FIBRATIONS OF LOW GENUS, I

By Fabrizio CATANESE and Roberto PIGNATELLI

ABSTRACT. – In the present paper we consider fibrations $f : S \to B$ of an algebraic surface over a curve B, with general fibre a curve of genus g. Our main results are:
 (1) A structure theorem for such fibrations in the case where $g = 2$.
 (2) A structure theorem for such fibrations in the case where $g = 3$, the general fibre is nonhyperelliptic, and each fibre is 2-connected.
 (3) A theorem giving a complete description of the moduli space of minimal surfaces of general type with $p_g = q = 1$, $K_S^2 = 3$, showing in particular that it has four unirational connected components.
 (4) Other applications of the two structure theorems.
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RÉSUMÉ. – Dans cet article nous considérons des fibrations $f : S \to B$ d'une surface algébrique S sur une courbe B, dont la fibre générale est une courbe de genre g. Nos résultats principaux sont les suivants :
 (1) Un théorème de structure pour de telles fibrations dans le cas $g = 2$.
 (2) Un théorème de structure pour de telles fibrations dans le cas où $g = 3$, la fibre générale est non hyperelliptique, et chaque fibre est 2-connexe.
 (3) Un théorème donnant une description complète de l'espace de modules des surfaces minimales de type général avec $p_g = q = 1$, $K_S^2 = 3$, en montrant en particulier qu'il comporte quatre composantes connexes qui sont unirationnelles.
 (4) D'autres applications de ces deux théorèmes de structure.
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Contents

1. Introduction

The study of fibrations $f : S \to B$ of an algebraic surface S over a curve B lies at the heart of surface classification (cf. e.g. [20,3]).

We denote by g the genus of a general fibre; any surface birational to S is called *a ruled surface* if $g = 0$, and *an elliptic surface* if $g = 1$. Establishing the existence of such fibrations with $g = 0, 1$ plays a prominent role in the Enriques classification of algebraic surfaces.

Genus 2 fibrations play a special role for surfaces of general type: the presence of a genus 2 fibration constitutes the so-called *standard exception* to the birationality of the bicanonical map (cf. [5,13,18]).

From a birational point of view, the datum of the fibration $f : S \to B$ is equivalent to the datum of a curve C of genus g over the function field $\mathbb{C}(B)$ of B.

Apart from complications coming from the fact that $\mathbb{C}(B)$ is not algebraically closed, curves of low genus $g \leqslant 5$ are easily described (mostly as complete intersection curves in a projective space), so one can hope to construct, resp. describe, such fibrations as complete intersections in some projective bundle over the base curve B.

To do this, one needs a fixed biregular model for the birational equivalence class of $f : S \to B$. The classical approach is to consider the so called *relative minimal model*, which is unique except in the case $g = 0$ (cf. [39]), where a relative minimal model is a $\mathbb{P}^1$ bundle, but the so called elementary transformations yield nonisomorphic surfaces out of the same birational genus 0 fibration.

When $g = 0$, all the fibres of a relative minimal fibration are isomorphic to $\mathbb{P}^1$; for $g = 1$ Kodaira gave, as a preliminary tool for his deep investigations of elliptic surfaces, a short complete list of the possible fibres (cf. [30]). In the $g = 2$ case a similar but overlong list was provided by Ogg in [35]. For example, Bombieri [6] was able to use Ogg's classification to prove that the genus of the bicanonical pencil of a numerical Godeaux surface cannot be 2. In this vein, we demonstrate in this paper the power of our new methods by providing a half-page proof of Bombieri's result.

To explain our new results, we first remark that the existing literature on fibrations with low g usually divides into two lines of research:

(1) Papers devoted to the necessary numerical restrictions that must be satisfied by surfaces admitting such a fibration (see e.g. [29,41,31]).

(2) Papers devoted to proving existence or devoted to the classification of such fibrations.

For instance, the classification of surfaces with irregularity $q = 1$ leads naturally to the above issue (2), because the Albanese map of such surfaces is a genus g fibration over a curve B of genus 1. Even when restricting to the case $p_g = q = 1$, where by general results such surfaces belong to a finite number of families, the classification has turned out to be quite a hard task. The first result is quite old, and due independently to Bombieri and the first named author and to Horikawa.

THEOREM [7,29]. – *A minimal surface with $K_S^2 = 2, p_g = q = 1$ is the double cover of the symmetric product $B^{(2)}$ of the elliptic curve $B := \mathrm{Alb}(S)$, with a branch divisor Δ which belongs to a fixed linear system of 6-sections of the $\mathbb{P}^1$-bundle $B^{(2)} \to B$.*

Their moduli space is unirational of dimension 7.

Recall in fact that the symmetric product $B^{(h)}$ of an elliptic curve parametrizes effective divisors of degree h on B, and, using the group structure on B, we get the Abel–Jacobi map $\pi : B^{(h)} \to B$, associating to the divisor $(P_1, \dots, P_h)$ the point $\pi(P_1, \dots, P_h) := P_1 + \cdots + P_h$. Abel's theorem shows that π makes $B^{(h)}$ a $\mathbb{P}^{h-1}$ bundle over B, and in fact we get the projectivization of an indecomposable vector bundle, denoted $E(h, 1)$ by Atiyah [2].

Ciliberto later proposed to the first author to construct surfaces with $p_g = q = 1$ and higher K_S^2 as complete intersections in projective bundles over elliptic curves.

Using Atiyah's classification of vector bundles over elliptic curves, and the representation theory of the Heisenberg groups, it was possible to obtain the following result.

THEOREM [11,12]. – *The Albanese fibre of a minimal surface with $K_S^2 = 3$, $p_g = q = 1$ has genus $g = 2$ or $g = 3$.*

Surfaces with $g = 3$, whose canonical model is a divisor Σ in the third symmetric product $B^{(3)}$, yield a unirational connected component of the moduli space of dimension 5.

A surface for which $g = 2$ is a double cover of the symmetric product $B^{(2)}$ branched in a divisor Δ belonging to a fixed algebraic system of 6-sections, and with two 4-uple singular points on the same fibre. These surfaces exist.

Catanese and Ciliberto [11] asked whether the second family is irreducible (and conjectured somewhat overhastily that the answer should be positive). A main application of our structure theorem for genus $g = 2$ is the complete classification of the moduli space of the above surfaces. This classification, while in particular answering the above question in the negative, shows however the existence of an irreducible *main stream family* of surfaces with $K_S^2 = 3$, $p_g = q = 1$, $g = 2$.

THEOREM 6.1. – *The moduli space $\mathcal{M}$ of minimal surfaces of general type with $K_S^2 = 3$, $p_g = q = 1$ has exactly four connected components, all (irreducible) unirational of dimension 5.*

Remark 1.1. – Catanese [10] shows, as a consequence of Seiberg Witten theory, that the genus g of the Albanese fibre is a differentiable invariant of the underlying complex surface.

Thus restricting ourselves to the case $g = 2$, we may ask about the existence of surfaces with $p_g = q = 1, K_S^2 > 3$ and $g = 2$. In this case, by a result of Xiao [40] one has the inequality $K_S^2 \leqslant 6$, and [9] proved the existence of surfaces with $p_g = q = 1, K_S^2 = 4, 5$ and $g = 2$.

We wish to propose the following questions as further applications of our methods.

Problem I. – Do there exist surfaces with $p_g = q = 1, K_S^2 = 6$ and $g = 2$?

Problem II. – How many connected and irreducible components do the moduli spaces corresponding to surfaces with $p_g = q = 1, K_S^2 = 4, 5, 6$ and $g = 2$ have?

We now introduce our structure theorems for fibrations of genus 2 and 3. We hope to treat the case $g = 4$ in a sequel to the present article (motivation here came from the classification problem of numerical Godeaux surfaces, cf. [37,16]).

Since the canonical map of a curve of genus 2 is a double cover of $\mathbb{P}^1$ branched in 6 points, one of the first geometric approaches, developed by Horikawa, was to study genus 2 fibrations $f : S \to B$ through a finite double cover $\phi : Y \to \mathbb{P}$ of a $\mathbb{P}^1$ bundle $\mathbb{P}$ over B, where Y is birational to S.

There are two drawbacks here: first, Y and $\mathbb{P}$ are not uniquely determined, second, the branch curve Δ has several *nonsimple* singularities, which are usually several pairs of 4-tuple points or of $(3,3)$ points occurring on the same fibre.

Our approach uses the geometry of the bicanonical map of a 1-connected divisor of genus 2, which is a morphism generically of degree 2 onto a plane conic Q which may be reducible or nonreduced.

In this way we obtain a unique birational model X of S, admitting a finite double cover $\psi : X \to \mathcal{C}$, where

  (i) $\mathcal{C}$ is a *conic bundle* over B;

  (ii) the branch curve $\Delta_{\mathcal{A}}$ has only *simple* singularities;

  (iii) X is the *relative canonical model* of f, and is obtained contracting the (-2)-curves contained in the fibres to singularities which are then Rational Double Points.

In order to better explain the meaning of (iii) above, and since our approach ultimately provides purely algebraic structure theorems, we need to recall the algebraic definition of the *relative*

canonical algebra $\mathcal{R}(f)$ of the fibration (its local structure was investigated for $g \leqslant 3$ by Mendes Lopes in [33] and its importance for general g was stressed in [38]).

To state our two main results we need some terminology: we consider the canonical ring of any fibre F_t of $f \colon S \to B$

$$\mathcal{R}(F_t) := \bigoplus_{n=0}^{\infty} V_n(t) := \bigoplus_{n=0}^{\infty} H^0\big(F_t, \mathcal{O}_{F_t}(nK_{F_t})\big),$$

where K denotes the canonical divisor. These vector spaces fit together, yielding vector bundles V_n and the relative canonical algebra $\mathcal{R}(f)$ on B, defined as follows:

$$\mathcal{R}(f) := \bigoplus_{n=0}^{\infty} V_n := \bigoplus_{n=0}^{\infty} f_*\big(\mathcal{O}_S\big(n\big(K_S - f^*(K_B)\big)\big)\big).$$

Write $\sigma_n \colon \mathrm{Sym}^n(V_1) \to V_n$ for the multiplication map. The sheaf $\mathcal{T}_2 := \mathrm{Coker}(\sigma_2)$ introduced in [11] also plays an important role, as does the class ξ of the extension

$$0 \to \mathrm{Sym}^2(V_1) \to V_2 \to \mathcal{T}_2 \to 0;$$

we show here that $\mathcal{T}_2$ is isomorphic to the structure sheaf $\mathcal{O}_\tau$ of an effective divisor τ on the curve B.

The geometry behind the above exact sequence is that σ_2 determines a (rational) relative quadratic Veronese map $\mathbb{P}(V_1) \dashrightarrow \mathbb{P}(V_2)$ whose image is the conic bundle $\mathcal{C}$ mentioned above. We denote by $\mathcal{A}$ the relative anticanonical algebra of the conic bundle $\mathcal{C}$, and we show that $\mathcal{R}$ is a locally free $\mathcal{A}$-module, $\mathcal{R} \cong \mathcal{A} \oplus (\mathcal{A}[-3] \otimes (\det V_1 \otimes \mathcal{O}_B(\tau))$.

The final datum, denoted by w, contains the most geometric meaning: when $g = 2$ it determines the branch divisor $\Delta_{\mathcal{A}}$ (intersecting the fibres of the conic bundle $\mathcal{C}$ in 6 points), when $g = 3$ it determines a divisor Σ in the $\mathbb{P}^2$ bundle $\mathbb{P}(V_1)$ intersecting a general fibre $\cong \mathbb{P}^2$ in the quartic curve Σ_t which is the canonical image of the nonhyperelliptic fibre F_t.

Algebraically, when $g = 2$, $w \in \mathbb{P}(\mathrm{Hom}((\det V_1 \otimes \mathcal{O}_B(\tau))^2, \mathcal{A}_6))$ gives the multiplication map $(\mathcal{A}[-3] \otimes (\det V_1 \otimes \mathcal{O}_B(\tau)))^{\oplus 2} \to \mathcal{A}$, whence it determines the algebra structure of $\mathcal{R}$.

Denote by b the genus of B, and define the condition of admissibility of a 5-tuple (B, V_1, τ, ξ, w) as above as the open condition which guarantees that the singularities of the relative canonical model $X := \mathbf{Proj}(\mathcal{R})$ are Rational Double Points. We can now give the exact statements of our two structure theorems.

THEOREM 4.13. – *Let f be a relatively minimal genus 2 fibration. Then its* associated 5-tuple (B, V_1, τ, ξ, w) *is admissible.*

Vice versa, every admissible genus two 5-tuple determines a sheaf of algebras $\mathcal{R} \cong \mathcal{A} \oplus (\mathcal{A}[-3] \otimes \det(V_1) \otimes \mathcal{O}_B(\tau))$ over B whose relative projective spectrum X is the relative canonical model of a relatively minimal genus 2 fibration $f \colon S \to B$ having the above as associated 5-tuple. Moreover, the surface S has the following invariants:

$$\chi(\mathcal{O}_S) = \deg(V_1) + (b - 1),$$
$$K_S^2 = 2\deg V_1 + \deg \tau + 8(b - 1).$$

We obtain thus a bijection between (isomorphism classes of) relatively minimal genus 2 fibrations and (isomorphism classes of) associated 5-tuples, which is functorial in the sense that to a flat family of fibrations corresponds a flat family of 5-tuples.

THEOREM 7.13. – *Let f be a relatively minimal genus 3 nonhyperelliptic fibration such that every fibre is 2-connected. Then its* associated 5-tuple (B, V_1, τ, ξ, w) *is admissible.*

Vice versa, every admissible genus three 5-tuple (B, V_1, τ, ξ, w) is the associated 5-tuple of a unique genus 3 nonhyperelliptic fibration $f \colon S \to B$ with the property that every fibre is 2-connected and with invariants $\chi(\mathcal{O}_S) = \deg V_1 + 2(b-1)$, $K_S^2 = 3 \deg V_1 + \deg \tau + 16(b-1)$. As in the case of genus 2, the bijection thus obtained is functorial.

In the somewhat shorter last section we give an application of the structure theorem for $g = 3$ in a case where the base curve B is $\mathbb{P}^1$. This case is easier than the case where the genus b is higher since by Grothendieck's theorem every vector bundle on $\mathbb{P}^1$ is a direct sum of line bundles. Our theorem, establishing a new result, namely the existence of nonhyperelliptic genus 3 fibrations for some subvarieties of the moduli spaces of surfaces with $p_g = 3$, $q = 0$ and $K_S^2 = 2, 3, 4, 5$, should be viewed as a guidebook to the use of our structure theorems for the case where $B = \mathbb{P}^1$.

As already mentioned, we plan in a sequel to this paper to describe the case of hyperelliptic fibrations of genus $g = 3$ and the case of nonhyperelliptic fibrations of genus $g = 4$, giving applications to other questions of surface theory.

We also hope that our present results may be found useful in developing the arithmetic theory of curves of genus $g = 2$, resp. of genus $g = 3$.

2. Generalities on fibrations of surfaces to curves

Throughout this paper $f \colon S \to B$ will be a relatively minimal fibration of a projective complex surface S onto a smooth projective complex curve B of genus b.

This means that f has connected fibres F, and that no fibre contains an exceptional curve of the first kind.

We denote by g the arithmetic genus of F, and we shall assume $g \geqslant 2$.

For every $p \in B$, we shall denote by F_p the fibre of f over p, i.e., the divisor $f^*(p)$. The canonical divisor K_S restricts on each fibre to the dualizing sheaf ω_{F_p}; recall that, f being a fibration, $\forall p \in B$, $h^0(\mathcal{O}_{F_p}) = 1$ (see [3, Lemma III 11.1]) and $h^0(\omega_{F_p}) = \text{genus}\,(F_p) =: g$.

Assume that S' is a smooth projective surface and that $f' \colon S' \dashrightarrow B$ is a rational map onto a smooth curve B.

f' is necessarily a morphism if $b := \text{genus}(B)$ is strictly positive: when b is equal to zero, we let $\beta \colon S \to S'$ be a minimal sequence of blow ups which yields a morphism $f \colon S \to B$ and the following commutative diagram

$$
\begin{array}{ccc}
 & S & \\
{\scriptstyle\beta}\Big\downarrow & & \searrow{\scriptstyle f} \\
S' & \overset{f'}{\dashrightarrow} & B
\end{array}
$$

If S' is a minimal surface of general type, f' is "almost always" a morphism, we have in fact (cf. Kodaira's lemma in [28] and [40, Proposition 4.1]).

LEMMA 2.1. – *A relatively minimal genus 2 fibration on a non minimal surface S of general type occurs only for the canonical pencil of a minimal surface S' with $K_{S'}^2 = 1, p_g(S') = 2$.*

Although less explicit, the next result also gives rise to a finite number of families.

LEMMA 2.2. – *A relatively minimal genus g fibration on a non minimal surface S of general type can only occur if its minimal model S' has*

$$K_{S'}^2 \leqslant (2g-3)^2.$$

Proof. – We have $K_S = \beta^* K_{S'} + \sum_{i=1,\dots,r} E_i$, where each E_i is an exceptional divisor of the first kind (i.e., the total transform of the maximal ideal of a smooth point). We have $2g - 2 = K_S F = F\beta^* K_{S'} + \sum_{i=1,\dots,r} F E_i$. Let M be the corresponding pencil on S': thus $F\beta^* K_{S'} = K_{S'} M \geqslant 1$. On the other hand, $F E_i \geqslant 1$, else E_i is vertical and the fibration is not relatively minimal. Set $\beta^*(M) = F - \sum_{i=1,\dots,r} m_i E_i$, so that $m := \sum_{i=1,\dots,r} F E_i = \sum_{i=1,\dots,r} m_i \leqslant 2g - 3$.

We obtain $M^2 \leqslant m^2$, and by the index theorem follows then $K_{S'}^2 \cdot M^2 \leqslant (K_{S'} M)^2$, whence $K_{S'}^2 \leqslant (2g - 2 - m)^2 \leqslant (2g - 3)^2$. $\square$

The canonical maps of the fibres can be combined into the *relative canonical map* (see e.g. [28]), given concretely as follows: let L be a divisor on B such that $H^0(S, \mathcal{O}_S(K_S + f^*L)) \to H^0(F_p, \omega_{F_p})$ is surjective $\forall p \in B$, and consider the rational map

$$h\colon S \dashrightarrow \mathbb{P}\big(H^0\big(S, \mathcal{O}_S(K_S + f^*L)\big)\big) \times B,$$

induced by the linear system $|\mathcal{O}_S(K_S + f^*L)|$ and the projection f.

h is a birational map unless every fibre is hyperelliptic: in this case h is a double cover of a surface Y ruled over B.

Example 2.3. – If $B = \mathbb{P}^1$ and $p_g(S) = q(S) = 0$, then, setting $L := F$, we obtain $h\colon S \to \mathbb{P}^{g-1} \times \mathbb{P}^1$, and every fibre of f is mapped to $\mathbb{P}^{g-1}$ via its canonical map.

Proof. – For every $p \in \mathbb{P}^1$, the exact sequence

$$0 \to \omega_S \to \mathcal{O}_S(K_S + F) \to \omega_{F_p} \to 0$$

yields an isomorphism $H^0(\mathcal{O}_S(K_S + F)) \to H^0(\omega_{F_p})$. $\square$

The previous example is particularly relevant in the case of numerical Godeaux surfaces, to which we are interested (cf. Theorem 5.1).

Let us first collect some known results on canonical maps of Gorenstein curves (cf. [14–16, 33]).

These are based on Franchetta's definition (cf. [22,23]).

DEFINITION 2.4. – An effective divisor D on a smooth algebraic surface S is said to be *k-connected* if, whenever we write $D = A + B$ as a sum of effective divisors $A, B > 0$, then $A \cdot B \geqslant k$.

LEMMA 2.5. – *Let F be a 2-connected curve of genus $g \geqslant 2$: then the canonical map of F is a morphism.*

LEMMA 2.6. – *Let C be a curve of genus $g \geqslant 3$, let ω be the dualizing sheaf of C. Then φ_ω embeds C if and only if C is 3-connected and C is not honestly hyperelliptic (i.e., a finite double cover of $\mathbb{P}^1$ induced by the canonical morphism).*

LEMMA 2.7. – *Let C be 2-connected of genus $g = 3$ with ω ample (e.g., if C is a fibre of the relative canonical model of a fibration of a surface of general type): then either φ_ω embeds C as a plane quartic, or φ_ω is a finite double cover of a plane conic, and more precisely C is a complete intersection of type $(2, 4)$ in the weighted projective space $\mathbb{P}(1, 1, 1, 2)$.*

LEMMA 2.8. – *Let C be a 3-connected genus 4 Gorenstein curve, which is not honestly hyperelliptic: then φ_ω embeds C as a complete intersection of type $(2, 3)$.*

Let us now recall an important definition of the relative canonical algebra $\mathcal{R}(f)$ of the fibration f (cf. [11,38]).

DEFINITION 2.9. – Consider the relative dualizing sheaf

$$\omega_{S|B} := \mathcal{O}_S(K_S - f^* K_B).$$

Its self-intersection defines the number

$$K_{S|B}^2 := (\omega_{S|B})^2 = K_S^2 - 8(b-1)(g-1)$$

and we further define $\chi(S|B) := \chi(\mathcal{O}_S) - (b-1)(g-1)$. The relative canonical algebra $\mathcal{R}(f)$ is the graded algebra

$$\mathcal{R}(f) := \bigoplus_0^\infty V_n,$$

where V_n is the vector bundle on B given by $V_n := f_*(\omega_{S|B}^{\otimes n})$.

Remark 2.10. – Since we have a fibration, $V_0 = \mathcal{O}_B$, V_1 has rank g, V_n has rank $(2n-1)(g-1)$.

By a theorem of Fujita [25,26] V_n is semipositive, meaning that every rank 1 locally free quotient of it has non negative degree, and V_1 is a direct sum $\mathcal{O}_B^{q(S)-b} \oplus V_1^a \oplus V_1^0$ where V_1^a is ample and V_1^0 is a direct sum of stable degree 0 bundles $\mathcal{E}_i$ with $H^0(\mathcal{E}_i) = 0$; if rank $\mathcal{E}_i = 1$, then $\mathcal{E}_i$ is a line bundle associated to a torsion divisor in $\mathrm{Pic}^0(B)$ (cf. [42]).

Notice that, by relative duality, $R^1 f_* \omega_{S|B} = \mathcal{O}_B$, while $R^1 f_* \omega_{S|B}^{\otimes n} = 0$ for $n \geqslant 2$ because the fibration is assumed to be relatively minimal.

It then follows by Riemann–Roch that for $n \geqslant 1$

$$\chi(V_n) = \chi\big(\omega_{S|B}^{\otimes n}\big) = \frac{1}{2} n(n-1) K_{S|B}^2 + \mathrm{rank}(V_n) \cdot \chi(\mathcal{O}_B) + \chi(S|B).$$

Remark 2.11. – Fujita's theorem shows that

$$\deg(V_n) = \frac{1}{2} n(n-1) K_{S|B}^2 + \chi(S|B) \geqslant 0.$$

The Arakelov inequality

$$K_{S|B}^2 = \big[K_S^2 - 8(g-1)(b-1)\big] \geqslant 0$$

follows as a corollary, together with the inequality

$$\chi(S|B) := \big\{\chi(\mathcal{O}_S) - (g-1)(b-1)\big\} \geqslant 0.$$

Moreover (cf. [21]) V_n is ample for $n \geqslant 2$ if the fibration is a nonconstant moduli fibration (i.e., the smooth fibres are not all isomorphic), and in this case

$$K_{S|B}^2 = \big[K_S^2 - 8(g-1)(b-1)\big] > 0.$$

Moreover, in the inequality

$$\chi(\mathcal{O}_S) - (g-1)(b-1) \geqslant 0,$$

equality holds if and only if f is a holomorphic bundle (by Noether's formula, cf. [4]).

Finally, we have the inequality $q(S) \leqslant b + g$, equality holding if and only if f is a product fibration $F \times B \to B$ (see [4]).

3. Invariants of the relative canonical algebra

Consider now the relative canonical algebra

$$\mathcal{R}(f) = \bigoplus_0^\infty V_n = \bigoplus_0^\infty f_*\big(\omega_{S|B}^{\otimes n}\big).$$

DEFINITION 3.1. – Denote by $\mu_{n,m} : V_n \otimes V_m \to V_{n+m}$, respectively by

$$\sigma_n : S^n(V_1) := \mathrm{Sym}^n(V_1) = S^n\big(f_*(\omega_{S|B})\big) \xrightarrow{\sigma_n} V_n = f_*\big(\omega_{S|B}^{\otimes n}\big),$$

the homomorphisms induced by multiplication.

We define $\mathcal{L}_n := \ker \sigma_n$ and $\mathcal{T}_n := \mathrm{coker}\, \sigma_n$.

Remark 3.2. – By Noether's theorem on canonical curves, $\mathcal{T}_n$ is a torsion sheaf if the general fibre of f is nonhyperelliptic, whereas more generally, if the general fibre is nonhyperelliptic, $\mathrm{coker}\, \mu_{n,m}$ is a torsion sheaf as long as $n, m \geqslant 1$.

A more precise result was recently proved by K. Konno and M. Franciosi [32,24]: if every fibre is 1-connected (equivalently, there is no multiple fibre), $\mathrm{coker}\, \mu_{n,m} = 0$ if $g \geqslant 2$, $n, m \geqslant 2, \max\{n, m\} \geqslant 3$.

When there are no multiple fibres, the relative canonical algebra is generated by elements of degree $\leqslant 3$, and in general it is generated by elements of degree $\leqslant 4$.

The previous remark shows that the two cases
  (I) the general fibre is nonhyperelliptic;
  (II) all the fibres are hyperelliptic
should be treated separately.

In the hyperelliptic case, one has the following useful method, of splitting the relative canonical algebra into the invariant, resp. anti-invariant part. Assume in fact that a general fibre is hyperelliptic: then there is a birational involution σ on S, whence also on S'. Assuming that S is not birationally ruled, σ acts biregularly on S'. Moreover, since σ preserves the rational map f', it preserves its indeterminacy locus, therefore the minimal sequence of blow-ups turning f' into a morphism is σ-equivariant, hence one concludes that σ acts biregularly on the fibration $f : S \to B$ (and trivially on the base B).

Therefore, each open set $U = f^{-1}U'$ is σ-invariant and σ acts linearly on the space of sections $\mathcal{O}_S(U, \omega_{S/B}^n)$, which splits as the direct sum of the $(+1)$-eigenspace and the (-1)-eigenspace.

Accordingly, we get direct sums:

$$V_n = V_n^+ \oplus V_n^- = f_*\big(\omega_{S|B}^{\otimes n}\big)^+ \oplus f_*\big(\omega_{S|B}^{\otimes n}\big)^-$$

and we can split the relative canonical algebra as:

$$\mathcal{R}(f) = \mathcal{R}(f)^+ \oplus \mathcal{R}(f)^-,$$

where we now observe that $\mathcal{R}(f)^+$ is a subalgebra and $\mathcal{R}(f)^-$ is an $\mathcal{R}(f)^+$-module.

Remark 3.3. – Let the general fibre of the fibration f be hyperelliptic: then $V_1 = V_1^-$, and the cokernels $\mathcal{T}_n$ will be bigger than in the nonhyperelliptic case.

We have in fact the following table for the ranks of V_n^+ and V_n^- (for $n \geqslant 2$):

n	rank V_n^+	rank V_n^-
even	$n(g-1)+1$	$(n-1)(g-1)-1$
odd	$(n-1)(g-1)-1$	$n(g-1)+1$

Then the sheaf maps

$$S^n(V_1) = S^n\big(f_*(\omega_{S|B})\big) \xrightarrow{\sigma_n} V_n = f_*\big(\omega_{S|B}^{\otimes n}\big)$$

are injective iff $g = 2$ and their image lies in V_n^+ for n even and in V_n^- for n odd.

So, if we define $\mathcal{T}_n^+ = \operatorname{coker} S^n(f_*(\omega_{S|B})) \to V_n^+$ for n even, and $\mathcal{T}_n^- = \operatorname{coker} S^n(f_*(\omega_{S|B})) \to V_n^-$ for n odd, the decomposition in invariant and anti-invariant part of the cokernels $\mathcal{T}_n$ is $\mathcal{T}_n = \mathcal{T}_n^+ \oplus V_n^-$ for n even, $\mathcal{T}_n = \mathcal{T}_n^- \oplus V_n^+$ for n odd, and the sheaves $\mathcal{T}_n^\pm$ are torsion sheaves.

We end this section by observing that

$$\deg\big(\operatorname{Sym}^n(V_1)\big) = \binom{n+g-1}{g}\deg(V_1) = \binom{n+g-1}{g}\chi(S|B).$$

4. Genus 2 fibrations: the structure theorem

Before describing the building data of the relative canonical algebra of a genus 2 fibration, it is convenient to explain the underlying geometry (cf. also [11,12]).

Let $f : S \to B$ be a genus 2 fibration. The rank 2 vector bundle $V_1 := f_*\omega_{S|B}$ induces a natural factorization of f as $\pi \circ \varphi$, where $\varphi : S \dashrightarrow \mathbb{P} := \mathbb{P}(V_1) = \mathbf{Proj}(\operatorname{Sym}(V_1))$ is a rational map of degree 2, and $\pi : \mathbb{P}(V_1) \to B$ is the natural projection.

The indeterminacy locus of φ is contained in the fibres of f that are not 2-connected, i.e., split as $F_p = \mathcal{E}_1 + \mathcal{E}_2$ with $\mathcal{E}_1\mathcal{E}_2 = 1$. Then $\mathcal{E}_i^2 = -1$, $\mathcal{E}_i$ has arithmetic genus 1 and is called an *elliptic cycle*. We will see that these fibres are exactly the inverse images of the points in $\operatorname{Supp}(\mathcal{T}_2)$.

The typical example is given by a fibre consisting of two smooth elliptic curves $\mathcal{E}_1, \mathcal{E}_2$ meeting transversally in a point P'. The blow-up of P' maps isomorphically to the fibre F'' of $\mathbb{P}$ over the point $P \in B$, while the elliptic curves $\mathcal{E}_1, \mathcal{E}_2$ are contracted to two distinct points of the fibre F''. The resolution $\tilde{\varphi}$ of φ is the composition of the contraction of $\mathcal{E}_1, \mathcal{E}_2$ to two simple -2-elliptic singularities, with a finite double cover where the branch curve Δ in $\mathbb{P}$ contains the fibre and has two distinct 4-tuple points on it. More complicated fibres containing elliptic tails can produce different configurations of singularities of the branching divisor of φ: complete lists are given by Ogg [35] and by Horikawa [28].

LEMMA 4.1. – *Let $f : S \to B$ be a genus 2 fibration. Then*
(1) *$\mathcal{T}_2$ is isomorphic to the structure sheaf of an effective divisor $\tau \in \operatorname{Div}_{\geqslant 0}(B)$, supported on the points of B corresponding to the fibres of f which are not 2-connected;*
(2) *τ determines all the (torsion) sheaves $\mathcal{T}_n$ as follows*:

$$\mathcal{T}_{2n}^+ \cong \mathcal{O}_{n\tau} \oplus \left(\bigoplus_{i=1}^{n-1} \mathcal{O}_{i\tau}\right)^{\oplus 2}, \qquad \mathcal{T}_{2n+1}^- \cong \left(\bigoplus_{i=1}^{n} \mathcal{O}_{i\tau}\right)^{\oplus 2};$$

In particular $\deg \mathcal{T}_{2n}^+ = n^2 \deg \tau$ *and* $\deg \mathcal{T}_{2n+1}^- = n(n+1)\deg \tau$.

Proof. – As shown in the thesis of M. Mendes Lopes (Theorem 3.7 of [33], p. 53), there are two possibilities for the canonical ring of a genus 2 fibre:

- either the curve is honestly hyperelliptic, i.e., the graded ring is isomorphic to

$$\mathbb{C}[x_0, x_1, z]/\big(z^2 - g_6(x_0, x_1)\big),$$

where $\deg x_0 = \deg x_1 = 1$, $\deg z = 3$, $\deg g_6 = 6$,
- or the fibre is not 2-connected and the ring is isomorphic to

$$\mathbb{C}[x_0, x_1, y, z]/(Q_2, Q_6),$$

where $\deg x_0 = \deg x_1 = 1$, $\deg y = 2$, $\deg z = 3$ and

$$Q_2 := x_0^2 - \lambda x_0 x_1,$$
$$Q_6 := z^2 - y^3 - x_1^2\big(\alpha_0 y^2 + \alpha_1 x_1^4\big).$$

The first case is the one where the fibre is 2-connected.

By Remark 3.3 x_0, x_1 are anti-invariant sections, y, z are invariant and the sheaves $\mathcal{T}_{2n}^+$, $\mathcal{T}_{2n+1}^-$ are zero away from the points P whose fibres are not 2-connected.

V_2^+ is locally generated by $x_0^2, x_0 x_1, x_1^2, y$.

By flatness, if t is a uniformizing parameter for $\mathcal{O}_{B,P}$, we can lift the relation $Q_2 := x_0^2 - \lambda x_0 x_1$ to

$$\big(x_0^2 - \lambda x_0 x_1\big) + t\mu(t)y + t\big[x_0^2 \psi_0(t) + x_1^2 \psi_1(t) + x_0 x_1 \phi(t)\big] := Q_2 + t\mu(t)y + tR(x,t).$$

$\mu(t)$ is not identically 0 since x_0 and x_1 are algebraically independent on a general fibre. Therefore, by a holomorphic change of coordinates in B, we may assume $\mu(t) = t^{s-1}$ for a suitable positive integer s: we will call this integer the "multiplicity" of our special point. The above relation shows that the stalk of $\mathcal{T}_2$ at a special point P is the principal module $\mathcal{O}_{B,P}/(t^s)$ generated by the class of y.

We can also choose a lifting $Q_6(t)$ of Q_6 of the form

$$Q_6(t) := z^2 - Q_6'(x_0, x_1, y, t):$$

since Q_6 is invariant, the lifting $Q_6(t)$ must be invariant too, otherwise a nontrivial antiinvariant relation would imply that z is identically zero, absurd.

By flatness these are all the relations of the stalk of $\mathcal{R}$ at P; we leave to the reader the straightforward computation showing that $(\mathcal{T}_{2n+1}^-)_P$ equals $\bigoplus_{i=1}^{n}(\mathcal{O}_{B,P}/(t^{is}) \oplus \mathcal{O}_{B,P}/(t^{is}))$ with minimal ordered system of generators

$$\big\{x_0 x_1^{2n-2}y, x_1^{2n-1}y, \ldots, x_0 y^n, x_1 y^n\big\},$$

and $(\mathcal{T}_{2n}^+)_P$ equals $(\bigoplus_{i=1}^{n-1}(\mathcal{O}_{B,P}/(t^{is}) \oplus \mathcal{O}_{B,P}/(t^{is}))) \oplus \mathcal{O}_{B,P}/(t^{ns})$ with minimal ordered system of generators

$$\big\{x_0 x_1^{2n-3}y, x_1^{2n-2}y, \ldots, x_0 x_1 y^{n-1}, x_1^2 y^{n-1}, y^n\big\}. \qquad \square$$

We now introduce a sheaf of graded algebras, whose **Proj** yields a conic bundle $\mathcal{C}$ over B admitting the relative canonical model X as a finite double cover.

DEFINITION 4.2. – Let $\mathcal{A}$ be the graded subalgebra of $\mathcal{R}$ generated by V_1 and V_2. $\mathcal{A}_n$ denotes its graded part of degree n, and write accordingly

$$\mathcal{A} = \mathcal{A}_{\text{even}} \oplus \mathcal{A}_{\text{odd}} := \left(\bigoplus_{n=0}^{\infty} \mathcal{A}_{2n} \right) \oplus \left(\bigoplus_{n=0}^{\infty} \mathcal{A}_{2n+1} \right).$$

We decompose similarly $\mathcal{R} = \mathcal{R}_{\text{even}} \oplus \mathcal{R}_{\text{odd}}$.

LEMMA 4.3. – $\mathcal{R}$ *is isomorphic to* $\mathcal{A} \oplus (\mathcal{A}[-3] \otimes V_3^+)$ *as a graded $\mathcal{A}$-module*; *moreover,* $\mathcal{A}_{\text{even}}$ *is the invariant part of* $\mathcal{R}_{\text{even}}$ *and* $\mathcal{A}_{\text{odd}}$ *is the antiinvariant part of* $\mathcal{R}_{\text{odd}}$.

Proof. – In the proof of Lemma 4.1 we wrote the stalk of $\mathcal{R}$ at a special point P as an $\mathcal{O}_{B,P}$-algebra generated by x_0, x_1, y and z, where the x_i's are antiinvariant of degree 1, y and z are invariant with respective degrees 2 and 3. We achieve a unified treatment of both cases writing the canonical ring of a honestly hyperelliptic fibre as

$$\mathbb{C}[x_0, x_1, y, z] / \big(y, z^2 - g_6 \big) := \mathbb{C}[x_0, x_1, y, z] / (Q_2, Q_6).$$

In both cases the stalk of $\mathcal{A}$ is the subalgebra generated by x_0, x_1 and y: therefore $\mathcal{A}_{\text{even}}$ is invariant and $\mathcal{A}_{\text{odd}}$ is antiinvariant.

Locally on B we have $\mathcal{R} \cong \mathcal{O}_B[x_0, x_1, y, z]/(Q_2(t), Q_6(t))$, and since $Q_6(t) = z^2 - Q_6'(x_0, x_1, y, t)$, it follows easily that locally
(1) $\mathcal{A} \cong \mathcal{O}_B[x_0, x_1, y]/(Q_2(t))$,
(2) $\mathcal{R} \cong \mathcal{A} \oplus z\mathcal{A}$.
Since z is a local generator of V_3^+, both assertions follow. $\square$

In the next lemma we give a more explicit description of $\mathcal{A}$, showing how we can construct $\mathcal{A}$ starting from σ_2.

LEMMA 4.4. – *There are exact sequences*

(1) $\qquad 0 \to (\det V_1)^2 \otimes S^{n-2}(V_2) \xrightarrow{i_n} S^n(V_2) \to \mathcal{A}_{2n} \to 0 \quad \forall n \geqslant 2;$

(2) $\qquad V_1 \otimes (\det V_1) \otimes \mathcal{A}_{2n-2} \xrightarrow{j_n} V_1 \otimes \mathcal{A}_{2n} \to \mathcal{A}_{2n+1} \to 0 \quad \forall n \geqslant 1;$

where

$$i_n\big((x_0 \wedge x_1)^{\otimes 2} \otimes q\big) := \big(\sigma_2(x_0^2)\sigma_2(x_1^2) - \sigma_2(x_0 x_1)^2\big)q,$$
$$j_n\big(l \otimes (x_0 \wedge x_1) \otimes q\big) := x_0 \otimes \big(\sigma_2(x_1 l)q\big) - x_1 \otimes \big(\sigma_2(x_0 l)q\big).$$

Proof. – The above maps $S^n(V_2) \to \mathcal{A}_{2n}$ and $V_1 \otimes \mathcal{A}_{2n} \to \mathcal{A}_{2n+1}$, induced by the ring structure of $\mathcal{A}$, are surjective because $\mathcal{A}$ is generated in degree $\leqslant 2$ by definition. Since $\mathcal{R}_n$ and $\mathcal{A}_n$ are locally free, the respective kernels are locally free sheaves on B.

Both sequences are complexes by virtue of associativity and commutativity of multiplication in $\mathcal{R}$.

To verify exactness, we first consider the stalk at a special point $P \in \text{Supp}(\mathcal{T})$. The sheaf homomorphisms i_n, j_n and σ_2 induce linear maps $i_{n,P}$, $j_{n,P}$, $\sigma_{2,P}$ on the fibres over P of the corresponding vector bundles, and ring homomorphisms $i_{n,(P)}$, $j_{n,(P)}$, $\sigma_{2,(P)}$ on the stalks of the corresponding sheaves.

In the proof of Lemma 4.1 we wrote the relation in degree 2 of $\mathcal{A}$ as $t^s y - Q'(x_0, x_1, t)$ with $Q'(x_0, x_1, 0) = -x_0^2 + \lambda x_0 x_1$.

Therefore $u_0 := y, u_1 := \sigma_{2,(P)}(x_0 x_1)$ and $u_2 := \sigma_{2,(P)}(x_1^2)$ yield a basis of the stalk (and by restriction of the fibre) of V_2 at P, and $\sigma_{2,(P)}(x_0^2) = -t^s u_0 + \lambda u_1 + t \cdots$.

In this basis $i_{n,P}((x_0 \wedge x_1)^2 \otimes q) = (\lambda u_1 u_2 - u_1^2)q$, thus $i_{n,P}$ is clearly injective. At a general point we choose

$$u_0 := \sigma_{2,(P)}(x_0^2), \qquad u_1 := \sigma_{2,(P)}(x_0 x_1) \quad \text{and} \quad u_2 := \sigma_{2,(P)}(x_1^2)$$

as basis and since $i_{n,P}((x_0 \wedge x_1)^2 \otimes q) = (u_0 u_2 - u_1^2)q$ we derive the same conclusion.

Since i_n injects $(\det V_1)^2 \otimes S^{n-2}(V_2)$ in $S^n(V_2)$ as a saturated subbundle, the exactness of the sequence (1) follows then from the equality

$$\operatorname{rank}\big((\det V_1)^2 \otimes S^{n-2}(V_2)\big) + \operatorname{rank}(\mathcal{A}_{2n}) = \binom{n}{2} + 2n + 1$$

$$= \operatorname{rank}\big(S^n(V_2)\big).$$

To show that the image of j_n contains the kernel of the projection on $\mathcal{A}_{2n+1}$, which is locally free by our previous remark, it is enough to work on the fibres of the associated vector bundles. We identify the fibre of $\mathcal{A}_k$ with a subspace of the canonical ring of the fibre curve. We must then show that, given p_0, p_1 of even degree with $x_0 p_1 = x_1 p_0$, there exists a q with $p_i = x_i q$ ($i = 0, 1$).

This is straightforward for the fibre of a general point, because this subring is the ring $\mathbb{C}[x_0, x_1]$.

On a special fibre our subring is the ring $\mathbb{C}[x_0, x_1, y]/(x_0^2 - \lambda x_0 x_1) = (\mathbb{C}[x_0, x_1]/(x_0^2 - \lambda x_0 x_1))[y]$. We can thus replace p_0, p_1 by their coefficients in $\mathbb{C}[x_0, x_1]/(x_0^2 - \lambda x_0 x_1)$.

In this ring, if $k \geqslant 2$ is the degree of the polynomials p_i, we can find uniquely determined constants a_i, b_i with $p_0 = a_0 x_1^{k-1} x_0 + b_0 x_1^k$, $p_1 = a_1 x_1^{k-1}(x_0 - \lambda x_1) + b_1 x_1^k$. With this expression of the polynomials the condition $x_0 p_1 - x_1 p_0$ becomes $b_0 = a_0 - b_1 = 0$ and it suffices to choose $q = a_0 x_1^{k-1} + a_1 x_1^{k-2}(x_0 - \lambda x_1)$. $\quad\square$

Remark 4.5. – The above exact sequences describe the subalgebra $\mathcal{A}_{\text{even}}$ as a quotient algebra of $\operatorname{Sym}(V_2)$ and $\mathcal{A}_{\text{odd}}$ as an $\mathcal{A}_{\text{even}}$ module. The multiplication map $\mathcal{A}_{\text{odd}} \times \mathcal{A}_{\text{odd}} \to \mathcal{A}_{\text{even}}$ is induced by

$$\mu_{1,1} : V_1 \otimes V_1 \to S^2(V_1) \xrightarrow{\sigma_2} V_2.$$

Thus $\mathcal{A}$ is completely determined as an $\mathcal{O}_B$-algebra by the map $\sigma_2 : S^2(V_1) \to V_2$.

The structure of quotient algebra of $\operatorname{Sym}(V_2)$ for $\mathcal{A}_{\text{even}}$ gives a canonical embedding of $\mathbf{Proj}(\mathcal{A})$ (canonically isomorphic to $\mathbf{Proj}(\mathcal{A}_{\text{even}})$) into the $\mathbb{P}^2$-bundle $\mathbb{P}(V_2)$.

Thus $\mathcal{C} := \mathbf{Proj}(\mathcal{A})$ is a conic bundle and we define $\pi_{\mathcal{A}} : \mathcal{C} \to B$ as the restriction of the natural projection $\pi_2 : \mathbb{P}(V_2) \to B$.

Then the natural morphism $\varphi_{\mathcal{A}} : S \to \mathcal{C} = \mathbf{Proj}(\mathcal{A})$ induced by the inclusion $\mathcal{A} \subset \mathcal{R}$ yields a factorization

$$f = \pi_{\mathcal{A}} \circ \varphi_{\mathcal{A}}.$$

Since the multiplication map from $V_3^+ \otimes V_3^+$ to $\mathcal{R}_6$ has image contained in $\mathcal{A}_6$ by Lemma 4.3, the ring structure on $\mathcal{R}$ induces a map

$$\delta : (V_3^+)^2 \to \mathcal{A}_6.$$

DEFINITION 4.6. – Let P be a point in the support of τ. We have seen in Lemma 4.1 that the map $\sigma_{2,P}$ has rank 2, therefore its image gives a pencil of lines in the plane which is the fibre of $\mathbb{P}(V_2)$ over this point.

This pencil of lines has a base point. Taking all the points thus associated to the points of $\operatorname{supp}(\tau)$ we get a subset of $\mathbb{P}(V_2)$ that we will denote by $\mathcal{P}$. Note that the projection onto B maps $\mathcal{P}$ bijectively onto $\operatorname{supp}(\tau)$.

THEOREM 4.7. – $\mathcal{C} = \mathbf{Proj}(\mathcal{A}) \subset \mathbb{P}(V_2)$ *is the divisor in the linear system*

$$\left| \mathcal{O}_{\mathbb{P}(V_2)}(2) \otimes \pi_2^*(\det V_1)^{-2} \right|$$

with equation induced by the map i_2 defined in Lemma 4.4.

$\mathcal{C}$ has at most rational double points as singularities.

$\varphi_{\mathcal{A}}$ is the minimal resolution of the singularities of the relative canonical model $X :=$ $\mathbf{Proj}(\mathcal{R})$, a double cover of $\mathcal{C}$ whose branch locus consists of the set of isolated points $\mathcal{P}$ together with the divisor $\Delta_{\mathcal{A}}$ in the linear system $|\mathcal{O}_{\mathcal{C}}(3) \otimes \pi_{\mathcal{A}}^*(V_3^+)^{-2}|$ determined by δ ($\Delta_{\mathcal{A}}$ is thus disjoint from $\mathcal{P}$).

Proof. – As observed in Remark 4.5, by the exact sequence (1) $\mathcal{A}_{\text{even}}$ is the quotient of the symmetric algebra $\mathrm{Sym}(V_2)$ by the sheaf of principal ideals image of the map $i_2 : (\det V_1)^2 \to$ $S^2(V_2)$, therefore its $\mathbf{Proj}$ is a Cartier divisor $\mathcal{C}$ in the corresponding linear system.

The map $\varphi_{\mathcal{A}}$ factors through $X = \mathbf{Proj}(\mathcal{R})$. The natural map $S \to X$ is the contraction of the (-2)-curves contained in a fibre of f; the map $X \to \mathcal{C}$ is finite of degree 2 by Lemma 4.3; in particular, because X has only rational double points as singularities, also the singularities of $\mathcal{C}$ must be isolated and rational; since $\mathcal{C}$ is a divisor of the smooth 3-fold $\mathbb{P}(V_2)$ they are rational hypersurface singularities, i.e., Rational Double Points (cf. [1]).

It only remains to compute the branch locus of the double cover $\psi : X \to \mathcal{C}$.

Since the question is local, we may assume that X is the subscheme of $\mathbb{P}(1,1,2,3) \times B_0$ (B_0 an affine open set) defined by the equations

$$q(x_0, x_1, y, t) = 0, \qquad z^2 = g_6(x_0, x_1, y, t).$$

Observe that $x_0 = x_1 = y = 0$ implies then $z = 0$.

At a point where $x_i \neq 0$ we simply localize the two equations dividing them by x_i^2, respectively by x_i^6. Hence, $z = 0$ is the ramification divisor and $g_6 = 0$ is the branch locus. At the points where $x_0 = x_1 = 0$, $y = 1$ we must have a point of $\mathcal{P}$. This point is fixed for the involution $z \to -z$ and $g_6 \equiv y^3 + x_0 \phi_5 + x_1 \psi_5 \pmod{t}$ does not vanish. $\quad\square$

Theorem 4.7 shows that a genus 2 fibration is determined by a sheaf of algebras $\mathcal{A}$ constructed as in Lemma 4.4, a line bundle V_3^+ and a map from its square to $\mathcal{A}_6$. These data are first of all related to the invariants of S because of the exact sequence

$$0 \to \mathrm{Sym}^2(V_1) \to V_2 \to \mathcal{T}_2 \to 0$$

which yields (see [38]) the Noether-type inequality

$$(\text{Horikawa}) \; K_S^2 - 6(b-1) + 2\chi = \deg(\tau) \geqslant 0.$$

The following proposition shows that these data are not independent even if we do not put restrictions on the invariants of S.

PROPOSITION 4.8. –

$$V_3^+ \cong \det V_1 \otimes \mathcal{O}_B(\tau).$$

Proof. – We have decomposed the fibration f as

$$S \xrightarrow{r} X \xrightarrow{\psi} \mathcal{C} \xrightarrow{\pi_{\mathcal{A}}} B,$$

where r is a contraction of (-2)-curves to Rational Double Points, and ψ is a finite double cover corresponding to the integral quadratic extension

$$\mathcal{A} \subset \mathcal{R} \cong \mathcal{A} \oplus \big(\mathcal{A}[-3] \otimes V_3^+\big).$$

Localizing, we obtain the splitting

$$\psi_* \mathcal{O}_X \cong \mathcal{O}_{\mathcal{C}} \oplus \big(\mathcal{O}_{\mathcal{C}}(-3) \otimes \pi_{\mathcal{A}}^* \big(V_3^+\big)\big).$$

Since X and $\mathcal{C}$ have only Rational Double Points as singularities, the standard adjunction formula for finite double covers yields

$$\omega_X \cong \psi^* \big(\omega_{\mathcal{C}}(3H) \otimes \pi_{\mathcal{A}}^* \big(V_3^+\big)^{-1}\big),$$

where H is a Weil divisor on $\mathcal{C}$ corresponding to $\mathcal{O}_{\mathcal{C}}(1)$. Therefore

$$\psi^* \big(\mathcal{O}_{\mathcal{C}}(1)\big) \cong \psi^* \big(\omega_{\mathcal{C}|B}(3H) \otimes \pi_{\mathcal{A}}^* \big(V_3^+\big)^{-1}\big)$$

and since by Theorem 4.7

$$\begin{aligned}
\omega_{\mathcal{C}|B} &= \pi_{\mathcal{A}}^* \det V_2 \otimes \mathcal{O}_{\mathcal{C}}(-6) \otimes \pi_{\mathcal{A}}^* (\det V_1)^{-2} \otimes \mathcal{O}_{\mathcal{C}}(4) \\
&= \mathcal{O}_{\mathcal{C}}(-2) \otimes \pi_{\mathcal{A}}^* \big(\det V_1 \otimes \mathcal{O}_B(\tau)\big),
\end{aligned}$$

we obtain the desired isomorphism by applying first $(\psi_*)^+$ and then $(\pi_{\mathcal{A}})_*$. □

Before stating the main theorem of this section we need to give an appropriate definition.

DEFINITION 4.9. – Given a genus 2 fibration $f \colon S \to B$ we define its *associated* 5-*tuple* (B, V_1, τ, ξ, w) as follows:
- B is the base curve;
- $V_1 = f_*(\omega_{S|B})$;
- τ is the effective divisor of B whose structure sheaf is isomorphic to $\mathcal{T}_2$;
- $\xi \in \mathrm{Ext}^1_{\mathcal{O}_B}(\mathcal{O}_\tau, S^2(V_1))/\mathrm{Aut}_{\mathcal{O}_B}(\mathcal{O}_\tau)$ is the isomorphism class of the pair V_2, σ_2

$$0 \to S^2(V_1) \xrightarrow{\sigma_2} V_2 \to \mathcal{O}_\tau \to 0;$$

- setting

$$\tilde{\mathcal{A}}_6 := \mathcal{A}_6 \otimes \big(V_3^+\big)^{-2},$$

$w \in \mathbb{P}(H^0(B, \tilde{\mathcal{A}}_6)) \cong |\mathcal{O}_{\mathcal{C}}(6) \otimes \pi_{\mathcal{A}}^*(V_3^+)^{-2}|$ is the class of a section with associated divisor $\Delta_{\mathcal{A}}$.

DEFINITION 4.10. – Let B be a smooth curve, V_1, V_2 two vector bundles on B of respective ranks 2 and 3. Let $\sigma_2 \colon S^2(V_1) \to V_2$ be an injective homomorphism whose cokernel is isomorphic to the structure sheaf of an effective divisor τ on B.
We define $\tilde{\mathcal{A}}_6$ to be the vector bundle

$$(\mathrm{coker}\, i_3) \otimes \big(\det V_1 \otimes \mathcal{O}_B(\tau)\big)^{-2},$$

where the map $i_3 \colon (\det V_1)^2 \otimes V_2 \to S^3(V_2)$ is the one induced by σ_2 as in Lemma 4.4 (see exact sequence (1)).

Remark 4.11. – The fact that $\tilde{\mathcal{A}}_6$ is automatically a vector bundle follows by the assumption that the rank of σ_2 drops at most by 1, which implies that the rank of the induced map i_3 never drops: this is a straightforward computation that we leave to the reader.

The previous results and definitions allow us to introduce the building package of a genus 2 fibration:

DEFINITION 4.12. – We shall say that a 5-tuple (B, V_1, τ, ξ, w) is an *admissible genus two 5-tuple* if
- B is a smooth curve;
- V_1 is a vector bundle on B of rank 2;
- $\tau \in \mathrm{Div}^+(B)$;
- $\xi \in \mathrm{Ext}^1_{\mathcal{O}_B}(\mathcal{O}_\tau, S^2(V_1))/\mathrm{Aut}_{\mathcal{O}_B}(\mathcal{O}_\tau)$ yields a vector bundle V_2;
- $w \in \mathbb{P}(H^0(B, \tilde{\mathcal{A}}_6))$, where $\tilde{\mathcal{A}}_6$ is the vector bundle determined by ξ as in Definition 4.10,

and moreover the following (open) conditions are satisfied:
  (i) Let $\mathcal{A}$ be the sheaf of algebras determined by B, V_1, τ, ξ as in Remark 4.5; then the conic bundle $\mathcal{C} := \mathbf{Proj}(\mathcal{A})$ has at most rational double points as singularities.
 (ii) Let $\Delta_{\mathcal{A}}$ be the divisor of w on $\mathcal{C}$; then $\Delta_{\mathcal{A}}$ does not contain any point of the set $\mathcal{P}$ defined as in 4.6.
(iii) At each point of $\Delta_{\mathcal{A}}$ the germ of the double cover X of $\mathcal{C}$ branched on $\Delta_{\mathcal{A}}$ has at most rational double points as singularities.

THEOREM 4.13. – *Let f be a relatively minimal genus 2 fibration. Then its associated 5-tuple (B, V_1, τ, ξ, w) is admissible.*

Vice versa, every admissible genus two 5-tuple determines a sheaf of algebras $\mathcal{R} \cong \mathcal{A} \oplus (\mathcal{A}[-3] \otimes \det(V_1) \otimes \mathcal{O}_B(\tau))$ over B whose relative projective spectrum X is the relative canonical model of a relatively minimal genus 2 fibration $f : S \to B$ having the above as associated 5-tuple. Moreover, the surface S has the following invariants:

$$\chi(\mathcal{O}_S) = \deg(V_1) + (b - 1),$$
$$K_S^2 = 2\deg V_1 + \deg \tau + 8(b - 1).$$

We obtain thus a bijection between (isomorphism classes of) relatively minimal genus 2 fibrations and (isomorphism classes of) associated 5-tuples, which is functorial in the sense that to a flat family of fibrations corresponds a flat family of 5-tuples.

Proof. – The associated 5-tuple of a genus 2 fibration is admissible by Theorem 4.7.

Vice versa, given an admissible (genus two) 5-tuple, let $\sigma_2 : S^2(V_1) \to V_2$ be the map induced by ξ and construct a sheaf of algebras $\mathcal{A}$ as in Remark 4.5.

Finally, a representative of w provides the sheaf of modules $\mathcal{A} \oplus (\mathcal{A}[-3] \otimes \det V_1 \otimes \mathcal{O}_B(\tau))$ with the structure of a sheaf of algebras $\mathcal{R}$.

We get then by construction a finite morphism of degree 2

$$\psi : X := \mathbf{Proj}(\mathcal{R}) \to \mathcal{C} := \mathbf{Proj}(\mathcal{A})$$

whose branch locus is the union of the divisor $\Delta_{\mathcal{A}}$ and of the finite set $\mathcal{P}$, which are disjoint by condition (ii), while condition (iii) ensures that X has at most rational double points as singularities. We finally take S to be a minimal resolution of the singularities of X.

The induced map $f : X \to B$ is clearly a genus 2 fibration whose associated 5-tuple coincides with the given admissible genus two 5-tuple: the only non trivial verification, that $f_* \omega_{S|B} = V_1$, can be done by the same arguments used in Proposition 4.8. f is relatively minimal since by construction the relative canonical bundle is f-nef.

If we start from relatively minimal genus 2 fibration $f : S \to B$ we get it back from the associated 5-tuple, and the invariants are as stated because of 2.11.

Finally, given a flat family $f_T : \mathcal{S} \to \mathcal{B} \to T$ of genus g fibrations, the sheaves $\mathcal{V}_{T,n} := (f_T)_*(\omega_{\mathcal{S}|\mathcal{B}}^{\otimes n})$ are vector bundles on $\mathcal{B}$ because of the base change theorem (Theorem 12.11 of [27]). For this reason the exact sequence

$$0 \to S^2(\mathcal{V}_{T,1}) \xrightarrow{\sigma_{T,2}} \mathcal{V}_{T,2} \to \mathcal{T}_{T,2} \to 0$$

remains exact when restricted to a fibre B_t, and since the rank of $\sigma_{T,2}$ drops at most by one, $\mathcal{T}_{T,2}$ is the structure sheaf of a Cartier divisor on $\mathcal{B}$. The rest follows now easily. □

Remark 4.14. – We discuss here the open conditions in Definition 4.12.

The fibres of the conic bundle $\mathcal{C}$ over a point not in $\text{supp}(\tau)$ are smooth conics, whereas the fibres over points of $\text{supp}(\tau)$ are singular: more precisely a singular reduced conic if the kernel of the map σ_2 on the corresponding fibre is not a square tensor, a double line otherwise.

Hence $\mathcal{C}$ is smooth on the complement of the special fibres and on the nonsingular locus of $\mathcal{C}$ the only condition to be fulfilled is that $\Delta_{\mathcal{A}}$ have only simple singularities (cf. [3]).

On a reduced singular fibre there is only one singular point, namely, the point $P(b) \in \mathcal{P}$, and a local computation shows that $P(b)$ is automatically a singularity of type A_{2s+1}: we must further check that $P(b) \notin \Delta_{\mathcal{A}}$.

The situation is slightly more complicated when $\mathcal{C}$ has a double fibre F_b: one has first to check that F_b is not contained in the singular locus. If this is the case, writing the equation locally one finds two possible cases:

Case (i) $\mathcal{C}$ has only a singular point (of type D_{2s}) in $P(b)$, which, as before, must not lie in $\Delta_{\mathcal{A}}$.

Case (ii) $P(b)$ is a singularity of type A_1, and $\mathcal{C}$ has on the fibre F_b a further singular point P', still of type A_1. The branch curve $\Delta_{\mathcal{A}}$ is allowed to pass through P', and if this is the case one has to verify that the double cover X has a Rational Double Point.

We have thus also seen that the only singularities that $\mathcal{C}$ can have are of type A_{2s+1} and of type D_{2s}.

Remark 4.15. – In [28] Horikawa gave a classification of the special fibres of a genus 2 fibration. Using the local equations we have used throughout all this section it is not difficult to recognize the classification of Horikawa, and more precisely how the parameters s and λ we have introduced determine the geometry of the fibre. We do not enter in the detail of the straightforward but long computation, but we think it can be of some interest for the reader to know that the special fibre is (in the notation of [28]) of type:

$$
\begin{array}{llll}
(3) & I_{\frac{s+1}{2}} & \text{if } s \text{ is odd and} & \lambda \neq 0; \\
 & II_{\frac{s}{2}} & \text{if } s \text{ is even and} & \lambda \neq 0; \\
 & III_1 \text{ or } V & \text{if } s = 1 \text{ and} & \lambda = 0; \\
 & III_{\frac{s+1}{2}} & \text{if } s \geqslant 3 \text{ is odd and} & \lambda = 0; \\
 & IV_{\frac{s}{2}} & \text{if } s \text{ is even and} & \lambda = 0.
\end{array}
$$

5. The genus of the bicanonical pencil of a Godeaux surface

We give an application of Theorem 4.13 to the classification of Godeaux surfaces. The following theorem was already proved by Bombieri ([6], cf. also footnote (1) on p. 494 of [5]) using Ogg's list of genus 2 fibres (cf. [35]).

THEOREM 5.1. – *Let S' be a minimal numerical Godeaux surface, i.e., the minimal model of a surface with $K_{S'}^2 = 1$, $p_g(S') = 0$, and let $f : S \to \mathbb{P}^1$ be the fibration induced by the bicanonical pencil of S'. Then the genus of the fibre can only be 3 or 4.*

Proof. – We already know (cf. [16] and 2.1) that it suffices to exclude the case where $S = S'$ and $|2K_S| = |M| + \Phi$ where $|M|$ is a base point free genus 2 pencil ($K_S\Phi = M^2 = 0$, $K_S M = 2$).

Let us argue by contradiction: we have $b = 0$, $K_S^2 = \chi(\mathcal{O}_S) = 1$. By the formulae in Theorem 4.13 we conclude $\deg V_1 = 2$, $\deg \tau = 5$. By Proposition 4.8 we get $V_3^+ = \mathcal{O}_{\mathbb{P}^1}(7)$.

In particular

$$h^0\big(\mathcal{O}_S(K_S + \Phi)\big) = h^0\big(\mathcal{O}_S(3K_S - M)\big) = h^0\big(f_*\mathcal{O}_S(3K_S - M)\big)$$
$$= h^0\big(f_*\big(\omega_{S|B}^3\big) \otimes \mathcal{O}_{\mathbb{P}^1}(-7)\big)$$
$$\geqslant h^0\big(V_3^+ \otimes \mathcal{O}_{\mathbb{P}^1}(-7)\big) = h^0(\mathcal{O}_{\mathbb{P}^1}) = 1.$$

By the long cohomology exact sequence associated to

$$0 \to \mathcal{O}_S(K_S) \to \mathcal{O}_S(K_S + \Phi) \to \mathcal{O}_\Phi(K_S + \Phi) \to 0,$$

and since $p_g(S) = q(S) = 0$, we conclude that $h^0(\mathcal{O}_\Phi(K_S + \Phi)) \geqslant 1$.

On the other hand the long cohomology exact sequence associated to

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(\Phi) \to \mathcal{O}_\Phi(\Phi) \to 0,$$

implies $h^0(\mathcal{O}_\Phi(\Phi)) = 0$ since $q(S) = 0$.

Since $K_S\Phi = 0$ we know (cf. [1]) that we have an isomorphism $\mathcal{O}_\Phi(K_S) \cong \mathcal{O}_\Phi$, and we derive a contradiction from $h^0(\mathcal{O}_\Phi(K_S + \Phi)) \geqslant 1$, $h^0(\mathcal{O}_\Phi(\Phi)) = 0$. □

Observe, by the way, that the above proof does not use the full strength of Theorem 4.13.

6. Surfaces with $p_g = q = 1$ and $K^2 = 3$

Let S be a minimal surface of general type with $p_g = q = 1$. The Noether inequality and the Bogomolov–Miyaoka–Yau inequality give us $2 \leqslant K_S^2 \leqslant 9$. The Albanese map is a morphism $f : S \to B$ where B is a smooth elliptic curve.

Theorem 4.13 can be used to study the case in which the genus g of the general fibre of f is 2. Xiao's inequality [40, Theorem 2.2] in this case yields $2 \leqslant K_S^2 \leqslant 6$. In fact, [7] proved that for $K_S^2 = 2$ the genus of the Albanese fibre is 2, the surface is a double cover of the second symmetric product $B^{(2)}$ of B, and the moduli space is irreducible, generically smooth, unirational of dimension 7.

The class of surfaces of general type with $K^2 = 3$, $p_g = q = 1$ was investigated in [11,12]. [11] proved that for this class of surfaces $g = 2$ or 3, and the second case is completely classified in [12], where it is shown that the corresponding component of the moduli space is generically smooth, irreducible, unirational of dimension 5.

Ref. [11] shows that the surfaces with $p_g = q = 1$, $K^2 = 3$ and genus 2 of the Albanese fibre are double covers of $B^{(2)}$. Given $t \in B$, let D_t be the curve of divisors in $B^{(2)}$ containing t, i.e., $D_t := \{(x, t) \mid x \in B\}$, and let E_t be the fibre over t ($\cong \mathbb{P}^1$) of the Abel Jacobi map $B^{(2)} \to B$. In order to show that this class of surfaces is not empty, it was shown in loc. cit. the existence of curves $C \sim 6D - E$ (C algebraically equivalent to $6D - E$) and with exactly two ordinary triple points on a given fibre E. By taking as branch locus the union $C \cup E$, and a corresponding double cover, one obtains surfaces with $p_g = q = 1$, $K^2 = 3$ and genus 2 of the Albanese fibre. It was conjectured there (see Problem 5.5) that this family of surfaces should form an irreducible family of the moduli space: we will now disprove this conjecture using Theorem 4.13, and showing moreover that the corresponding locus of the moduli space is indeed disconnected.

THEOREM 6.1. – *Let $\mathcal{M}$ be the moduli space of the minimal surfaces of general type S with $p_g(S) = q(S) = 1$, $K_S^2 = 3$. Then $\mathcal{M}$ has 4 connected components, all irreducible and unirational of dimension 5.*

Remark 6.2. – The unirational family of dimension 5 studied in [12] is a connected component of $\mathcal{M}$ by virtue of the differentiable invariance of the genus g (cf. [10]).

Let $\mathcal{M}'$ be the (open and closed) subset of $\mathcal{M}$ corresponding to the surfaces whose Albanese fibres have genus 2; Theorem 6.1 follows then from the following:

PROPOSITION 6.3. – *$\mathcal{M}'$ has 3 connected components, all irreducible and unirational of dimension 5.*

To prove this we define a stratification of $\mathcal{M}$; we first describe in geometric terms one of the strata, that we call *of the main stream*, because it is defined by generality assumptions on the branch curve.

The conjecture of [11] was in fact related to the existence and density of this main stream family; however, we will later show that the surfaces constructed in [11] were not of the main stream!

The main stream family corresponds to double covers of $B^{(2)}$ branched on curves $C + E$ belonging to the 5-dimensional family (here B also varies) described as follows

THEOREM 6.4. – *Let $p : B^{(2)} \to B$ be the Albanese (Abel–Jacobi) map, and let E be a fixed fibre of p. Let D_t be a natural section of p given by the set of pairs (of points) on B which contain a given point $t \in B$.*

Let P_1, P_2 be general points of E and let C be a divisor algebraically equivalent to $6D - E$ having P_1, P_2 as points of multiplicity at least 3.

Then, for C general in the algebraic equivalence class, the linear system $|C|$ has dimension 1, the general curve C inside the system is irreducible and its proper transform after the blow-up of P_1 and P_2 has singularities at most double points.

Proof. – Recall (cf. e.g. [7,11,12]) that on $B^{(2)}$ the bianticanonical system $|-2K|$ is a base point free pencil of elliptic curves $\Delta_s = \{\{x, x + s\}\}$ where $s \in B$. Δ_s is an irreducible smooth elliptic curve isomorphic to B, except when s is a non trivial 2-torsion point η, and then $\Delta_\eta = 2T_\eta$, where T_η is the smooth elliptic curve $\cong B/\langle \eta \rangle$. Observe moreover that $-K$ is algebraically equivalent to $2D - E$.

CLAIM 1. – *Let C be algebraically equivalent to $6D - E$: then*

$$H^1\big(\mathcal{O}_{B^{(2)}}(C)\big) = 0.$$

Proof. – By Serre duality it suffices to show the vanishing of $H^1(\mathcal{O}_{B^{(2)}}(K - C))$.

Writing $H^1(\mathcal{O}_{B^{(2)}}(K - C)) = H^1(\mathcal{O}_{B^{(2)}}(-(C - K)))$ we infer the vanishing from the Ramanujam vanishing theorem ([36, Theorem 2, p. 48], see also [5]). In fact, we have the following algebraic equivalences: $C - K \sim (-2K) + 4D \sim \Delta_s + D_{t_1} + \cdots + D_{t_4}$ which show that indeed $C - K$ is linearly equivalent to a reduced and connected effective divisor which is not composed of an irrational pencil. $\quad\square$

By Riemann–Roch we obtain $\dim|C| = \frac{1}{2}C(C - K) - 1 = 13$, whence for any choice of distinct points $P_1, P_2 \in B^{(2)}$ we have:

(2) $\dim|C - 3P_1 - 3P_2| \geqslant 1$.

(3) Let P_3 be a third point of the fibre E, distinct from P_1, P_2; a curve in $|C - 3P_1 - 3P_2|$ passing through P_3 must contain the whole fibre E, whence the system $|C - 3P_1 - 3P_2|$ contains $E + |C - E - 2P_1 - 2P_2|$ as a linear subsystem of codimension at most 1.

(4) Observe that $\dim|C - E| = 6$ follows by the same argument as in (1). Instead, we claim that $C - 2E \sim -3K$ is empty if the linear equivalence class of C is general. In fact, if $M \sim (C - 2E)$ is an effective divisor, then $M\Delta_s = 0$, thus M consists of a fibre Δ_s of the bianticanonical pencil plus a curve T_η.

(5) Assume now that the general curve in $|C - 3P_1 - 3P_2|$ is always reducible, i.e., that generally there exists a fixed part Φ.

There are two cases:

(5.1) $\Phi \not\geqslant E$.

(5.2) $\Phi \geqslant E$.

CLAIM I. – *Case* (5.1) *is not possible.*

Proof. – If $\Phi \not\geqslant E$, by (3) the system $|C - 3P_1 - 3P_2|$ would contain a reducible curve of the form $E + \Phi + M$, where $C' := \Phi + M$ is in $|C' - 2P_1 - 2P_2|$. Now let the general pair specialize to a pair of distinct points contained in the complete intersection $\Delta_s \cap E$.

The intersection number $C'\Delta_s$ equals $\Delta_s \cdot 2D = 4$, and therefore the system $|C' - 2P_1 - 2P_2|$ contains the subsystem $\Delta_s + |C' - \Delta_s - P_1 - P_2|$ as a linear subsystem of codimension $\leqslant 1$.

Since $C' - \Delta_s \sim 2D$, and $|2D - E| = \emptyset$ for general choice of the linear equivalence classes of D, E, we see that $\dim|2D| = 2$ always, and that for general choice of the linear equivalence class of C', the subsystem $|C' - \Delta_s - P_1 - P_2|$ has dimension zero.

MAIN CLAIM. – *For special choice of* $\{P_1, P_2\} \subset \Delta_s \cap E$ *and* C, Δ_s *general,* $|C' - 2P_1 - 2P_2|$ *equals the subsystem* $\Delta_s + |C' - \Delta_s - P_1 - P_2|$. *Moreover, for a general choice of* $\{P_1, P_2\}$ *and of* C', $|C' - 2P_1 - 2P_2|$ *has dimension* 0.

Proof. – For otherwise, $|C' - 2P_1 - 2P_2|$ always has dimension $\geqslant 1$ for a special choice of the points $\{P_1, P_2\} \subset \Delta_s \cap E$, and by (4) this dimension is generally equal to 1. Varying E, Δ_s, and the linear equivalence class, we obtain an irreducible family of dimension 4 which dominates a subvariety of dimension 2 of the symmetric square of $B^{(2)}$.

Thus, it suffices to show that at some curve Z of the 4-dimensional family given by curves in $\Delta_s + G$ with $G \in |C' - \Delta_s - P_1 - P_2|$, the equisingular deformations of Z dominate the symmetric square of $B^{(2)}$ and the dimension of the fibre is at most 1. In fact we then obtain a contradiction which proves the first assertion. Since the family has dimension $\geqslant 5$, the second assertion also follows.

Let us choose Z of the form $\Delta_s + D_1 + D_2$, where $P_i \in D_i$.

Observe that D_1 and D_2 meet transversally at a point P, while $\Delta_s \cap D_i = \{P_i, Q_i\}$.

We make everything explicit, writing $E = \{(y, -y) \mid y \in B\}$ for the fibre of pairs that add to 0, and letting $P_1 = (a, -a)$, $P_2 = (b, -b)$. Set $D_1 = \{(a, x) \mid x \in B\}$, $D_2 = \{(b, z) \mid z \in B\}$, $s = 2a$. Then $\Delta_s = \{(u, u + 2a)\}$. Without loss of generality we may assume that $2a = s = 2b$. Then one computes that $Q_1 = (a, 3a)$, while $Q_2 = (b, 3b)$.

We conclude that on the elliptic curve Δ_s the divisor $P_1 + P_2$ is not linearly equivalent to $Q_1 + Q_2$ for a general choice of s, since $-a - b \neq a + b$.

Consider now the curve Z and the normal sheaf $\mathcal{N}'_Z$ of deformations that are equisingular at P_1, P_2.

Its space of global sections fits into the exact sequence:

$$0 \to H^0(\mathcal{N}'_Z)$$
$$\to H^0\big(\mathcal{N}_{D_1}(Q_1 + P)\big) \oplus H^0\big(\mathcal{N}_{D_2}(Q_2 + P)\big) \oplus H^0\big(\mathcal{O}_\Delta(Q_1 + Q_2)\big)$$
$$\to \mathbb{C}_{Q_1} \oplus \mathbb{C}_{Q_2} \oplus \mathbb{C}_P \to \dots .$$

Since the normal sheaf $\mathcal{N}_{D_i}$ has degree 1, each summand $H^0(\mathcal{N}_{D_i}(Q_i + P))$ in the middle surjects onto the direct sum $\mathbb{C}_{Q_i} \oplus \mathbb{C}_P$ of the two corresponding summands on the right. It follows that we have surjectivity at the right term, and hence that $\dim(H^0(\mathcal{N}'_Z)) = 5$.

Since we saw that $P_1 + P_2$ is not linearly equivalent to $Q_1 + Q_2$ on Δ_s, we obtain that $H^0(\mathcal{N}'_Z)$ surjects onto the direct sum of the cotangent spaces of the surface at P_1, P_2, since $H^0(\mathcal{N}_{D_1}(Q_1 + P - P_1)) \oplus H^0(\mathcal{N}_{D_2}(Q_2 + P - P_2)) \oplus H^0(\mathcal{O}_\Delta(Q_1 + Q_2 - P_1 - P_2))$ has dimension 4 and surjects onto $\mathbb{C}_{Q_1} \oplus \mathbb{C}_{Q_2} \oplus \mathbb{C}_P$.

We have therefore shown that the tangent dimension to the family is 5 at Z, the differential of the map onto the symmetric square of $B^{(2)}$ is surjective, whence the family is smooth at Z of dimension 5 and the fibre is smooth of dimension 1. $\square$

We now proceed to prove Claim I: we let the pair P_1, P_2 tend to a special pair: then the reducible curve $\Phi + M$ has as limit a curve in the linear system $|C' - 2P_1 - 2P_2| = \Delta_s + G$. By a general choice of the class of C' we obtain that $G \sim 2D$ is irreducible. It follows that, modulo exchanging the roles of Φ and M, we may assume that Φ tends to Δ_s, in particular Φ is algebraically equivalent to Δ_s and thus belongs to the antibicanonical pencil. This however shows that the points P_1, P_2 must always be special, a contradiction. $\square$

We finally observe that also case (5.2) is impossible: in this case we would have $|C - 3P_1 - 3P_2| = E + |C - E - 2P_1 - 2P_2|$ and then $\dim |C - E - 2P_1 - 2P_2|$ is always $\geqslant 1$, contradicting the Main Claim.

Hence, we have proved that the general curve C which has multiplicity at least 3 in each point P_1, P_2 is generally irreducible, and for a fixed choice of P_1, P_2 we have a linear pencil.

Observe however that the intersection number of C with E is 6, whence it follows that each point P_i is of multiplicity exactly 3, and that the fibre is not tangent. After blowing up the two points, the proper transform of C, call it $\tilde{C}$, does not intersect the proper transform of the fibre E, and moreover $\tilde{C}^2 = 6$, whence we obtain a pencil of curves $\tilde{C}$ which generically have at most double points as singularities. $\square$

We have just shown how to construct the main stream family of surfaces, i.e., the double covers for which all the choices made (elliptic curve, fibre E, two points on the fibre, linear equivalence class) can be taken general. However, for special choices, it can happen that the dimension of the corresponding linear system of curves jumps.

It is for instance clear that this happens when the points are very special, i.e., on a curve T_η, but to analyse all the possible cases, we need to make full use of the theory developed in the previous section.

We need now to recall some standard notation and results about vector bundles over elliptic curves.

DEFINITION 6.5. – Given a point u of an elliptic curve B, and integers r, d with $r > 0$, $(r, d) = 1$, we will denote by $E_u(r, d)$ (following [2]) the only indecomposable vector bundle of rank r on B with $\det E_u(r, d) = \mathcal{O}_B(u)^{\otimes d}$.

LEMMA 6.6 [11, p. 76]. – *Let S be a minimal surface of general type with $p_g(S) = q(S) = 1$ whose Albanese map $f : S \to B$ is a genus 2 fibration.*

If $K_S^2 \leqslant 3$, then there is a point $u \in B$ such that $f_*\omega_{S|B} = E_u(2,1)$.

If instead $K_S^2 \geqslant 4$, either $f_*\omega_{S|B} = E_u(2,1)$ or $f_*\omega_{S|B} = L \oplus \mathcal{O}_B(u)$ with $u \in B$ and L a non-trivial torsion line bundle.

From this lemma and Theorem 4.13 we immediately get the following

Remark 6.7. – Let S be a minimal surface of general type with $p_g(S) = q(S) = 1$ and $K_S^2 = 3$ whose Albanese map $f : S \to B$ is a genus 2 fibration. Let (B, V_1, τ, ξ, w) be the associated 5-tuple: then we may assume w.l.o.g. that $V_1 = E_{[0]}(2,1)$.

Moreover, $\deg \tau = K_S^2 - 2 = 1$, i.e., τ is a point of B.

To apply Theorem 4.13, we need to compute $S^2(V_1) = S^2(E_{[0]}(2,1))$; this is a computation based on the results in [2]: the interested reader can find the computation in [12].

Remark 6.8 (Cf. [12]). – Given an elliptic curve B, let us denote by L_i, $i \in \{1,2,3\}$ the three line bundles on B with $L_i \not\cong \mathcal{O}_B$, $L_i^2 \cong \mathcal{O}_B$. Then

$$S^2\big(E_{[0]}(2,1)\big) = \bigoplus_{i=1}^{3} L_i\big([0]\big).$$

The 4-th element ξ of the associated 5-tuple belongs to

$$\mathrm{Ext}^1_{\mathcal{O}_B}\left(\mathcal{O}_\tau, \bigoplus_{i=1}^{3} L_i\big([0]\big)\right) / \mathrm{Aut}_{\mathcal{O}_B}(\mathcal{O}_\tau).$$

Note that ξ cannot be the class of 0, since the extension must yield a vector bundle. Since τ is a point, $\mathrm{Aut}_{\mathcal{O}_B}(\mathcal{O}_\tau) \cong \mathbb{C}^*$, acting on $\mathrm{Ext}^1_{\mathcal{O}_B}(\mathcal{O}_\tau, S^2(V_1)) \setminus \{0\}$ by scalar multiplication: therefore the space to which ξ belongs is the projective space $\mathbb{P}(\mathrm{Ext}^1_{\mathcal{O}_B}(\mathcal{O}_\tau, S^2(V_1)))$.

To compute this space we fix a section $f_0 \in H^0(\mathcal{O}_B(\tau)) \setminus \{0\}$; applying the functor $\mathrm{Hom}_{\mathcal{O}_B}(\cdot, S^2(V_1))$ to the exact sequence

$$0 \to \mathcal{O}_B\big([0] - \tau\big) \xrightarrow{\cdot (-f_0)} \mathcal{O}_B\big([0]\big) \to \mathcal{O}_\tau \to 0,$$

we get isomorphisms

$$(4) \qquad \mathrm{Ext}^1_{\mathcal{O}_B}\left(\mathcal{O}_\tau, \bigoplus_{i=1}^{3} L_i\big([0]\big)\right) \cong \mathrm{Hom}_{\mathcal{O}_B}\left(\mathcal{O}_B\big([0] - \tau\big), \bigoplus_{i=1}^{3} L_i\big([0]\big)\right)$$

$$\cong H^0\left(\bigoplus_1^{3} L_i(\tau)\right) \cong \mathbb{C}^3.$$

Therefore the associated projective space is a $\mathbb{P}^2$: we have parametrized the first 4 data of our 5-tuples via a unirational parameter space of dimension 4 (a $\mathbb{P}^2$-bundle over a universal family of elliptic curves).

To complete the 5-tuple we need to compute $H^0(\tilde{\mathcal{A}}_6)$ (in terms of B, τ, ξ). First we describe V_2 explicitly as follows

Remark 6.9. – Let (f_1, f_2, f_3) be an element of $H^0(\bigoplus_1^3 L_i(\tau))$. Then the corresponding extension class ξ induced by f_0 as above yields a sheaf V_2 that is the cokernel of the map from $\mathcal{O}_B([0] - \tau)$ to $\mathcal{O}_B([0]) \oplus (\bigoplus_1^3 L_i([0]))$ induced by the matrix ${}^t(f_0, f_1, f_2, f_3)$.

In fact, by standard homological algebra we have a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_B\big([0]-\tau\big) & \xrightarrow{\cdot(-f_0)} & \mathcal{O}_B\big([0]\big) & \longrightarrow & \mathcal{O}_\tau & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \bigoplus_1^3 L_i\big([0]\big) & \xrightarrow{\sigma_2} & V_2 & \longrightarrow & \mathcal{O}_\tau & \longrightarrow & 0
\end{array}
$$

where the vertical map on the left is the one given by $^t(f_1, f_2, f_3)$. This diagram clearly induces an exact sequence

$$
(5) \qquad 0 \to \mathcal{O}_B\big([0]-\tau\big) \to \mathcal{O}_B\big([0]\big) \oplus \left(\bigoplus_1^3 L_i\big([0]\big) \right) \to V_2 \to 0.
$$

LEMMA 6.10. – V_2 is determined by the f_i's as follows:

$$(I) \quad f_i \neq 0 \; \forall i \in \{1,2,3\} \qquad \Leftrightarrow \qquad V_2\big(-[0]\big) \cong E_\tau(3,1),$$

$$(II) \quad \exists! i \in \{1,2,3\} \text{ with } f_i = 0 \quad \Leftrightarrow \quad V_2\big(-[0]\big) \cong E_{\tau_i}(2,1) \oplus L_i,$$

$$(III) \quad \exists! i \in \{1,2,3\} \text{ with } f_i \neq 0 \quad \Leftrightarrow \quad V_2\big(-[0]\big) \cong L_i(\tau) \oplus L_j \oplus L_k,$$

where the point $\tau_i \in B$ is the divisor of a non trivial section of $L_i(\tau)$, and whenever j and k appear $\{i,j,k\} = \{1,2,3\}$.

Proof. – Let m be the cardinality of the set $\{i \mid f_i = 0\}$. Note that $m \leqslant 3$ because f_0 is different from zero by assumption.

By the exact sequence (5), it follows that V_2 is a vector bundle if and only if the f_i's have no common zeroes, i.e., if and only if $m \leqslant 2$ (since the points τ_i are distinct from τ).

Tensoring the exact sequence (5) by $\mathcal{O}_B(-[0])$ and since

$$
H^1\big(\mathcal{O}_B(-\tau)\big) \xrightarrow{\cdot f_0} H^1\big(\mathcal{O}_B\big)
$$

is an isomorphism, we conclude that $H^1(V_2(-[0])) = 0$.

Further twisting the exact sequence (5) by any degree 0 line bundle L and repeating the argument, we see that $H^1(V_2(-[0]) \otimes L)$ vanishes unless $L = L_i$ for some i and $f_i = 0$: in this last case $H^1(V_2(-[0]) \otimes L) \cong \mathbb{C}$.

Therefore $V_2(-[0])$ is a vector bundle of rank 3 and determinant $\mathcal{O}_B(\tau)$ having the property that there are exactly m line bundles L of degree zero with $H^1(V_2(-[0]) \otimes L) \neq 0$.

If $V_2(-[0])$ is indecomposable, it is $E_\tau(3,1)$ by Atiyah's classification. This sheaf has trivial cohomology when twisted with any degree 0 line bundle, therefore $m = 0$: we are in case (I).

If on the contrary $V_2(-[0])$ is a sum of three line bundles, being a quotient of $\mathcal{O}_B \oplus (\bigoplus L_i)$ of degree 1, we see that two summands have degree 0: it is clear that in this case $m = 2$ and that $V_2(-[0])$ is the one described in the statement (case (III)).

Else $V_2(-[0]) = W \oplus L$ is the sum of two indecomposable vector bundles of respective ranks 2 and 1.

First we exclude the case $\deg W = 0$, $\deg L = 1$. Recall that, by Theorem 5 of [2], part (ii), for each indecomposable vector bundle W of degree zero there is exactly one line bundle L with

$H^1(W \otimes L) \neq 0$. Therefore, if W had degree 0, we would get $m = 1$. But then, if f_i is the vanishing section, L_i is a direct summand of $V_2(-[0])$, a contradiction.

Then (by semipositivity) $\deg L = 0$ and $\deg W = 1$; every twist of W by a degree 0 line bundle has trivial first cohomology group, therefore the corresponding twist of $V_2(-[0])$ has nontrivial first cohomology group if and only if we twist by L^{-1}: whence $m = 1$ and we are in case (II). $\quad\square$

The preceding result suggests to consider a stratification of our moduli space of surfaces:

DEFINITION 6.11. – We stratify $\mathcal{M}'$ as $\mathcal{M}' = \mathcal{M}_I \cup \mathcal{M}_{II} \cup \mathcal{M}_{III}$ according to the number of indecomposable summands for $V_2 = f_*(\omega^2_{S|B})$, as in Lemma 6.10.

We decompose $\mathcal{M}_I$ further as $\mathcal{M}_I^\circ \cup \mathcal{M}_{I,3}$ where $\mathcal{M}_{I,3}$ consists of the surfaces with $3\tau \equiv 3[0]$ and $\mathcal{M}_I^\circ$ is the rest.

Remark 6.12. – By a theorem of Clemens ([19], see [17] for the way it is applied) the dimension of an irreducible component of the moduli space of minimal surfaces of general type with $p_g = q = 1$ is at least $10\chi - 2K^2 + 1$, that is, 5 in our case. In particular each stratum of smaller dimension cannot contain an open set and can be disregarded for the determination of the irreducible components.

Studying the 4 strata introduced above we will find that $\mathcal{M}_{III}$ consists of two not empty unirational families of dimension 5, $\mathcal{M}_I^\circ$ is the unirational main stream family, while the other strata have smaller dimension.

Finally we will study how the closures of these strata intersect.

To simplify the exposition, we first treat together the cases in which V_2 is decomposable: $\mathcal{M}_{II}$ and $\mathcal{M}_{III}$. We need the following

Remark 6.13. – We view the natural map

$$(\det V_1)^2 \to S^2\big(S^2(V_1)\big) \quad \text{given by} \quad (x_0 \wedge x_1)^2 \to x_0^2 x_1^2 - (x_0 x_1)^2$$

(because $V_1 \cong E_{[0]}(2,1)$) as a map

$$\mathcal{O}_B\big(2[0]\big) \to \big(\mathcal{O}_B^3 \oplus L_1 \oplus L_2 \oplus L_3\big)\big(2[0]\big),$$

given through a vector with 6 entries, whose last 3 are necessarily zero. Moreover, each of the first 3 is nonzero, since the Segre image of $\mathbb{P}^1$ in $\mathbb{P}^2$ is a smooth conic, i.e., a conic of rank 3.

In particular (up to automorphisms) we can assume this vector to be the transpose of $(1,1,1,0,0,0)$.

LEMMA 6.14. – *If $V_2(-[0])$ splits as $W \oplus L$, with $L \otimes L \cong \mathcal{O}_B$ and W a vector bundle of rank 2, then*

$$\tilde{\mathcal{A}}_6 \cong \big(S^3(W) \oplus \big(S^2(W) \otimes L\big)\big)([0] - 2\tau).$$

Proof. – Applying Definition 4.10 in this case, we get that $\tilde{\mathcal{A}}_6$ is the cokernel of an injective map

$$(W \oplus L)\big([0] - 2\tau\big) \hookrightarrow \big(\big[S^3(W) \oplus \big(S^2(W) \otimes L\big)\big] \oplus \{W \oplus L\}\big)\big([0] - 2\tau\big).$$

Applying Remark 6.13 to the definition of the exact sequence (1) for $n = 2$, one easily sees that the second component of this map is the identity, thus the above sequence splits. $\quad\square$

We can now discuss the families $\mathcal{M}_{II}$ and $\mathcal{M}_{III}$ separately.

PROPOSITION 6.15. – $\mathcal{M}_{II}$ *is either empty or it has dimension* 4.

Proof. – By Lemma 6.10, in case II, ξ varies in a 1-parameter family.
By Lemma 6.14 and by the cited result of Atiyah [2]

$$\tilde{\mathcal{A}}_6 \cong E_{\tau_i}(2,1)\big([0]-\tau\big) \oplus E_{\tau_i}(2,1)\big([0]-\tau\big) \oplus \left(\left(\bigoplus_{j=1}^3 L_j\right)\big([0]-\tau\big)\right),$$

and we obtain $h^0(\tilde{\mathcal{A}}_6) = 2$ unless $\mathcal{O}_B([0]-\tau)$ is a nontrivial 2-torsion bundle: in this last case $h^0(\tilde{\mathcal{A}}_6) = 3$. Each of the two cases gives a (possibly empty) unirational family of dimension 4. $\square$

PROPOSITION 6.16. – $\mathcal{M}_{III}$ *has two connected components. Each is non-empty, unirational of dimension* 5. *For the first component* $\tau = [0]$, *whereas for the second* τ *is a 2-torsion point. In both cases the branch curve* $C = C_1 \cup C_2 \subset \mathcal{C}$ *is disconnected.*

Proof. – By Lemma 6.10 we can assume $f_j = f_k = 0$; applying Lemma 6.14 we get

$$\tilde{\mathcal{A}}_6 \cong L_i\big([0]+\tau\big) \oplus L_j\big([0]\big) \oplus L_i\big([0]-\tau\big) \oplus L_j\big([0]-2\tau\big)$$
$$\oplus L_k\big([0]\big) \oplus \mathcal{O}_B\big([0]-\tau\big) \oplus L_k\big([0]-2\tau\big).$$

Therefore either $h^0(\tilde{\mathcal{A}}_6) = 4$ or $h^0(\tilde{\mathcal{A}}_6) = 5$, the last case occurring when $\tau = [0]$ or when $\mathcal{O}_B([0]-\tau) \cong L_i$.

The decomposition $V_2(-[0]) = L_i(\tau) \oplus L_j \oplus L_k$ yields natural coordinates y_i, y_j, y_k on $\mathbb{P}(V_2)$. In these coordinates our conic bundle has equation $f_0^2 y_i^2 + y_j^2 + y_k^2$: we note that it has only one singular point (of type A_1).

Each line bundle summand of $S^3(V_2)$ corresponds to a monomial $y_i y_j y_k$. Therefore the relative cubic given by the corresponding section of $H^0(\tilde{\mathcal{A}}_6)$ is cut by a relative cubic on $\mathbb{P}(V_2)$. If $h^0(\tilde{\mathcal{A}}_6) = 4$ we see that the only monomials allowed to have nonzero coefficient are the monomials $y_i^3, y_i^2 y_j, y_i^2 y_k$: in particular the branch curve is not reduced (it contains $\{y_i = 0\}$ twice) and the generality conditions in Definition 4.12 are never fulfilled: this case does not occur.

Therefore $h^0(\tilde{\mathcal{A}}_6) = 5$ and either $\tau = [0]$ or $\mathcal{O}_B(\tau - [0]) = L_i$.

If $\mathcal{O}_B(\tau - [0]) = L_i$ we have a relative cubic of the form $a y_i^3 + b y_i^2 y_j + c y_i^2 y_k + d y_i y_j^2$, where a, b, c and d are global sections of the respective line bundles.

This cubic is clearly reducible and we write the corresponding curve on the conic bundle as $C_1 \cup C_2$ where $C_1 = \{y_i = 0\}$ and $C_2 = \{a y_i^2 + b y_i y_j + c y_i y_k + d y_j^2 = 0\}$. Here d is a section of the trivial bundle and it is nonzero (or we would get the same contradiction as in the previous case): whence $C_1 \cap C_2 = \emptyset$.

C_1 is obviously smooth, while C_2 varies in a linear system that has no fixed points on the conic bundle, therefore by Bertini's theorem its general element is smooth. If we moreover assume that a (section of $\mathcal{O}_B(2[0])$) does not vanish in $[0]$, we ensure that the curve does not pass through the singular point of the conic bundle: therefore there exists an element in $H^0(\tilde{\mathcal{A}}_6)$ fulfilling the open conditions listed in Definition 4.12.

If $[0] = \tau$ we have the relative cubic of equation $a y_i^3 + b y_i^2 y_j + c y_i^2 y_k + d y_i y_j y_k$: the proof of the existence of a 'good' cubic is identical to the previous one and again the branch curve is disconnected.

The computation of the "number of parameters" is easy: 1 parameter for B, no parameters for τ and ξ, 4 from $H^0(\tilde{\mathcal{A}}_6)$. $\square$

Remark 6.17. – The example constructed in [11] belongs to the second connected component of M_{III} ($\tau \neq [0]$). The fact that the two singular points of the branch curve in $B^{(2)}$, blown up by the rational map $\mathbb{P}(V_1) \dashrightarrow \mathcal{C}$, are contained in one of the curves T_η is equivalent to the fact that the extension class ξ is as in case III of Lemma 6.10.

This holds true because the surface is a minimal resolution of the singularities of a double cover of $B^{(2)}$ branched on

$$T_\eta + D_t + D_{t+\eta} + D_{t+\eta'} + D_{t+\eta+\eta'} + E_{2t+\eta},$$

for some $t \in B$ and a pair of distinct non trivial 2-torsion elements η, η'. This curve has in fact two quadruple points at the complete intersection $T_\eta \cap E_{2t+\eta} = \{t, t+\eta\} \cup \{t+\eta', t+\eta'+\eta\}$.

The above mentioned branch curve, after removing the fibre $E_{2t+\eta}$, stays in $|6D_0 - E_p|$ for $p = \eta - 4t$, and it is therefore given by a map $\mathcal{O}_B(p) \to S^6(E_{[0]}(2,1))$; so the branch curve in $\mathcal{C}$ comes from a map $\mathcal{O}_B(p+3\tau) \cong L_i(3[0]+\tau) \to \mathcal{A}_6$. Since $\mathcal{A}_6$ splits (following Lemma 6.14) as $(S^3(L_i(\tau) \oplus L_j) \oplus (S^2(L_i(\tau) \oplus L_j) \otimes L_k))(3[0])$, we find $L_i(3[0]+\tau)$ as a direct summand in the left summand, and this characterises the second case of Proposition 6.16.

We note now that Theorem 6.4 shows that $\mathcal{M}_I^\circ \neq \emptyset$: in fact it shows that there is a component of the moduli space where the degree 0 line bundle $\det V_1(-\tau)$ can be chosen general, whereas for the stratum $\mathcal{M}_{I,3}$ we have that $\det V_1(-\tau)$ is a 3-torsion line bundle, and we have just shown that for each nonempty irreducible component contained in $\mathcal{M}_{II} \cup \mathcal{M}_{III}$, $\det V_1(-\tau)$ is a 2-torsion line bundle. In the next proposition we analyse this stratum.

PROPOSITION 6.18. – $\mathcal{M}_I^\circ$ *forms an irreducible unirational family of dimension* 5.

Proof. – By Lemma 6.10 all f_i's are different from zero, and $V_2(-[0]) \cong E_\tau(3,1)$.
By Theorem 14′ of [2] we have that

$$S^2(E_\tau(3,1)) \oplus \Lambda^2 E_\tau(3,1) = E_\tau(3,1) \otimes E_\tau(3,1) \cong E_\tau(3,2)^{\oplus 3},$$

whence $S^2(E_\tau(3,1)) \cong E_\tau(3,2)^{\oplus 2}$, $\Lambda^2(E_\tau(3,1)) \cong E_\tau(3,2)$, and that

$$E_\tau(3,1) \otimes E_\tau(3,2) \cong \mathcal{O}_B(\tau) \oplus \left(\bigoplus_1^8 M_i(\tau) \right),$$

where the M_i's are the line bundles with $M_i^3 \cong \mathcal{O}_B$, $M_i \not\cong \mathcal{O}_B$.

Consider the Eagon–Northcott exact sequence

$$0 \to \det E_\tau(3,1) \to \Lambda^2 E_\tau(3,1) \otimes E_\tau(3,1)$$
$$\to E_\tau(3,1) \otimes S^2(E_\tau(3,1)) \to S^3(E_\tau(3,1)) \to 0.$$

We have shown that the first 3 bundles are direct sums of line bundles of degree 1: by cancellation we get

$$S^3(E_\tau(3,1)) = \left(\bigoplus_1^2 \mathcal{O}_B(\tau) \right) \oplus \left(\bigoplus_1^8 M_i(\tau) \right).$$

By Definition 4.10 we have an exact sequence

$$(6) \quad 0 \to E_\tau(3,1) \otimes \mathcal{O}_B([0]-2\tau) \to \left(\left(\bigoplus_1^2 \mathcal{O}_B \right) \oplus \left(\bigoplus_1^8 M_i \right) \right)([0]-\tau) \to \tilde{\mathcal{A}}_6 \to 0.$$

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Since $3\tau \neq [0]$, $H^0(\tilde{\mathcal{A}}_6) \cong H^1(E_\tau(3,1) \otimes \mathcal{O}_B([0] - 2\tau)) \cong \mathbb{C}^2$.

Summing up, we can parametrize $\mathcal{M}_I^\circ$ via a rational 5-dimensional family (1 parameter for w, 2 for ξ, 1 for τ and 1 for B). $\square$

We are left with the stratum $\mathcal{M}_{I,3}$, computationally more complicated. We will show that it has dimension $\leqslant 4$.

We will need the following algebraic lemma.

LEMMA 6.19. – *Let* B *be a smooth elliptic curve,* $\tau \in B$ *a point,* $V_1 = E_0(2,1)$. *Fix* $f_0 \in H^0(\mathcal{O}_B(\tau)) \setminus \{0\}$, *and* $\forall i \in \{1,2,3\}$ $f_i \in H^0(\bigoplus_1^3 L_i(\tau))$.
Let then

$$0 \to S^2(V_1) \xrightarrow{\sigma_2} V_2 \to \mathcal{O}_\tau \to 0$$

be the extension associated to (f_0, f_1, f_2, f_3), *and let* $\tilde{\mathcal{A}}_6$ *be the vector bundle associated to* σ_2 (*as in Definition* 4.10).
Then $\tilde{\mathcal{A}}_6$ *is isomorphic to the cokernel of the map* F *from*

$$(\mathcal{O}_B \oplus L_1 \oplus L_2 \oplus L_3 \oplus \mathcal{O}_B \oplus L_3 \oplus L_2 \oplus \mathcal{O}_B \oplus L_1)\big([0] - 3\tau\big)$$

to

$$(\mathcal{O}_B \oplus L_1 \oplus L_2 \oplus L_3 \oplus \mathcal{O}_B \oplus L_3 \oplus L_2 \oplus \mathcal{O}_B \oplus L_1$$
$$\oplus L_2 \oplus L_3 \oplus L_1 \oplus \mathcal{O}_B \oplus L_1 \oplus L_3 \oplus L_2)\big([0] - 2\tau\big)$$

given by the transpose of the matrix

$$M = \begin{pmatrix} f_0 & f_1 & f_2 & f_3 & & & & & & & & & & & & \\ & f_0 & & & f_1 & f_2 & f_3 & & & & & & & & & \\ & & f_0 & & & f_1 & & f_2 & f_3 & & & & & & & \\ & & & f_0 & -f_3 & & f_1 & -f_3 & f_2 & & & & & & & \\ & & & & f_0 & & & & & f_2 & f_3 & -f_1 & & -f_1 & & \\ & & & & & f_0 & & & & f_1 & & f_2 & f_3 & & & \\ & & & & & & f_0 & & & & f_1 & & f_2 & f_3 & & \\ & & & & & & & f_0 & & -f_2 & & f_1 & & & f_3 & -f_2 \\ & & & & & & & & f_0 & & & & f_1 & & f_2 & f_3 \end{pmatrix}.$$

In particular $H^0(\tilde{\mathcal{A}}_6)$ *is isomorphic to the kernel of* $H^1(F)$.

Proof. – To keep all the diagrams on the page, we denote by $\mathcal{B}$ the vector bundle $\mathrm{End}(V_1) = \mathcal{O}_B \oplus L_1 \oplus L_2 \oplus L_3$.

Our by now familiar exact sequence

$$0 \to \mathcal{O}_B\big([0] - \tau\big) \to \mathcal{B}\big([0]\big) \to V_2 \to 0$$

induces the exact sequence

$$0 \to S^2(\mathcal{B})\big(3[0] - \tau\big) \to S^3(\mathcal{B})\big(3[0]\big) \to S^3(V_2) \to 0.$$

We have the following diagram with exact rows and columns (cf. Definition 4.10)

$$
\begin{array}{ccccccccc}
& & & & & & 0 & & \\
& & & & & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_B(3[0]-\tau) & \longrightarrow & \mathcal{B}(3[0]) & \longrightarrow & \det(V_1)^2 \otimes V_2 & \longrightarrow & 0 \\
& & & & & & \downarrow {\scriptstyle i_3} & & \\
(7) \qquad 0 & \longrightarrow & S^2(\mathcal{B})(3[0]-\tau) & \longrightarrow & S^3(\mathcal{B})(3[0]) & \longrightarrow & S^3(V_2) & \longrightarrow & 0 \\
& & & & & & \downarrow & & \\
& & & & & & \tilde{\mathcal{A}}_6(2[0]+2\tau) & & \\
& & & & & & \downarrow & & \\
& & & & & & 0 & &
\end{array}
$$

Note that all the vector bundles outside the last column are direct sum of line bundles. We now give explicit 'vertical' maps

$$
\alpha : \mathcal{O}_B\big(3[0]-\tau\big) \to S^2(\mathcal{B})\big(3[0]-\tau\big), \qquad \beta : \mathcal{B}\big(3[0]\big) \to S^3(\mathcal{B})\big(3[0]\big)
$$

to be inserted in (7) to enlarge it to a bigger commutative diagram.

Recall that i_n is defined by the formula $i_n((x_0 \wedge x_1)^{\otimes 2} \otimes q) = (\sigma_2(x_0^2)\sigma_2(x_1^2) - \sigma_2(x_0 x_1)^2)q$, for $q \in S^{n-2}(V_2)$. In particular i_2 factors as

$$
(\det V_1)^2 \to S^2\big(S^2(V_1)\big) \to S^2(V_2).
$$

Remark 6.13 gave us an explicit form of the above (injective) map on the left between $\det V_1^2 \cong \mathcal{O}_B(2[0])$, and

$$
S^2\big(S^2(V_1)\big) \cong S^2\big((L_1 \oplus L_2 \oplus L_3)\big([0]\big)\big) \cong (\mathcal{O} \oplus L_3 \oplus L_2 \oplus \mathcal{O} \oplus L_1 \oplus \mathcal{O})\big(2[0]\big).
$$

It has the matrix ${}^t(1,0,0,1,0,1)$ (it is different from the previous one because now we use the lexicographic order).

We now define β as the composition of the natural maps

$$
\begin{aligned}
(\det V_1)^2 \otimes \mathcal{B}\big([0]\big) &\to S^2\big(S^2(V_1)\big) \otimes \mathcal{B}\big([0]\big) \\
&\to S^2\big(\mathcal{B}\big([0]\big)\big) \otimes \mathcal{B}\big([0]\big) \to S^3\big(\mathcal{B}\big([0]\big)\big)
\end{aligned}
$$

where the first map is the one we have just described (tensored by $\mathcal{B}([0])$), and the second map is induced by the inclusion $S^2(V_1) \hookrightarrow \mathcal{O}_B([0]) \oplus S^2(V_1) = \mathcal{B}([0])$.

A similar splitting can be done for α: using again the lexicographic order we get

$$
S^2(\mathcal{B})\big(3[0]-\tau\big) = (\mathcal{O}_B \oplus L_1 \oplus L_2 \oplus L_3 \oplus \mathcal{O}_B \oplus L_3 \oplus L_2 \oplus \mathcal{O}_B \oplus L_1 \oplus \mathcal{O}_B)\big(3[0]-\tau\big)
$$

and $\alpha = {}^t(0,0,0,0,1,0,0,1,0,1)$.

Taking the mapping cone we obtain a free resolution of $\tilde{\mathcal{A}}_6$ as follows:

$$0 \rightarrow \mathcal{O}_B\big(3[0] - \tau\big) \xrightarrow{\tilde{\alpha}} \mathcal{B}\big([3[0]\big) \oplus S^2(\mathcal{B})\big(3[0] - \tau\big)$$

$$\xrightarrow{\tilde{\beta}} S^3(\mathcal{B})\big(3[0]\big) \rightarrow \tilde{\mathcal{A}}_6\big(2[0] + 2\tau\big) \rightarrow 0$$

with

$$\tilde{\alpha} = \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ -1 \\ 0 \\ -1 \end{pmatrix}, \qquad \tilde{\beta} = \begin{pmatrix} & & & f_0 & & & & & & & & \\ & & & f_1 & f_0 & & & & & & & \\ & & & f_2 & & f_0 & & & & & & \\ & & & f_3 & & & f_0 & & & & & \\ 1 & & & & f_1 & & & f_0 & & & & \\ & & & & f_2 & f_1 & & & f_0 & & & \\ & & & & f_3 & & f_1 & & & f_0 & & \\ 1 & & & & & f_2 & & & & & f_0 & \\ & & & & & f_3 & f_2 & & & & & f_0 \\ 1 & & & & & & f_3 & & & & & & f_0 \\ & 1 & & & & & & f_1 & & & & \\ & & 1 & & & & & f_2 & f_1 & & & \\ 1 & & & & & & & f_3 & & f_1 & & \\ & & 1 & & & & & & f_2 & & f_1 & \\ & & & & & & & & f_3 & f_2 & & f_1 \\ 1 & & & & & & & & & f_3 & & f_1 \\ & 1 & & & & & & & & & f_2 & \\ & & 1 & & & & & & & & f_3 & f_2 \\ & & 1 & & & & & & & & & f_3 & f_2 \\ & & 1 & & & & & & & & & & f_3 \end{pmatrix}.$$

The reader can now easily simplify this resolution and get the minimal one that is given by the above matrix ${}^tM$. □

We now consider the missing case $\mathcal{M}_{I,3}$: V_2 is indecomposable, and τ is a 3-torsion point. Exact sequence (6) still holds but the vector bundle in the middle has nontrivial cohomology: both cohomology groups have the same dimension: 1 if $\tau \neq [0]$, and 2 if $\tau = [0]$.

PROPOSITION 6.20. – $\mathcal{M}_{I,3}$ *has dimension at most* 4.

Proof. – We start with the case where τ is a 3-torsion point, $\tau \neq [0]$ and assume by contradiction that we have then an irreducible family $\mathcal{S}$ of dimension at least 5. The data (B, τ, ξ) give a map from $\mathcal{S}$ to an irreducible family $\mathcal{Y}$ of dimension 3 (1-parameter for B, no parameters for τ, 2 parameters for ξ) whose fibre is the projective space $\mathbb{P}(H^0(\tilde{\mathcal{A}}_6))$ which has dimension 2.

We know in fact that $h^0(\tilde{\mathcal{A}}_6) \leqslant 3$, that the map dominates $\mathcal{Y}$, whence the fibre has dimension exactly two.

In other terms, for any triple (B, τ, ξ) with $\tau \neq [0]$ of 3-torsion we have $h^0(\tilde{\mathcal{A}}_6) = 3$.

Fix then a general pair (B, τ) with τ a nontrivial 3-torsion point. In Lemma 6.19 we have shown that $h^0(\tilde{\mathcal{A}}_6)$ equals the dimension of the kernel of $H^1(F)$, where F is determined by f_0, f_1, f_2, f_3: therefore we are assuming that for our choices of B, τ and general choice of the f_i's $H^1(F)$ has a kernel of dimension 3. Letting now f_1 tend to zero we obtain, in the limit, case $\mathcal{M}_{II}$ where, as shown in the proof of Proposition 6.15 (since in this case τ is not a 2-torsion), $h^0(\tilde{\mathcal{A}}_6) = 2$: this contradicts the semicontinuity theorem.

The case $\tau = [0]$ is more difficult, since in this case we only know that $2 \leqslant h^0(\tilde{\mathcal{A}}_6) \leqslant 4$.

Also in this case we use Lemma 6.19. Dualizing, we see that $H^0(\tilde{\mathcal{A}}_6)$ is isomorphic to the cokernel of the map F' from

$$H^0\big((\mathcal{O}_B \oplus L_1 \oplus L_2 \oplus L_3 \oplus \mathcal{O}_B \oplus L_3 \oplus L_2 \oplus \mathcal{O}_B \oplus L_1$$
$$\oplus L_2 \oplus L_3 \oplus L_1 \oplus \mathcal{O}_B \oplus L_1 \oplus L_3 \oplus L_2)\big([0]\big)\big)$$

to

$$H^0(\mathcal{O}_B \oplus L_1 \oplus L_2 \oplus L_3 \oplus \mathcal{O}_B \oplus L_3 \oplus L_2 \oplus \mathcal{O}_B \oplus L_1)\big(2[0]\big)$$

given by the matrix M.

Since we are assuming V_2 to be indecomposable, all the f_i's are not zero and therefore each f_i generates the corresponding H^0. We can then write explicitly generators of the image of F' as vectors whose entries have the form $f_i f_j$.

The pairs $\{f_0 f_i, f_j f_k\}$ give bases of $H^0(L_i(2[0]))$ (for $\{i,j,k\} = \{1,2,3\}$); the remaining 4 elements $(f_0^2, f_1^2, f_2^2, f_3^2)$ generate $H^0(\mathcal{O}_B(2[0]))$: therefore there are two independent relations among them.

We identify (f_0, f_1, f_2, f_3) to scalar multiples of the functions

$$(\theta_{11}, \theta_{10}, \theta_{01}, \theta_{00}),$$

where the θ_{ij}'s are the half-integer theta functions $\theta_{ij}(z,\mu)$ (cf. e.g. [34, p. 17]: here μ is a point of Poincaré's upper half plane).

Therefore the relations between the f_i's come from the relations between the θ_{ij}'s; using the relations (E_1) and (E_2) in [34, p. 23], we can write them as $f_2^2 = a f_0^2 + b f_1^2$, $f_3^2 = c f_0^2 + d f_1^2$, with a,b,c,d that vary freely in an open set of $\mathbb{C}^4$ according to the choice of B and of the f_i's.

We have now given a basis of each respective space of global sections, and we may write explicitly (depending on the parameters a,b,c,d) the corresponding matrix of F'. We wrote a Macaulay 2 script (available upon request) that told us that

- F' is generically injective ($h^O(\tilde{\mathcal{A}}_6) = 2$),
- F' acquires a kernel of dimension 1 ($h^O(\tilde{\mathcal{A}}_6) = 3$) on a subvariety of codimension 1 of the space of parameters,
- F' acquires a kernel of dimension 2 in codimension 4 (therefore never, the parameters a,b,c,d giving only three moduli up to automorphisms).

We get then two (possibly empty) families:

(i) $h^O(\tilde{\mathcal{A}}_6) = 2$ gives a family of dimension at most $3 + 1 = 4$;
(ii) $h^O(\tilde{\mathcal{A}}_6) = 3$ gives a family of dimension at most $3 - 1 + 2 = 4$. $\square$

We can now conclude our classification theorem

End of the proof of Theorem 6.1. – We have shown the existence of three irreducible components of dimension 5: the main stream component whose general point is contained in $\mathcal{M}_I^\circ$, and the two ones contained in the closed set $\mathcal{M}_{III}$, according to Proposition 6.16.

We have also seen that every other family in the moduli space has dimension at most 4; by Remark 6.12 we conclude that $\mathcal{M}'$ has three irreducible components of dimension 5.

It remains to show that they do not intersect. The two components in $\mathcal{M}_{III}$ do not intersect, since (Proposition 6.16) in one case τ is $[0]$, in the other case it is a non trivial 2-torsion point.

Since a general point of the main stream component is contained in $\mathcal{M}_I^\circ$ it is then enough to show that $\overline{\mathcal{M}_I^\circ} \cap \mathcal{M}_{III} = \emptyset$. It will be then enough to show that there is no flat family on a disc whose central fibre is a surface in $\mathcal{M}_{III}$ and whose general fibre is a general surface in M_I°.

We borrow an argument used often in Horikawa's work. We note that by Theorem 6.4 a general surface of type M_I° is a double cover of $\mathcal{C}$ with irreducible branch curve. On the contrary we have shown in Proposition 6.16 that the same double cover, for a surface in $\mathcal{M}_{III}$, has a disconnected

branch curve $C_1 \cup C_2$. Therefore we would have a family of double covers with a connected general branch curve and with a disconnected special branch curve, a contradiction. $\quad\square$

7. Fibrations with non hyperelliptic general fibre: the case $g = 3$

In the rest of the paper we will concentrate on the case when the general fibre of $f : S \to B$ is nonhyperelliptic of genus $g = 3$.

The natural morphism of graded $\mathcal{O}_B$-algebras $\sigma : \mathrm{Sym}(V_1) \to \mathcal{R}(f)$ has as kernel the graded sheaf of ideals $\mathcal{L}$ and we denote as usual by $\mathcal{T}$ its (graded) cokernel.

Remark 7.1. – Letting $X := \mathbf{Proj}\,(\mathcal{R}(f))$ the rational map

$$\psi_1 : X \dashrightarrow \Sigma \subset \mathbf{Proj}(\mathrm{Sym}\,V_1) = \mathbb{P}(V_1)$$

allows to factor the relative canonical map $\varphi : S \dashrightarrow \Sigma$ as $\psi_1 \circ r$, where $r : S \to X$ is a minimal resolution of singularities (RDP's).

Observe that, if the general fibre of f is nonhyperelliptic, all the above maps are birational and $\mathcal{L}$ is the ideal sheaf of the canonical image $\Sigma := \varphi(S)$.

The next lemma investigates the sheaf $\mathcal{T}$: assuming the general fibre to be nonhyperelliptic, we know that the $\mathcal{T}_n$ are all torsion sheaves on B.

Main assumption: We will assume (often without explicit mention) in the rest of the section that every fibre of the genus 3 fibration f is 2-connected.

LEMMA 7.2. – *Let f be a genus 3 fibration whose general fibre is nonhyperelliptic and such that every fibre is 2-connected. Then*:
  (1) *the sheaf $\mathcal{T}_2$ is isomorphic to the structure sheaf $\mathcal{O}_\tau$ of an effective divisor τ on B*;
  (2) *the sheaves $\mathcal{T}_n$ are isomorphic to free $\mathcal{O}_\tau$-modules of rank $2n - 3$*;
  (3) *the multiplication map $V_1 \otimes V_2 \to V_3$ induces an isomorphism $V_1 \otimes \mathcal{T}_2 \cong \mathcal{T}_3$.*

Proof. – The argument is similar to the one given in Lemma 4.1.

By the classification of genus 3 fibres due to M. Mendes Lopes (cf. [33]), and by the hypothesis of 2-connectedness, a fibre F is either nonhyperelliptic, i.e., it has a canonical ring of the form

$$\mathbb{C}[x_1, x_2, x_3]/\langle F_4(x_i)\rangle,$$

or it is honestly hyperelliptic, i.e., it has a canonical ring of the form

$$R = \mathbb{C}[x_1, x_2, x_3, y]/\langle r_1 := Q(x_i),\ r_2 := y^2 - G(x_i)\rangle,$$

where $\deg x_i = 1$, $\deg y = 2$, $\deg Q = 2$, $\deg G = 4$.

If t is a local parameter in B such that the point $p := \{t = 0\}$ is the image of a hyperelliptic fibre, the relation r_1 lifts to a relation

$$\overline{r}_1 = \overline{Q}(x_i, t) + \mu(t)y,$$

where $\overline{Q}(x_i, 0) = Q(x_i)$, and $\mu(0) = 0$. The assumption that the generic fibre is nonhyperelliptic imposes $\mu \not\equiv 0$, whence we may assume as in 4.1 $\mu(t) = t^s$ and then $(\mathcal{T}_2)_p = \mathcal{O}_{B,P}/t^s$, proving the first part of the statement.

Using the lift of r_2 to eliminate the multiples of y^2, we see that, chosen a basis $\{q_j\}$ for the homogeneous part of degree $k - 2$ of the quotient ring $\mathbb{C}[x_1, x_2, x_3]/(Q)$, the set $\{t^i q_j y \mid i < s\}$

is a basis for the complex vector space, stalk of $\mathcal{T}_k$ at p. It follows then that $\mathcal{T}_{k,p}$ is a free $\mathcal{O}_{B,p}$-module with basis $\{q_j y\}$.

For $k = 3$ the above basis is $\{x_1 y, x_2 y, x_3 y\}$, i.e., exactly the image of the natural basis of $V_1 \otimes \mathcal{T}_2$, and the lemma is proven.   □

The next lemma investigates the sheaf $\mathcal{L}$. Write $\pi : \mathbb{P}(V_1) \to B$ for the canonical projection.

LEMMA 7.3. – *If f is a genus 3 fibration with nonhyperelliptic general fibre and such that every fibre is 2-connected, then*
  (1) $\mathcal{L}_2 = \mathcal{L}_3 = 0$.
  (2) $\deg V_1 = \chi - 2(b-1)$, $\deg \tau = K^2 - 3\chi(\mathcal{O}_S) - 10(b-1)$.
  (3) $\mathcal{L}_4$ *is a line bundle of degree* $\deg V_1 - \deg \tau$.
  (4) *The maps $\mathcal{L}_4 \otimes S^{n-4}(V_1) \hookrightarrow \mathcal{L}_n$ are isomorphisms: in particular the "relative canonical image" Σ is a divisor on $\mathbb{P}(V_1)$ belonging to the linear system $|\mathcal{O}_{\mathbb{P}(V_1)}(4) \otimes \pi^* \mathcal{L}_4^\vee|$.*

Proof. – We have already noted that all the maps σ_n are generically surjective, so that their kernels are vector bundles of rank $\binom{n+2}{2} - 4n + 2$ for $n \geqslant 2$. We get
  • $\mathcal{L}_2 = \mathcal{L}_3 = 0$;
  • $\mathcal{L}_4$ is a line bundle;
  • $\forall n \geqslant 4$, rank $\mathcal{L}_n =$ rank $S^{n-4}(V_1)$.
By Remark 2.11 $\deg V_1 = \chi(\mathcal{O}_S) - 2(b-1)$, $\deg V_2 = \chi(\mathcal{O}_S) + K^2 - 18(b-1)$ and once more the exact sequence $0 \to S^2(V_1) \to V_2 \to \mathcal{O}_\tau \to 0$ yields $\deg \tau = \deg V_2 - 4 \deg V_1 = K^2 - 3\chi(\mathcal{O}_S) - 10(b-1)$.

By Remark 2.11 $\deg V_2 = 6K^2 + \chi(\mathcal{O}_S) - 98(b-1)$. Moreover, $\deg S^4(V_1) = 20 \deg V_1 = 20\chi(\mathcal{O}_S) - 40(b-1)$. By Lemma 7.2 $\deg \mathcal{T}_4 = 5 \deg \tau = 5K^2 - 15\chi(\mathcal{O}_S) - 50(b-1)$, and we conclude

$$\begin{aligned}
\deg \mathcal{L}_4 &= \deg S^4(V_1) + \deg \mathcal{T}_4 - \deg V_4 \\
&= 5\big(\chi(\mathcal{O}_S) + K^2\big) - 90(b-1) - 6K^2 - \chi(\mathcal{O}_S) + 98(b-1) \\
&= 4\chi(\mathcal{O}_S) - K_S^2 + 8(b-1) = \deg V_1 - \deg \tau.
\end{aligned}$$

(4) is obvious since Σ is a Cartier divisor in $\mathbb{P}(V_1)$.   □

Now, with the informations on $\mathcal{T}$ and $\mathcal{L}$ provided by Lemmas 7.2 and 7.3, we can investigate the $\mathcal{O}_B$-algebra structure of $\mathcal{R}(f)$, i.e., the multiplication maps $\mu_{i,j} : V_i \otimes V_j \to V_{i+j}$.

For $i = j = 1$ we have the composition $V_1 \otimes V_1 \to S^2(V_1) \xrightarrow{\sigma_2} V_2$.

The next proposition shows that the triple (V_1, V_2, σ_2) determines $\mathcal{R}(f)$ in degree $\leqslant 3$.

DEFINITION 7.4. – Let $A : V_1 \otimes \Lambda^2(V_1) \to S^2(V_1) \otimes V_1$ be defined by

$$A\big(c \otimes (a \wedge b)\big) := bc \otimes a - ac \otimes b$$

and $B : \Lambda^3(V_1) \to V_1 \otimes \Lambda^2(V_1)$ be defined by

$$B(a \wedge b \wedge c) := a \otimes (b \wedge c) + b \otimes (c \wedge a) + c \otimes (a \wedge b).$$

PROPOSITION 7.5. – $V_3 = \mathrm{coker}(((\sigma_2 \otimes \mathrm{Id}) \circ A) : V_1 \otimes \Lambda^2(V_1) \to V_2 \otimes V_1)$ *and $\mu_{2,1}$ is given by the projection onto the cokernel.*

Proof. – Consider the diagram

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & S^3(V_1) & \xrightarrow{\ \sigma_3\ } & V_3 & \longrightarrow & \mathcal{T}_3 & \longrightarrow & 0 \\
& & \uparrow & & \uparrow{\scriptstyle \mu_{2,1}} & & \uparrow & & \\
0 & \longrightarrow & S^2(V_1)\otimes V_1 & \xrightarrow{\ \sigma_2\otimes\mathrm{Id}\ } & V_2\otimes V_1 & \longrightarrow & \mathcal{T}_2\otimes V_1 & \longrightarrow & 0 \\
& & \uparrow{\scriptstyle A} & & \uparrow{\scriptstyle (\sigma_2\otimes\mathrm{Id})\circ A} & & \uparrow & & \\
0 & \longrightarrow & V_1\otimes\Lambda^2(V_1) & == & V_1\otimes\Lambda^2(V_1) & \longrightarrow & 0 & & \\
& & \uparrow{\scriptstyle B} & & \uparrow & & & & \\
& & \Lambda^3(V_1) & == & \Lambda^3(V_1) & & & & \\
& & \uparrow & & \uparrow & & & & \\
& & 0 & & 0 & & & &
\end{array}
$$

where the map $S^2(V_1)\otimes V_1 \to S^3(V_1)$ is the natural one in $\mathrm{Sym}(V_1)$ while the map $\mathcal{T}_2\otimes V_1 \to \mathcal{T}_3$ is the push forward of $\mu_{2,1}$ (it is an isomorphism by Lemma 7.2).

It is then easy to check that the diagram commutes with exact rows and columns. This proves our statement. $\square$

We now analyse the algebra $\mathcal{R}(f)$ in degree 4.

DEFINITION 7.6. – Let $C\colon S^2(\Lambda^2(V_1)) \to S^2(S^2(V_1))$ be defined by

$$
C\big((a\wedge b)(c\wedge d)\big) := (ac)(bd) - (ad)(bc).
$$

LEMMA 7.7. – *The map $S^2(\sigma_2)\circ C\colon S^2(\Lambda^2(V_1)) \to S^2(V_2)$ is injective with locally free cokernel.*

Proof. – We must show that the map C is injective on every fibre. If ϕ is a map between vector bundles on B, for every $p\in B$ we will denote with $(\phi)_{(p)}$ the corresponding linear map between the fibres over p.

C is injective on every fibre because rank $V_1 = 3$. Note that injectivity fails for higher rank since, if $x_1,\dots,x_4$ are linearly independent, then

$$
\sum_{\sigma(1)=1,\,\mathrm{sign}(\sigma)=1} (x_1\wedge x_{\sigma(2)})(x_{\sigma(3)}\wedge x_{\sigma(4)})
$$

$$
\mapsto \sum_{\sigma(1)=1,\,\mathrm{sign}(\sigma)=1} (x_1 x_{\sigma(3)})(x_{\sigma(2)}x_{\sigma(4)}) - (x_1 x_{\sigma(4)})(x_{\sigma(2)}x_{\sigma(3)}) = 0.
$$

We saw in the proof of Lemma 7.2 that, if F_p is not hyperelliptic, $(\sigma_2)_{(p)}$ is an isomorphism, so $(S^2(\sigma_2)\circ C)_{(p)}$ is injective.

At points p of B for which the fibre F_p is hyperelliptic, the kernel of $(\sigma_2)_{(p)}$ has rank 1. We can choose a basis x_0, x_1, x_2 for the fibre of V_1 over p so that this kernel is generated either by x_0^2, or by $x_0^2 + x_1^2$, or by $x_0^2 + x_1^2 + x_2^2$, according to the rank of the conic q canonical image of the corresponding hyperelliptic fibre.

This choice induces a basis of $S^2(\Lambda^2(V_1))$ given by three vectors of the form $(x_i \wedge x_j)(x_i \wedge x_j)$ and three of the form $(x_i \wedge x_j)(x_i \wedge x_k)$.

By definition:

$$C\big((x_i \wedge x_j)(x_i \wedge x_j)\big) = \big(x_i^2\big)\big(x_j^2\big) - (x_i x_j)(x_i x_j),$$
$$C\big((x_i \wedge x_j)(x_i \wedge x_k)\big) = \big(x_i^2\big)(x_j x_k) - (x_i x_j)(x_i x_k).$$

The 6 image vectors of the basis elements are linearly independent. It remains to show that the subspace they span intersects transversally the subspace of $S^2(S^2(V_1))$ consisting of multiples of q. This is straightforward: it suffices to send to zero the subspace spanned by the elements $(x_i^2)(x_j x_k)$ (for all i, j, k). $\quad\square$

DEFINITION 7.8. –
(1) Given a curve B, a rank 3 vector bundle V_1 on B, an effective divisor τ, and an extension

$$0 \to S^2(V_1) \to V_2 \to \mathcal{O}_\tau \to 0$$

where V_2 is still a vector bundle, set

$$\tilde{V}_4 := S^2(V_2)/S^2\big(\Lambda_2(V_1)\big) = \operatorname{coker}\big(S^2(\sigma_2) \circ C\big).$$

Lemma 7.7 shows that $\tilde{V}_4$ is a locally free vector bundle of rank 15.

(2) Note that $\operatorname{coker} C = S^4(V_1)$, so the image of $S^2(\sigma_2) \circ C$ is in the kernel of the map $S^2(V_2) \to V_4$, inducing a map $\tilde{V}_4 \to V_4$: we denote by $\mathcal{L}_4'$ its kernel.

PROPOSITION 7.9. –
(1) *The maps $S^2(V_2) \to V_4$ and $\tilde{V}_4 \to V_4$ are surjective.*
(2) $\mathcal{L}_4 \cong \det V_1 \otimes \mathcal{O}_B(-\tau)$.
(3) $\mathcal{L}_4' \cong \det V_1 \otimes \mathcal{O}_B(\tau)$.
(4) *Consider the two linear maps induced, on the fibres over a point $p \in \operatorname{Supp}(\tau)$, by the embedding $\mathcal{L}_4' \hookrightarrow \tilde{V}_4$ and by the composition map $V_2 \otimes S^2(V_1) \to S^2(V_2) \to \tilde{V}_4$. Their images are vector subspaces intersecting only in zero.*

Proof. – The first part is an immediate consequence of the fact that $\forall p$ the map $S^2(H^0(F_p, \omega_{F_p}^2)) \to H^0(F_p, \omega_{F_p}^4)$ is surjective, as made clear by the explicit description of the canonical rings given in the proof of Lemma 7.2.

Recall that the map $\mathcal{L}_4 \hookrightarrow S^4(V_1)$ defines Σ as a divisor in $|\mathcal{O}_{\mathbb{P}(V_1)}(4) \otimes \pi^*(\mathcal{L}_4)^{\vee}|$, as shown in Lemma 7.3.

The dualizing sheaf of $\mathbb{P}(V_1)$ is $\omega_{\mathbb{P}(V_1)} = \mathcal{O}_{\mathbb{P}(V_1)}(-3) \otimes \pi^*(\det V_1 \otimes \omega_B)$ (cf. [27, ex. III.8.4.(b)]), therefore the dualizing sheaf of Σ is the restriction of $\mathcal{O}_{\mathbb{P}(V_1)}(1) \otimes \pi^*(\mathcal{L}_4^{\vee} \otimes \det V_1 \otimes \omega_B)$.

The morphism (cf. Remark 7.1) $\psi_1 : X \to \Sigma$ is an isomorphism when restricted to the preimage of $B \setminus \operatorname{Supp}(\tau)$, and since the relative dualizing sheaf $\omega_{S|B}$ is on the one side

$$\omega_{S|B} = \varphi^* \mathcal{O}_{\mathbb{P}}(1),$$

on the other side it equals

$$\mathcal{I}\omega_{\Sigma|B} \cong \omega_{S|B} \cong \mathcal{I}\varphi^* \mathcal{O}_{\mathbb{P}}(1) \otimes f^*\big((\mathcal{L}_4)^{\vee} \otimes \det V_1\big),$$

where $\mathcal{I}$ is the conductor ideal, (2) follows if we prove that $\mathcal{I}$ is the principal ideal associated to the divisor τ.

We will now use the local equations for X in a neighborhood of a fibre F_p with $p \in \text{Supp}(\tau)$.

Choose a small open neighborhood U of p such that X is the subvariety of $U \times \mathbb{P}(1,1,1,2)$ defined by ideal generated by $t^s y - \bar{Q}(x,t)$ and $y^2 - \bar{G}(x,t)$. The above equations show that $\mathcal{O}_X$ is locally generated by $\{1,y\}$ as an $\mathcal{O}_\Sigma$-module and the rows of the following matrix are relations for $\{1,y\}$:

$$\begin{pmatrix} -t^s \bar{G} & \bar{Q} \\ -\bar{Q} & t^s \end{pmatrix}.$$

Since the determinant of the above matrix is a generator of the ideal of Σ in $U \times \mathbb{P}^2$, it follows that the above is the full matrix of relations, and we conclude that the conductor ideal is generated by $\bar{Q}, t^s$ as an ideal in $\mathcal{O}_\Sigma$ and by t^s as an ideal in $\mathcal{O}_X$.

We have the commutative diagram with exact rows and columns:

$$
\begin{array}{ccccccccc}
& & & & & 0 & & 0 & \\
& & & & & \uparrow & & \uparrow & \\
0 & \to & \mathcal{L}_4 & \to & S^4(V_1) & \overset{\sigma_4}{\longrightarrow} & V_4 & \to & \mathcal{T}_4 & \to & 0 \\
& & & & \| & & \uparrow & & \uparrow & \\
0 & \to & \frac{S^2(S^2(V_1))}{S^2(\Lambda^2 V_1)} & \overset{S^2(\sigma_2)/S^2(\Lambda^2 V_1)}{\longrightarrow} & \tilde{V}_4 & \to & \mathcal{K} & \to & 0 \\
& & & & & \uparrow & & & \\
& & & & & \mathcal{L}'_4 & & & \\
& & & & & \uparrow & & & \\
& & & & & 0 & & & \\
\end{array}
$$

By Lemma 7.2, $\mathcal{T}_4 = \mathcal{O}_\tau{}^{\oplus 5}$, and $\mathcal{K}$ is isomorphic to $\text{coker}\, S^2(\sigma_2)$, whence

$$\mathcal{K} \cong \mathcal{O}_\tau{}^{\oplus 5} \oplus \mathcal{O}_{2\tau}.$$

Computing determinants in the three exact sequences of the diagram above, one finds:

$$\det V_4 \otimes \mathcal{L}'_4 = \det \tilde{V}_4 = \det S^4(V_1) \otimes \mathcal{O}_B(7\tau),$$

$$\det V_4 \otimes \mathcal{L}_4 = \det S^4(V_1) \otimes \mathcal{O}_B(5\tau)$$

therefore $\mathcal{L}_4 = \mathcal{L}'_4(-2\tau)$ and also (3) is proven.

Finally we want to prove that the embedding $\mathcal{L}'_4 \hookrightarrow \tilde{V}_4$ satisfies (4). Consider $p \in \text{supp}(\tau)$. The fibre over p of $\mathcal{R}(f)$, i.e., the stalk modulo the maximal ideal of p, is the canonical ring of the corresponding fibre F_p. In degree $\leqslant 4$ we can now see that it has three generators x_1, x_2, x_3 in degree 1 given by V_1, one generator y in degree 2, generating $\mathcal{T}_2$, one relation, $Q(x_i)$, generating the kernel of the map induced by σ_2 at the level of the fibres, and one further relation in degree 4, given by a representative in $S^2(V_2)$ of the image of a generator of $\mathcal{L}'_4$. We know by the 2-connectedness hypothesis that this relation has the form $y^2 = G(x_i)$, whence the image of $V_2 \otimes S^2(V_1)$ is given by the polynomials without the term y^2. $\quad\square$

DEFINITION 7.10. – Define the *genus three* 5-*tuple* associated to a genus 3 fibration with nonhyperelliptic general fibre and 2-connected fibres as (B, V_1, τ, ξ, w), where

- ξ is the element $\xi \in \mathrm{Ext}^1(\mathcal{O}_\tau, S^2(V_1))/\mathrm{Aut}(\mathcal{O}_\tau)$ yielding the exact sequence $0 \to S^2(V_1) \to f_*(\omega_{S|B}^2) \to \mathcal{O}_\tau \to 0$,
- w is the element in $\mathbb{P}(H^0(\tilde{V}_4 \otimes \mathcal{L}_4'^{-1}))$ corresponding to the embedding $i: \mathcal{L}_4' \hookrightarrow \tilde{V}_4 = S^2(V_2)/S^2(\Lambda^2(V_1))$.

We have seen that
  (i) ξ yields a vector bundle (V_2);
  (ii) $\mathcal{L}_4' \cong \mathcal{O}_B(\tau) \otimes \det V_1$;
  (iii) i has a locally free cokernel;
  (iv) the image of the fibre map corresponding to i at a point in $\mathrm{supp}(\tau)$ is not contained in the image of $V_2 \otimes S^2(V_1)$.
The next theorem shows that these data determine the fibration.

DEFINITION 7.11. – (1) Define a *genus three 5-tuple* (B, V_1, τ, ξ, w) as follows:
- B is a smooth curve,
- V_1 is a rank 3 vector bundle,
- τ is an effective divisor on B,
- ξ is an element of $\mathrm{Ext}^1(\mathcal{O}_\tau, S^2(V_1))/\mathrm{Aut}(\mathcal{O}_\tau)$ yielding a vector bundle V_2,
- if $\tilde{V}_4$ is the vector bundle constructed via σ_2 as in Definition 7.8, then $w = \mathbb{P}(i)$, where i is an embedding $\mathcal{O}_B(\tau) \otimes \det V_1 \hookrightarrow \tilde{V}_4$ such that (iii) and (iv) above hold.

(2) Define its *associated relative canonical model* X as follows: let $W \subset \mathbb{P}(V_2)$ be the image of the rational Veronese map $\mathbb{P}(V_1) \dashrightarrow \mathbb{P}(V_2)$ induced by σ_2, and let X be the relative quadric divisor on W cut by the principal ideal generated by the image of i.

(3) Define a genus three 5-tuple to be *admissible* if its associated relative canonical model X has only Rational Double Points as singularities.

Remark 7.12. – One can explicitly, cf. the proof of 7.13, define a sheaf of graded $\mathcal{O}_B$-algebras $\mathcal{R}$, generated by $V_1 \oplus V_2$ such that $X = \mathbf{Proj}(\mathcal{R})$. The geometric procedure is more suitable to determine the singularities of X.

THEOREM 7.13. – *Let f be a relatively minimal genus 3 nonhyperelliptic fibration such that every fibre is 2-connected. Then its associated 5-tuple (B, V_1, τ, ξ, w) is admissible.*

Vice versa, every admissible genus three 5-tuple (B, V_1, τ, ξ, w) is the associated 5-tuple of a unique genus 3 nonhyperelliptic fibration $f: S \to B$ with the property that every fibre is 2-connected and with invariants $\chi(\mathcal{O}_S) = \deg V_1 + 2(b-1)$, $K_S^2 = 3 \deg V_1 + \deg \tau + 16(b-1)$. As in the case of genus 2, the bijection thus obtained is functorial.

Proof. – The 5-uple of such a fibration is admissible by definition.

Conversely, as in the proof of Theorem 4.13, we construct V_2 via the extension class ξ, $\tilde{V}_4$ via σ_2 as in Definition 7.8, and finally X as in Definition 7.11.

By our assumption, X has at most rational double points as singularities and a minimal resolution S of the singularities of X yields a genus 3 fibration $f: S \to B$ such that the general fibre is nonhyperelliptic.

Let us show that each fibre F of f is 2-connected: otherwise one of following holds:
  (0) F is not 1-connected, i.e., by Zariski's lemma $F = 2F'$, where F' has arithmetic genus 2, or
  (1) $F = A + B$, $A \cdot B = 1$.
In case (0), since $\mathcal{O}_{F'}(2K_S) = \mathcal{O}_{F'}(2K_{F'})$, the relative bicanonical map is not birational, a contradiction.

In case (1), the relative canonical map has base points, contradicting the fact that, by (iv), the coefficient of y^2 in the local equations is equal to 1.

The rest of the proof is analogous to the proof of Theorem 4.13. $\square$

8. Genus three fibrations on some surfaces with $p_g = 3$, $q = 0$

The present section is meant to show, via a concrete example, how the structure theorem for genus three fibrations 7.13 can be applied.

Assume that S is a minimal surface of general type with $p_g = 3$, $q = 0$, and with $K_S^2 = 2 + d$. We make several simplifying assumptions:

Assumption I: we have a genus 3 fibration $f : S \to \mathbb{P}^1$ ($B \cong \mathbb{P}^1$ since $q = 0$) such that, for a general fibre F, the restriction map $H^0(K_S) \to H^0(\omega_F)$ is surjective.

It follows then that $V_1 \cong \bigoplus_1^3 \mathcal{O}_{\mathbb{P}^1}(2)$. Our standard formulae read out as

$$K_S^2 = 2 + \deg(\tau) \implies d = \deg(\tau).$$

Assumption II: $d \leqslant 6$ and σ_2 yields a general extension class

$$(8) \qquad 0 \to \bigoplus_1^6 \mathcal{O}_{\mathbb{P}^1}(4) \to \left(\bigoplus_1^d \mathcal{O}_{\mathbb{P}^1}(5) \right) \oplus \left(\bigoplus_1^{6-d} \mathcal{O}_{\mathbb{P}^1}(4) \right) \to \mathcal{O}_\tau \to 0.$$

It follows that $S^2 \Lambda^2 V_1 \cong \bigoplus_1^6 \mathcal{O}_{\mathbb{P}^1}(8)$, $\mathcal{L}_4' \cong \mathcal{O}_{\mathbb{P}^1}(d+6)$ and

$$(9) \qquad S^2(V_2) = \binom{6-d}{2} \mathcal{O}_{\mathbb{P}^1}(8) \oplus d(6-d) \mathcal{O}_{\mathbb{P}^1}(9) \oplus \binom{d}{2} \mathcal{O}_{\mathbb{P}^1}(10).$$

PROPOSITION 8.1. – *Assume $d \leqslant 3$. Then X is the complete intersection of W with a relative quadric $Z \in |\mathcal{O}_{\mathbb{P}(V_2)}(2) \otimes \pi^*(\mathcal{O}_{\mathbb{P}^1}(-6-d))|$, $\pi : \mathbb{P}(V_2) \to \mathbb{P}^1$ being the natural projection.*

Proof. – Tensoring the exact sequence

$$0 \to S^2\left(\Lambda^2(V_1)\right) \to S^2(V_2) \to \tilde{V}_4 \to 0$$

by $\mathcal{L}_4'^{-1}$ we see that the restriction map

$$H^0\left(S^2(V_2) \otimes \mathcal{L}_4'^{-1}\right) \to H^0\left(\tilde{V}_4 \otimes \mathcal{L}_4'^{-1}\right)$$

is surjective as soon as $H^1(S^2(\Lambda^2(V_1)) \otimes \mathcal{L}_4'^{-1}) = 0$. The result follows then from $S^2(\Lambda^2(V_1)) \otimes \mathcal{L}_4'^{-1} \cong \bigoplus_1^6 \mathcal{O}_{\mathbb{P}^1}(2-d)$. $\square$

THEOREM 8.2. – *Let $d \leqslant 3$. Let σ_2 be a general homomorphism as in the exact sequence (8) and let ζ be a general section of the vector bundle $S^2(V_2) \otimes \mathcal{L}_4'^{-1}$ (cf. (9)). Denote by Z the divisor of ζ on $\mathbb{P}(V_2)$ and set $X = W \cap Z$. Then X is a smooth minimal surface of general type with $p_g = 3$, $q = 0$ and $K_X^2 = 2 + d$. We obtain in this way a subvariety of the moduli space of dimension $33 - 2d$.*

Proof. – Let $U_1 = \mathbb{C}x_1 \oplus \mathbb{C}x_2 \oplus \mathbb{C}x_3$ and $V_1 = U_1 \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^1}(2)$. Let $q_1, \ldots, q_6$ be a general basis of $\mathrm{Sym}^2(U_1)$. Let σ_2 be given, in the corresponding basis of $\mathrm{Sym}^2(V_1)$, by the diagonal matrix whose entries are the first 6 elements of a sequence constituted by 1 repeated $6 - d$ times followed by $t_0, t_1, t_0 - t_1$.

If q_4, q_5, q_6 yield irreducible conics, W is a $\mathbb{P}^2$ bundle except possibly over the points $(0, 1, \infty)$ where the fibre can be the cone over a rational normal quartic curve. The singular points of W are then exactly the cited vertices.

For $d \leqslant 2$, $S^2(V_2) \otimes \mathcal{L}_4'^{-1}$ is globally generated and therefore the linear system cut by Z on W has no base points. So the smoothness assertion is proved.

If $d = 3$ we set $U_2 := \mathbb{C}w_1 \oplus \mathbb{C}w_2 \oplus \mathbb{C}w_3$, $U_2' := \mathbb{C}z_1 \oplus \mathbb{C}z_2 \oplus \mathbb{C}z_3$ and $V_2 = (U_2 \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^1}(4)) \oplus (U_2' \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^1}(5))$. It follows easily that in this case the base locus of the linear system of Z is the $\mathbb{P}^2$ bundle of equations $\{z_1 = z_2 = z_3 = 0\}$.

Assume now that q_4, q_5, q_6 are smooth conics and that $q_4 \cap q_5 \cap q_6 = \emptyset$. Then we see that W does not intersect the base locus of Z, and that X is smooth.

The minimality of X follows since $H^0(K_X)$ is given by x_1, x_2, x_3 and restricts to the canonical system of each fibre. By the local equation that we have one sees that this last is base point free.

We compute the dimension of the subvariety of the moduli space we have constructed. The choice of τ gives d parameters, the extension class ξ (yielding σ_2) gives $6d - d = 5d$ parameters since we have to quotient by $\mathrm{Aut}(\mathcal{O}_\tau)$, the linear system cut by Z on W has dimension $44 - 8d$. We have therefore $44 - 2d$ parameters and we have to subtract 3 for the automorphisms of $\mathbb{P}^1$ and 8 for those of V_1. $\square$

Remark 8.3. – Minimal surfaces with $K_S^2 = 2, p_g = 3$ are well known since the time of Noether to be the double covers of $\mathbb{P}^2$ branched on a curve D of degree 8. Hence, their moduli space is unirational of dimension $44 - 8 = 36$. The existence of such a genus three fibration is equivalent to the existence of a one dimensional family of quartic curves C_t (images of the fibres) which are everywhere tangent to D. Thus D is a square modulo each C_t, equivalently the equation of D can be written as a symmetrical determinant $\det\left(\begin{smallmatrix} A & B \\ B & C \end{smallmatrix}\right) = 0$. We obtain in this way only a 33-dimensional family, since $3 \times 15 - \dim GL(2) - \dim \mathbb{P}GL(3) = 33$.

This shows how our above examples yield proper subvarieties of the moduli space.

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Fabrizio CATANESE
Lehrstuhl Mathematik VIII,
Universität Bayreuth,
Universitätsstr. 30,
D-95447 Bayreuth, Germany
E-mail: fabrizio-catanese@uni-bayreuth.de

Roberto PIGNATELLI
Dipartimento di Matematica,
Università di Trento,
Via Sommarive, 14,
I-38050 Povo (TN), Italy
E-mail: pignatel@science.unitn.it