, so $p^r A_1 = 0$. Therefore, if we repeat the same argument for the lowest term of $p^{ir} A$ for $i < i_0$, then we have $p^{\lambda_2} A_i = 0$ for $i < i_0$. Now suppose that $p^{\lambda_2} A_{i_1} \frac{u^{i_1}}{e(i_1)!}$ is the lowest term in $p^{\lambda_2} A$. Consider the lowest term in $p^{\lambda_2 - r} X \varphi(A) Y_0$. We claim that

$$p^{\lambda_2 - r} \varphi(A_i) \frac{u^{pi}}{e(i)!} = 0 \quad \text{for all } i < i_1, \tag{7.3.3}$$

so the lowest possible term of $p^{\lambda_2 - r} \varphi(A)$ is $p^{\lambda_2 - r} \varphi(A_{i_1}) \frac{u^{p i_1}}{e(i_1)!}$. Comparing the lowest term of $p^{\lambda_2} A$ with that of $p^{\lambda_2 - r} X \varphi(A) Y_0$, we see that $p^{\lambda_2} A_{i_1} = 0$, and hence $p^{\lambda_2} A = 0$. It remains to prove claim (7.3.3). We have seen that $p^{(i_0 - 1) r} A_i = 0$ for $i < i_0$, so $i_0 \leqslant i_1$. For $i_0 \leqslant i < i_1$, note that

$$p^{\lambda_2 - r} \varphi(A_i) \frac{u^{pi}}{e(i)!} = p^{(i_0 - 1) r} \frac{e(pi)!}{e(i)!} \varphi(A_i) \frac{u^{pi}}{e(i)!}.$$

By definition of i_0, we see that $v_p(p^{(i_0-1)r} \frac{e(pi)!}{e(i)!}) \geqslant i_0 r = \lambda_2$. Since $p^{\lambda_2} A_i = 0$ for all $i < i_1$, we see that $p^{(i_0-1)r} \frac{e(pi)!}{e(i)!} \varphi(A_i) = 0$. This proves the claim. □

8. The case $p = 2$

Recall that $K_\infty = \bigcup_{n \geqslant 0} K(\pi_n)$ and $K_{p^\infty} = \bigcup_{n \geqslant 0} K(\zeta_{p^n})$ with $\pi_{n+1}^p = \pi$ and ζ_{p^n} primitive p^n-th root of unity. We have proved Conjecture 1.0.1 in previous sections under Assumption 5.2.1, that is, $p \geqslant 3$ or $K_{p^\infty} \cap K_\infty = K$ if $p = 2$. In this section, we prove Conjecture 1.0.1 for $p = 2$ and we assume $p = 2$ throughout this section.

LEMMA 8.0.4. – *Let $\tilde{K} = \mathbb{Q}_2(\zeta_8) \cap K$. If $[\tilde{K} : \mathbb{Q}_2] > 1$, then $K_\infty \cap K_{2^\infty} = K$.*

Proof. – $\mathrm{Gal}(\mathbb{Q}_2(\zeta_8)/\mathbb{Q}_2) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Therefore, $\mathbb{Q}_2(\zeta_8)$ contains three quadratic extensions over $\mathbb{Q}_2$: $\mathbb{Q}_2(\sqrt{-1})$, $\mathbb{Q}_2(\sqrt{2})$ and $\mathbb{Q}_2(\sqrt{-2})$. Since $[\tilde{K} : \mathbb{Q}_2] > 1$, K must contain one of the above three quadratic extensions. If K contains $\zeta_4 = \sqrt{-1}$, then the proof of Lemma 5.1.2 in [18] (where we proved the case $p \geqslant 3$) also works here. So we may assume that $K \cap \mathbb{Q}_2(\zeta_8) = \mathbb{Q}_2(\sqrt{2})$ or $K \cap \mathbb{Q}_2(\zeta_8) = (\sqrt{-2})$. We only prove the case that $K \cap \mathbb{Q}_2(\zeta_8) = \mathbb{Q}_2(\sqrt{2})$ because another case is totally symmetric. Now we prove that $F_n := K(\pi_n) \cap K_{2^\infty} = K$ by induction on n. The case that $n = 0$ is trivial. Now assume that $F_n = K$ but $F_{n+1} \neq K$. Then $[F_{n+1} \cdot K(\pi_n) : K(\pi_n)]$ is nontrivial. So $[F_{n+1} \cdot K(\pi_n) : K(\pi_n)] = 2$ and $F_{n+1} \cdot K(\pi_n) = K(\pi_{n+1})$. Note that $F_{n+1} \cap K(\pi_n) \subset F_n = K$; we have $\mathrm{Gal}(K(\pi_{n+1})/K(\pi_n)) \simeq \mathrm{Gal}(F_{n+1}/F_n)$ and $[F_{n+1} : F_n] = [F_{n+1} : K] = 2$. Now we claim that F_{n+1} has to be $K(\zeta_8)$. Let us accept the claim for a while. Now $\zeta_8 \in \mathcal{O}_{K(\pi_{n+1})}$, we may write $\zeta_8 = a + b\pi_{n+1}$ with $a,\ b \in \mathcal{O}_{K(\pi_n)}$. Let $\sigma \in \mathrm{Gal}(K(\pi_{n+1})/K(\pi_n))$ be the nontrivial element; we have $\sigma(\zeta_8) = a + b\sigma(\pi_{n+1}) = a - b\pi_{n+1}$. Since $\mathrm{Gal}(K(\pi_{n+1})/K(\pi_n)) \simeq \mathrm{Gal}(F_{n+1}/K)$, we have $\sigma(\zeta_8) = -\zeta_8 = -a - b\pi_{n+1}$. Therefore $a = 0$ and $\zeta_8 = b\pi_{n+1}$. This contradicts that ζ_8 is a unit. Thus F_{n+1} has to be K. Now it suffices to show that $F_{n+1} = K(\zeta_8)$. Let $K' := K \cap \mathbb{Q}_{2^\infty}$ and $F = \mathbb{Q}_2(\sqrt{2})$. We claim that $K' = F$. In fact, $\mathrm{Gal}(\mathbb{Q}_{2^\infty}/F) \simeq 1 + 2\mathbb{Z}_2$ which is a procyclic 2-group. If $[K' : F] > 1$, then K' must contain $\mathbb{Q}_2(\zeta_8)$ and this contradicts the fact that $K \cap \mathbb{Q}_2(\zeta_8) = \mathbb{Q}_2(\sqrt{2})$. Thus we must have $K' = F$. Therefore, $\mathrm{Gal}(K_{2^\infty}/K) \simeq \mathrm{Gal}(\mathbb{Q}_{2^\infty}/F) \simeq 1 + 2\mathbb{Z}_2$. Since $[F_{n+1} : K] = 2$ and $F_{n+1} \subset K_{2^\infty}$, F_{n+1} must be $K(\zeta_8)$. $\square$

Now to complete the proof of Conjecture 1.0.1, we only need to consider the case that $\mathbb{Q}_2(\zeta_8) \cap K = \mathbb{Q}_2$. Let $K_1 = K(\sqrt{-1})$ and $K_2 = K(\sqrt{2})$. Clearly, $K_1 \cap K_2 = K$. Recall that T is the $\mathbb{Z}_p$-representation in Conjecture 1.0.1 and set $V := T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. From the above discussion, we see that V restricted to $\mathrm{Gal}(\bar{K}/K_1)$ and to $\mathrm{Gal}(\bar{K}/K_2)$ is semi-stable (resp. crystalline) with Hodge–Tate weights in $\{0, \dots, r\}$. Now Let $D := (V \otimes_{\mathbb{Q}_p} B_{\mathrm{st}})^{\mathrm{Gal}(\bar{K}/K(\zeta_8))}$ (resp. $D := (V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{\mathrm{Gal}(\bar{K}/K(\zeta_8))}$). Then $\dim_{K_0} D = \dim_{\mathbb{Q}_p} V$ and $\mathrm{Gal}(K(\zeta_8)/K)$ acts on D. Now it suffices to show that $\mathrm{Gal}(K(\zeta_8)/K)$ acts on D trivially. Since V is semi-stable (resp. crystalline) over K_1 and K_2, $\mathrm{Gal}(K(\zeta_8)/K_1)$ and $\mathrm{Gal}(K(\zeta_8)/K_2)$ act trivially on D. Therefore, $\mathrm{Gal}(K(\zeta_8)/K)$ acts on D trivially.

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