

THE TAMAGAWA NUMBER CONJECTURE OF ADJOINT MOTIVES OF MODULAR FORMS

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ABSTRACT. – Let f be a newform of weight $k \geq 2$, level N with coefficients in a number field K , and A the adjoint motive of the motive M associated to f . We carefully discuss the construction of the realisations of M and A , as well as natural integral structures in these realisations. We then use the method of Taylor and Wiles to verify the λ -part of the Tamagawa number conjecture of Bloch and Kato for $L(A, 0)$ and $L(A, 1)$. Here λ is any prime of K not dividing $Nk!$, and so that the mod λ representation associated to f is absolutely irreducible when restricted to the Galois group over $\mathbb{Q}(\sqrt{(-1)^{(\ell-1)/2}\ell})$ where $\lambda \mid \ell$. The method also establishes modularity of all lifts of the mod λ representation which are crystalline of Hodge–Tate type $(0, k - 1)$.

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RÉSUMÉ. – Soient f une forme nouvelle de poids k , de conducteur N , à coefficients dans un corps de nombres K , et A le motif adjoint du motif M associé à f . Nous présentons en détail les réalisations des motifs M et A avec leurs réseaux entiers naturels. En utilisant les méthodes de Taylor–Wiles nous prouvons la partie λ -primaire de la conjecture de Bloch–Kato pour $L(A, 0)$ et $L(A, 1)$. Ici λ est une place de K ne divisant pas $Nk!$ et telle que la représentation modulo λ associée à f , restreinte au groupe de Galois du corps $\mathbb{Q}(\sqrt{(-1)^{(\ell-1)/2}\ell})$ avec $\lambda \mid \ell$, est irréductible. Notre méthode démontre aussi la modularité de toutes les représentations λ -adiques cristallines de type de Hodge–Tate $(0, k - 1)$ congrues à la représentation associée à f modulo λ .

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0. Introduction

This paper concerns the Tamagawa number conjecture of Bloch and Kato [4] for adjoint motives of modular forms of weight $k \geq 2$. The conjecture relates the value at 0 of the associated L -function to arithmetic invariants of the motive. We prove that it holds up to powers of certain “bad primes”. The strategy for achieving this is essentially due to Wiles [88], as completed with Taylor in [86]. The Taylor–Wiles construction yields a formula relating the size of a certain module measuring congruences between modular forms to that of a certain Galois cohomology group. This was carried out in [88] and [86] in the context of modular forms of weight 2, where it was used to prove results in the direction of the Fontaine–Mazur conjecture [40]. While it was no surprise that the method could be generalized to higher weight modular forms and that the resulting formula would be related to the Bloch–Kato conjecture, there remained many technical details to verify in order to accomplish this. In particular, the very formulation of the conjecture relies on a comparison isomorphism between the ℓ -adic and de Rham realizations of the motive provided by theorems of Faltings [31] or Tsuji [87], and verification of the conjecture requires

the careful application of such a theorem. We also need to generalize results on congruences between modular forms to higher weight, and to compute certain local Tamagawa numbers.

0.1. Some history

Special values of L -functions have long played an important role in number theory. The underlying principle is that the values of L -functions at integers reflect arithmetic properties of the object used to define them. A prime example of this is Dirichlet's class number formula; another is the Birch and Swinnerton–Dyer conjecture. The Tamagawa number conjecture of Bloch and Kato [4], refined by Fontaine, Kato and Perrin-Riou [55,41,38], is a vast generalization of these. Roughly speaking, they predict the precise value of the first non-vanishing derivative of the L -function at zero (hence any integer) for every motive over \mathbb{Q} . This was already done up to a rational multiple by conjectures of Deligne and Beilinson; the additional precision of the Bloch–Kato conjecture can be thought of as a generalized class number formula, where ideal class groups are replaced by groups defined using Galois cohomology.

Dirichlet's class number formula amounts to the conjecture for the Dedekind zeta function for a number field at $s = 0$ or 1 . The conjecture is also known for Dirichlet L -functions (including the Riemann zeta function) at any integer [62,4,52,8,35]. It is known up to an explicit set of bad primes for the L -function of a CM elliptic curve at $s = 1$ if the order of vanishing is ≤ 1 [11,69,58]. There are also partial results for L -functions of other modular forms at the central critical value [45,56,59,64,89] and for values of certain Hecke L -functions [49,48,57]. For a more detailed survey of known results we refer to [35].

Here we consider the adjoint L -function of a modular form of weight $k \geq 2$ at $s = 0$ and 1 . Special values of the L -function associated to the adjoint of a modular form, and more generally, twists of its symmetric square, have been studied by many mathematicians. A method of Rankin relates the values to Petersson inner products, and this was used by Ogg [65], Shimura [81], Sturm [84,85], Coates and Schmidt [10,73] to obtain nonvanishing results and rationality results along the lines of Deligne's conjecture. Hida [51] related the precise value to a number measuring congruences between modular forms. In the case of forms corresponding to (modular) elliptic curves, results relating the value to certain Galois cohomology groups (Selmer groups) were obtained by Coates and Schmidt in the context of Iwasawa theory, and by one of the authors, who in [34] obtained results in the direction of the Bloch–Kato conjecture.

A key point of Wiles' paper [88] is that for many elliptic curves, modularity could be deduced from a formula relating congruences and Galois cohomology [88]. This formula could be regarded as a primitive form of the Bloch–Kato conjecture for the adjoint motive of a modular form. His attempt to prove it using the Euler system method introduced in [34] was not successful except in the CM case using generalizations of results in [47] and [70]. Wiles, in work completed with Taylor [86], eventually proved his formula using a new construction which could be viewed as a kind of "horizontal Iwasawa theory".

In this paper, we refine the method of [88] and [86], generalize it to higher weight modular forms and relate the result to the Bloch–Kato conjecture. Ultimately, we prove the conjecture for the adjoint of an arbitrary newform of weight $k \geq 2$ up to an explicit finite set of bad primes. We should stress the importance of making this set as small and explicit as possible; indeed the refinements in [22,13,5] which completed the proof of the Shimura–Taniyama–Weil conjecture can be viewed as work in this direction for weight two modular forms. In this paper, we make use of some of the techniques introduced in [22] and [13], as well as the modification of Taylor–Wiles construction in [24] and [42]. One should be able to improve our results using current technology in the weight two case (using [13,13,72]), and in the ordinary case (using [28,82]); one just has to relate the results in those papers to the Bloch–Kato conjecture. Finally we remark

that Wiles' method has been generalized to the setting of Hilbert modular forms by Fujiwara [42], Skinner–Wiles [83] (using Shimura curves) and Dimitrov [29] (using Hilbert modular varieties). Dimitrov's work allows to relate the Selmer group with the special value of the adjoint L-function of the Hilbert modular form but to verify the Bloch–Kato conjecture it remains to relate the appearing period with a motivic period.

0.2. The framework

The Bloch–Kato conjecture is formulated in terms of “motivic structures,” a term referring to the usual collection of cohomological data associated to a motive. This data consists of:

- vector spaces $M_?$, called realizations, for $? = B, dR$ and ℓ for each rational prime ℓ , each with extra structure (involution, filtration or Galois action);
- comparison isomorphisms relating the realizations;
- a weight filtration.

Suppose that f is a newform of weight $k \geq 2$ and level N . Much of the paper is devoted to the construction of the motivic structure A_f for which we prove the conjecture. This construction is not new; it is due for the most part to Eichler, Shimura, Deligne, Jannsen, Scholl and Faltings [79,15,53,75,31]. We review it however in order to collect the facts we need and set things up in a way suited to the formulation of the Bloch–Kato conjecture. For proofs of results not readily found in the literature, we direct the reader to [25].

Let us briefly recall here how the construction works. We start with the modular curve X_N parametrizing elliptic curves with level N structure. Then one takes the Betti, de Rham and ℓ -adic cohomology of X_N with coefficients in a sheaf defined as the $(k - 2)$ nd symmetric power of the relative cohomology of the universal elliptic curve over X_N . These come with the additional structure and comparison isomorphisms needed to define a motivic structure $M_{N,k}$, the comparison between ℓ -adic and de Rham cohomology being provided by a theorem of Faltings [31]. The structures $M_{N,k}$ can also be defined as in [75] using Kuga-Sato varieties; this has the advantage of showing they arise from “motives” and provides the option of applying Tsuji's comparison theorem [87]. However the construction using “coefficient sheaves” is better suited to defining and comparing lattices in the realizations which play a key role in the proof.

The structures $M_{N,k}$ also come with an action of the Hecke operators and a perfect pairing. The Hecke action is used to “cut out” a piece M_f , which corresponds to the newform f and has rank two over the field generated by the coefficients of f . The pairing comes from Poincaré duality, is related to the Petersson inner product and restricts to a perfect pairing on M_f . We finally take the trace zero endomorphisms of M_f to obtain the motivic structure $A_f = \text{ad}^0 M_f$. The construction also yields integral structures \mathcal{M}_f and \mathcal{A}_f , consisting of lattices in the various realizations and integral comparison isomorphisms outside a set of bad primes.

Our presentation of the Bloch–Kato conjecture is much influenced by its reformulation and generalization due to Fontaine and Perrin-Riou [41]. Their version assumes the existence of a category of motives with conjectural properties. Without assuming conjectures however, they define a category $\text{SPM}_{\mathbb{Q}}(\mathbb{Q})$ of pre-motivic structures whose objects consist of realizations with additional structure and comparison isomorphisms. The category of mixed motives is supposed to admit a fully faithful functor to it, and a motivic structure is an object of the essential image. Their version of the Bloch–Kato conjecture is then stated in terms of Ext groups of motivic structures, but whenever there is an explicit “motivic” construction of (conjecturally) all the relevant extensions, the conjecture can be formulated entirely in terms of pre-motivic structures. This happens in our case, for all the relevant Ext's conjecturally vanish. There will therefore be no further mention of motives in this paper. We make several other slight modifications to the framework of [41]:

- We use premotivic structures with coefficients in a number field K , as in [38].
- We forget about the ℓ -adic realization and comparison isomorphisms at a finite set of “bad” primes S .
- We work with S -integral premotivic structures.

This yields a version of the conjecture which predicts the value of $L(A_f, 0)$ up to an S -unit in K . The conjecture is independent of the choice of integral structures, but the formalism is convenient and certain lattices arise naturally in the proof.

We make our set S explicit: Let S_f be the set of finite primes λ in K such that either:

- $\lambda \mid Nk!$, or
- the two-dimensional residual Galois representation $\mathcal{M}_{f,\lambda}/\lambda\mathcal{M}_{f,\lambda}$ is not absolutely irreducible when restricted to G_F , where $F = \mathbb{Q}(\sqrt{(-1)^{(\ell-1)/2}\ell})$ and $\lambda \mid \ell$.

Note that since S_f includes the set of primes dividing $Nk!$, we will only be applying Faltings’ comparison theorem in the “easy” case of crystalline representations whose associated Dieudonné module has short filtration length.

0.3. The main theorems

Our main result can be stated as follows.

THEOREM 0.1 (= Theorem 2.15). – *Let f be a newform of weight $k \geq 2$ and level N with coefficients in K . If λ is not in S_f , then the λ -part of the Bloch–Kato conjecture holds for A_f and $A_f(1)$.*

The main tool in the proof is the construction of Taylor and Wiles, which we axiomatize (Theorem 3.1), and apply to higher weight forms to obtain the following generalization of their class number formula.

THEOREM 0.2 (= Theorem 3.7). – *Let f be a newform of weight $k \geq 2$ and level N with coefficients in K . Suppose Σ is a finite set of rational primes containing those dividing N . Suppose that λ is a prime of K which is not in S_f and does not divide any prime in Σ . Then the $\mathcal{O}_{K,\lambda}$ -module*

$$H^1_{\Sigma}(G_{\mathbb{Q}}, A_{f,\lambda}/\mathcal{A}_{f,\lambda})$$

has length $v_{\lambda}(\eta_f^{\Sigma})$.

Here η_f^{Σ} , defined in Section 1.7.3, is a generalization of the congruence ideal of Hida and Wiles; it can also be viewed as measuring the failure of the pairing on M_f to be perfect on \mathcal{M}_f .

Another consequence of Theorem 0.2, is the following result in the direction of Fontaine–Mazur conjecture [40].

THEOREM 0.3 (= Theorem 3.6). – *Suppose $\rho: G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(K_{\lambda})$ is a continuous geometric representation whose restriction to G_{ℓ} is ramified and crystalline and its associated Dieudonné module has filtration length less than $\ell - 1$. If its residual representation is modular and absolutely irreducible restricted to $\mathbb{Q}(\sqrt{(-1)^{(\ell-1)/2}\ell})$ where $\lambda \mid \ell$, then ρ is modular.*

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1. The adjoint motive of a modular form

1.1. Generalities and examples of premotivic structures

1.1.1. Premotivic structures

For a field F , \bar{F} will denote an algebraic closure, and $G_F = \text{Gal}(\bar{F}/F)$. We fix an embedding $\bar{\mathbb{Q}} \rightarrow \bar{\mathbb{Q}}_p$ for each prime p , and an embedding $\bar{\mathbb{Q}} \rightarrow \mathbb{C}$. If F is a number field, we let \mathbf{I}_F denote the set of embeddings $F \rightarrow \bar{\mathbb{Q}}$, which we identify with the set of embeddings $F \rightarrow \mathbb{C}$ via our fixed one of $\bar{\mathbb{Q}}$ in \mathbb{C} .

We write G_p for $G_{\mathbb{Q}_p}$, I_p for the inertia subgroup of G_p , and Frob_p for the geometric Frobenius element in $G_p/I_p \cong G_{\mathbb{F}_p}$. We identify $I_p \subset G_p$ with their images in $G_{\mathbb{Q}}$.

If K is a number field, then $S_f(K)$ denotes the set of finite places of K . Suppose that $\lambda \in S_f(K)$ divides $\ell \in S_f(\mathbb{Q})$. Let $B_{\text{dR}} = B_{\text{dR},\ell}$ and $B_{\text{crys}} = B_{\text{crys},\ell}$ be the rings defined by Fontaine [37, §2], [41, I.2.1]. Suppose that V is a finite-dimensional vector space over K_λ with a continuous action of G_ℓ (i.e., a λ -adic representation of G_ℓ). Then $D_{\text{dR}}(V) = (B_{\text{dR}} \otimes_{\mathbb{Q}_\ell} V)^{G_\ell}$ is a filtered finite-dimensional vector space over K_λ , and $D_{\text{crys}}(V) = (B_{\text{crys}} \otimes_{\mathbb{Q}_\ell} V)^{G_\ell}$ is a filtered finite-dimensional vector space over K_λ equipped with a K_λ -linear endomorphism ϕ . The representation V is called *de Rham* if $\dim_{K_\lambda} D_{\text{dR}}(V) = \dim_{K_\lambda} V$, and V is called *crystalline* if $\dim_{K_\lambda} D_{\text{crys}}(V) = \dim_{K_\lambda} V$. We recall that if V is crystalline, then V is de Rham.

A λ -adic representation V of $G_{\mathbb{Q}}$ is *pseudo-geometric* [41, II.2] if it is unramified outside of a finite number of places of \mathbb{Q} and its restriction to G_ℓ is de Rham. The representation V is said to have *good reduction at p* if its restriction to G_p is crystalline (resp. unramified) if $p = \ell$ (resp. $p \neq \ell$).

We work with categories of premotivic structures based on notions from [41,38,4].

For a number field K , we let \mathbf{PM}_K denote the category of premotivic structures over \mathbb{Q} with coefficients in K . In the notation of [41, III.2.1], this is the category $\mathbf{SPM}_{\mathbb{Q}}(\mathbb{Q}) \otimes K$ of K -modules in $\mathbf{SPM}_{\mathbb{Q}}(\mathbb{Q})$. Thus an object M of \mathbf{PM}_K consists of the following data:

- a finite-dimensional K -vector space M_B with an action of $G_{\mathbb{R}}$;
- a finite-dimensional K -vector space M_{dR} with a finite decreasing filtration Fil^i , called the Hodge filtration;
- for each $\lambda \in S_f(K)$, a finite-dimensional K_λ vector space M_λ with a continuous pseudo-geometric action of $G_{\mathbb{Q}}$;
- a $\mathbb{C} \otimes K$ -linear isomorphism

$$I^\infty : \mathbb{C} \otimes M_{\text{dR}} \rightarrow \mathbb{C} \otimes M_B$$

respecting the action of $G_{\mathbb{R}}$ (where $G_{\mathbb{R}}$ acts on $\mathbb{C} \otimes M_B$ diagonally and acts on $\mathbb{C} \otimes M_{\text{dR}}$ via the first factor);

- for each $\lambda \in S_f(K)$, a K_λ -linear isomorphism

$$I_B^\lambda : K_\lambda \otimes_K M_B \rightarrow M_\lambda$$

respecting the action of $G_{\mathbb{R}}$ (where the action on M_λ is via the restriction $G_{\mathbb{R}} \rightarrow G_{\mathbb{Q}}$ determined by our choice of embedding $\bar{\mathbb{Q}} \rightarrow \mathbb{C}$);

- for each $\lambda \in S_f(K)$, a $B_{\text{dR},\ell} \otimes_{\mathbb{Q}_\ell} K_\lambda$ -linear isomorphism

$$I^\lambda : B_{\text{dR},\ell} \otimes_{\mathbb{Q}_\ell} K_\lambda \otimes_K M_{\text{dR}} \rightarrow B_{\text{dR},\ell} \otimes_{\mathbb{Q}_\ell} M_\lambda$$

respecting filtrations and the action of $G_{\mathbb{Q}_\ell}$ (where ℓ is the prime which λ divides, K_λ and M_λ are given the degree-0 filtration, K_λ and M_{dR} are given the trivial $G_{\mathbb{Q}_\ell}$ -action and the action on M_λ is determined by our choice of embedding $\bar{\mathbb{Q}} \rightarrow \bar{\mathbb{Q}}_\ell$);

- increasing weight filtrations W^i on M_B, M_{dR} and each M_λ respecting all of the above data, and such that $\mathbb{R} \otimes M_B$ with its Galois action and weight filtration, together with the Hodge filtration on $\mathbb{C} \otimes M_B$ defined via I_B , defines a mixed Hodge structure over \mathbb{R} (see [41, III.1]).

If $S \subsetneq S_f(K)$ is a set of primes of K , we let \mathbf{PM}_K^S denote the category defined in exactly the same way, but with $S_f(K)$ replaced by the complement of S . If $S \subseteq S'$, we use $\cdot^{S'}$ to denote the forgetful functor from \mathbf{PM}_K^S to $\mathbf{PM}_K^{S'}$.

The category \mathbf{PM}_K^S is equipped with a tensor product, which we denote \otimes_K , and an internal hom, which we denote Hom_K . There is also a unit object, which we denote simply by K . These are defined in the obvious way; for example, $(M \otimes_K N)_B$ is the $K[G_{\mathbb{R}}]$ -module $M_B \otimes_K N_B$. If $K \subseteq K'$, we let $S^{K'}$ denote the set of primes in $S_f(K')$ lying over those in S , then $K' \otimes_K \cdot$ defines a functor from \mathbf{PM}_K^S to $\mathbf{PM}_{K'}^{S^{K'}}$.

If M is an object of \mathbf{PM}_K^S , then for each prime p and each $\lambda \notin S$, we can associate a representation of the Deligne–Weil group of \mathbb{Q}_p (see [16, §8]), which we denote by $WD_p(M_\lambda)$. For λ not dividing p , the representation is over K_λ ; for λ dividing p , we have that $M_\lambda|_{G_p}$ is potentially semistable [3, Theorem 0.7], so the construction in [41, I.2.2] yields a representation over $\mathbb{Q}_p^{\text{ur}} \otimes_{\mathbb{Q}_p} K_\lambda$. We recall that $M_\lambda|_{G_p}$ is crystalline if and only if $WD_p(M_\lambda)$ is unramified (in the sense that the monodromy operator and the inertia group act trivially), in which case $WD_p(M_\lambda) = \mathbb{Q}_p^{\text{ur}} \otimes_{\mathbb{Q}_p} D_{\text{crys}}(M_\lambda)$ with Frob_p acting via $1 \otimes \phi^{-1}$. An object M of \mathbf{PM}_K^S

- has *good reduction* at p if $WD_p(M_\lambda)$ is unramified for all $\lambda \notin S$;
- is *L-admissible* at p if the Frobenius semisimplifications of $WD_p(M_\lambda)$, for $\lambda \notin S$, form a compatible system of K -rational representations of the Deligne–Weil group of \mathbb{Q}_p (see [16, §8]);
- is *L-admissible everywhere* if it is *L-admissible* at p for all primes p .

If M is *L-admissible* at p , then the local factor associated to $WD_p(M_\lambda)$ is of the form $P(p^{-s})^{-1}$ for some polynomial $P(u) \in K[u]$ independent of λ not in S . For an embedding $\tau : K \rightarrow \mathbb{C}$ we put $L_p(M, \tau, s) = \tau P(p^{-s})$ and we regard the collection $\{L_p(M, \tau, s)\}_{\tau \in \mathbf{I}_K}$ as a meromorphic function on \mathbb{C} with values in $\mathbb{C}^{\mathbf{I}_K} \cong K \otimes \mathbb{C}$. If S is finite and M is *L-admissible everywhere*, then its *L-function*

$$L(M, s) := \prod_p L_p(M, s)$$

is a holomorphic $K \otimes \mathbb{C}$ -valued function in some right half plane $\text{Re}(s) > r$ (with components $L(M, \tau, s) = \prod_p L_p(M, \tau, s)$).

1.1.2. Integral premotivic structures

Before introducing integral premotivic structures, we recall some of the theory of Fontaine and Laffaille [39]. We let \mathcal{MF} denote the category whose objects are finitely generated \mathbb{Z}_ℓ -modules equipped with

- a decreasing filtration such that $\text{Fil}^a A = A$ and $\text{Fil}^b A = 0$ for some $a, b \in \mathbb{Z}$, and for each $i \in \mathbb{Z}$, $\text{Fil}^i A$ is a direct summand of A ;
- \mathbb{Z}_ℓ -linear maps $\phi^i : \text{Fil}^i A \rightarrow A$ for $i \in \mathbb{Z}$ satisfying $\phi^i|_{\text{Fil}^{i+1} A} = \ell \phi^{i+1}$ and $A = \sum \text{Im } \phi^i$.

It follows from [39, 1.8] that \mathcal{MF} is an abelian category. Let \mathcal{MF}^a denote the full subcategory of objects A satisfying $\text{Fil}^a A = A$ and $\text{Fil}^{a+\ell} A = 0$ and having no non-trivial quotients A' such that $\text{Fil}^{a+\ell-1} A' = A'$, and let $\mathcal{MF}_{\text{tor}}^a$ denote the full subcategory of \mathcal{MF}^a consisting of objects

of finite length. So $\mathcal{MF}_{\text{tor}}^0$ is the category denoted $\underline{\mathcal{MF}}_{\text{tor}}^{f,\ell'}$ in [39], and it follows from [39, 6.1] that \mathcal{MF}^a and $\mathcal{MF}_{\text{tor}}^a$ are abelian categories, stable under taking subobjects, quotients, direct products and extensions in \mathcal{MF} .

Fontaine and Laffaille define a contravariant functor \underline{U}_S from $\mathcal{MF}_{\text{tor}}^0$ to the category of finite continuous $\mathbb{Z}_\ell[G_\ell]$ -modules and they prove it is fully faithful [39, 6.1]. We let \mathbb{V} denote the functor defined by $\mathbb{V}(A) = \text{Hom}(\underline{U}_S(A), \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ and we extend it to a fully faithful functor on \mathcal{MF}^0 by setting $\mathbb{V}(A) = \text{projlim } \mathbb{V}(A/\ell^n A)$. Then \mathbb{V} defines an equivalence between \mathcal{MF}^0 and the full subcategory of $\mathbb{Z}_\ell[G_\ell]$ -modules whose objects are isomorphic to quotients of the form L_1/L_2 , where $L_2 \subset L_1$ are finitely generated submodules of short crystalline representations. Here we define a crystalline representation V to be *short* if the following hold

- $\text{Fil}^0 D = D$ and $\text{Fil}^\ell D = 0$, where $D = (B_{\text{crys}} \otimes_{\mathbb{Q}_\ell} V)^{G_\ell}$;
- if V' is a nonzero quotient of V , then $V' \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell(\ell - 1)$ is ramified.

In particular, the essential image of \mathbb{V} is closed under taking subobjects, quotients and finite direct sums. Furthermore, one sees from [39, 8.4] that the natural transformations

$$(1) \quad \begin{aligned} \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} A &\rightarrow (B_{\text{crys}} \otimes_{\mathbb{Z}_\ell} \mathbb{V}(A))^{G_\ell}, \\ B_{\text{crys}} \otimes_{\mathbb{Z}_\ell} A &\rightarrow B_{\text{crys}} \otimes_{\mathbb{Z}_\ell} \mathbb{V}(A) \quad \text{and} \\ \text{Fil}^0(B_{\text{crys}} \otimes_{\mathbb{Z}_\ell} A)^{\phi=1} &\rightarrow \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{V}(A) \end{aligned}$$

are isomorphisms.

If K is a number field and $\lambda \in S_f(K)$ is a prime over ℓ , we let $\mathcal{O}_\lambda = \mathcal{O}_{K,\lambda}$ and let $\mathcal{O}_\lambda\text{-}\mathcal{MF}^a$ denote the category of \mathcal{O}_λ -modules in \mathcal{MF}^a . We can regard \mathbb{V} as a functor from $\mathcal{O}_\lambda\text{-}\mathcal{MF}^0$ to the category of $\mathcal{O}_\lambda[G_\ell]$ -modules.

If A and A' are objects of $\mathcal{O}_\lambda\text{-}\mathcal{MF}^0$ such that $A \otimes_{\mathcal{O}_\lambda} A'$ defines an object of $\mathcal{O}_\lambda\text{-}\mathcal{MF}^0$, then there is a canonical isomorphism

$$\mathbb{V}(A \otimes_{\mathcal{O}_\lambda} A') \cong \mathbb{V}(A) \otimes_{\mathcal{O}_\lambda} \mathbb{V}(A').$$

Analogous assertions hold for $\text{Hom}_{\mathcal{O}_\lambda}(A, A')$.

We now define a category \mathcal{PM}_K^S of *S-integral premotivic structures* as follows. We let $\mathcal{O}_S = \mathcal{O}_{K,S}$ denote the set of $x \in K$ with $v_\lambda(x) \geq 0$ for all $\lambda \notin S$. An object \mathcal{M} of \mathcal{PM}_K^S consists of the following data:

- a finitely generated \mathcal{O}_K -module \mathcal{M}_B with an action of $G_{\mathbb{R}}$;
- a finitely generated \mathcal{O}_S -module \mathcal{M}_{dR} with a finite decreasing filtration Fil^i , called the Hodge filtration;
- for each $\lambda \in S_f(K)$, a finitely generated \mathcal{O}_λ -module \mathcal{M}_λ with continuous action of $G_{\mathbb{Q}}$ inducing a pseudo-geometric action on $\mathcal{M}_\lambda \otimes_{\mathcal{O}_\lambda} K_\lambda$;
- for each $\lambda \notin S$, an object $\mathcal{M}_{\lambda\text{-crys}}$ of $\mathcal{O}_\lambda\text{-}\mathcal{MF}^0$;
- an $\mathbb{R} \otimes \mathcal{O}_K$ -linear isomorphism

$$I^\infty : \mathbb{C} \otimes \mathcal{M}_{\text{dR}} \rightarrow \mathbb{C} \otimes \mathcal{M}_B$$

respecting the action of $G_{\mathbb{R}}$;

- for each λ in $S_f(K)$, an isomorphism

$$I_B^\lambda : \mathcal{M}_B \otimes_{\mathcal{O}_K} \mathcal{O}_\lambda \cong \mathcal{M}_\lambda$$

respecting the action of $G_{\mathbb{R}}$;

- for each $\lambda \notin S$, an \mathcal{O}_λ -linear isomorphism

$$I_{\text{dR}}^\lambda : \mathcal{M}_{\text{dR}} \otimes_{\mathcal{O}} \mathcal{O}_\lambda \cong \mathcal{M}_{\lambda\text{-crys}}$$

respecting filtrations;

- for each $\lambda \notin S$, an \mathcal{O}_λ -linear isomorphism

$$I^\lambda : \mathbb{V}(\mathcal{M}_{\lambda\text{-crys}}) \rightarrow \mathcal{M}_\lambda$$

respecting the action of $G_{\mathbb{Q}_\ell}$, where ℓ is the prime which λ divides;

- increasing weight filtrations W^i on $\mathbb{Q} \otimes \mathcal{M}_B$, $\mathbb{Q} \otimes \mathcal{M}_{\text{dR}}$ and each $\mathbb{Q} \otimes \mathcal{M}_\lambda$ respecting all of the above data and giving rise to a mixed Hodge structure.

With the evident notion of morphism this becomes an \mathcal{O}_K -linear abelian category. Note also that there is a natural functor $\mathbb{Q} \otimes \cdot$ from $\mathcal{P}\mathcal{M}_K^S$ to \mathbf{PM}_K^S , where we set $(\mathbb{Q} \otimes \mathcal{M})_? = \mathbb{Q} \otimes \mathcal{M}_?$ for $? = B, \text{dR}$ and λ for $\lambda \notin S$, with induced additional structure and comparison isomorphisms. (The comparison I^λ for $\mathbb{Q} \otimes \mathcal{M}$ is defined as the composite

$$B_{\text{dR},\ell} \otimes_{\mathbb{Z}_\ell} \mathcal{O}_\lambda \otimes_{\mathcal{O}} \mathcal{M}_{\text{dR}} \cong B_{\text{dR},\ell} \otimes_{\mathbb{Z}_\ell} \mathcal{M}_{\lambda\text{-crys}} \rightarrow B_{\text{dR},\ell} \otimes_{\mathbb{Z}_\ell} \mathbb{V}(\mathcal{M}_{\lambda\text{-crys}}) \rightarrow B_{\text{dR},\ell} \otimes_{\mathbb{Z}_\ell} \mathcal{M}_\lambda,$$

where the maps are respectively, I_{dR}^λ , the canonical map (1) and I^λ , each with scalars extended to $B_{\text{dR},\ell}$.)

If $S \subset S'$, we define a functor $\cdot^{S'}$ from $\mathcal{P}\mathcal{M}_K^S$ to $\mathcal{P}\mathcal{M}_K^{S'}$ in the obvious way. We say that \mathcal{M} is S' -flat if $\mathcal{M}_{\text{dR}}^{S'} = \mathcal{M}_{\text{dR}} \otimes_{\mathcal{O}_{K,S}} \mathcal{O}_{K,S'}$ is flat over $\mathcal{O}_{K,S'}$. Note that if \mathcal{M} is S' -flat, then so is any subobject of \mathcal{M} . Let K' be a finite extension of K . We also have a natural functor $\mathcal{O}_{K'} \otimes_{\mathcal{O}_K} \cdot$ from $\mathcal{P}\mathcal{M}_K^S$ to $\mathcal{P}\mathcal{M}_{K'}^{S'}$.

We say that \mathcal{M} has *good reduction at p* , is *L -admissible at p* or is *L -admissible everywhere* according to whether the same is true for $\mathbb{Q} \otimes \mathcal{M}$. Note that if \mathcal{M} is L -admissible at p and p is not invertible in \mathcal{O} , then \mathcal{M} necessarily has good reduction at p .

For objects \mathcal{M} and \mathcal{M}' of $\mathcal{P}\mathcal{M}_K^S$, we can form $\mathcal{M} \otimes_{\mathcal{O}_K} \mathcal{M}'$ in $\mathcal{P}\mathcal{M}_K^S$ provided $\mathcal{M}_{\lambda\text{-crys}} \otimes_{\mathcal{O}_\lambda} \mathcal{M}'_{\lambda\text{-crys}}$ defines an object of \mathcal{MF}_λ^0 for all $\lambda \notin S$. In particular this holds if there exist positive integers a, a' such that $\text{Fil}^a \mathcal{M}_{\text{dR}} = 0$, $\text{Fil}^{a'} \mathcal{M}'_{\text{dR}} = 0$ and $a + a' < \ell$ for all primes ℓ not invertible in \mathcal{O} . If \mathcal{N} and \mathcal{N}' are objects of $\mathcal{P}\mathcal{M}_K^S$ such that $\mathcal{N} \otimes_{\mathcal{O}_K} \mathcal{N}'$ is as well, and if $\alpha : \mathcal{M} \rightarrow \mathcal{N}$ and $\alpha' : \mathcal{M}' \rightarrow \mathcal{N}'$ are morphisms in $\mathcal{P}\mathcal{M}_K^S$, then there is a well-defined morphism $\alpha \otimes \alpha' : \mathcal{M} \otimes_{\mathcal{O}_K} \mathcal{N} \rightarrow \mathcal{M}' \otimes_{\mathcal{O}_K} \mathcal{N}'$ in $\mathcal{P}\mathcal{M}_K^S$. Analogous assertions hold for the formation and properties of $\text{Hom}_{\mathcal{O}_K}(\mathcal{M}, \mathcal{M}')$.

Note that if \mathcal{M} is an object of $\mathcal{P}\mathcal{M}_K^S$, then $\text{End } \mathcal{M}$ is a finitely generated \mathcal{O}_K -module. If I is an \mathcal{O}_K -submodule of $\text{End } \mathcal{M}$, then we define an object $\mathcal{M}[I]$ of $\mathcal{P}\mathcal{M}_K^S$ as the kernel of

$$(x_1, \dots, x_r) : \mathcal{M} \rightarrow \mathcal{M}^r$$

where x_1, \dots, x_r generate I . This is independent of the choice of generators. This applies in particular when I is the image in $\text{End } \mathcal{M}$ of an ideal in a commutative \mathcal{O}_K -algebra R mapping to $\text{End } \mathcal{M}$, or the augmentation ideal in $\mathcal{O}_K[G]$ where G is a group acting on \mathcal{M} . In the latter case, we write \mathcal{M}^G instead of $\mathcal{M}[I]$.

1.1.3. Basic examples

The object $\mathbb{Q}(-1)$ in $\mathbf{PM}_\mathbb{Q}$ is the weight two premotivic structure defined by $H^1(\mathbb{G}_m)$. To give an explicit description, let ε denote the generator of $\mathbb{Z}_\ell(1) = \varprojlim \mu_{\ell^n}(\mathbb{Q})$ defined by $(e^{2\pi i/\ell^n})_n$ via our fixed embedding $\bar{\mathbb{Q}} \rightarrow \mathbb{C}$.

- Let $\mathcal{T}_B = H_B^1(G_{\mathbf{m}}(\mathbb{C}), \mathbb{Z}) \cong (2\pi i)^{-1} \mathbb{Z} \subset \mathbb{C}$ with complex conjugation in $G_{\mathbb{R}}$ acting by -1 , and let $\mathbb{Q}(-1)_B = \mathbb{Q} \otimes \mathcal{T}_B$.
- Let $\mathcal{T}_{dR} = H_{dR}^1(G_{\mathbf{m}}/\mathbb{Z})$, which with its Hodge filtration is isomorphic to $\mathbb{Z}[-1]$ (where $[n]$ denotes a shift by n in the filtration, so $\text{Fil}^i V[n] = \text{Fil}^{i+n} V$). Write ι for the canonical basis $\frac{dx}{x}$ of $\mathcal{T}_{dR} \cong \mathbb{Z}[-1]$ and let $\mathbb{Q}(-1)_{dR} = \mathbb{Q} \otimes \mathcal{T}_{dR}$, so $\text{Fil}^1 \mathbb{Q}(-1)_{dR} = \mathbb{Q}\iota$ and $\text{Fil}^2 \mathbb{Q}(-1)_{dR} = 0$.
- Let $\mathcal{T}_{\ell} = H_{\text{et}}^1(G_{\mathbf{m}, \mathbb{Q}}, \mathbb{Z}_{\ell}) \cong \text{Hom}_{\mathbb{Z}_{\ell}}(\mathbb{Z}_{\ell}(1), \mathbb{Z}_{\ell}) = \mathbb{Z}_{\ell}\delta$ where $\delta(\varepsilon) = 1$, and let $\mathbb{Q}(-1)_{\ell} = \mathbb{Q} \otimes \mathcal{T}_{\ell}$.
- Let $\mathcal{T}_{\ell\text{-crys}}$ denote the object of \mathcal{MF}_{ℓ} defined by $\mathbb{Z}_{\ell} \otimes \mathcal{T}_{dR} = \mathbb{Z}_{\ell}\iota$ with $\phi^1(\iota) = \iota$.
- $I^{\infty} : \mathbb{C} \otimes_{\mathbb{Z}} \mathcal{T}_{dR} \rightarrow \mathbb{C} \otimes_{\mathbb{Z}} \mathcal{T}_B$ is defined by $1 \otimes \iota \mapsto 2\pi i \otimes (2\pi i)^{-1}$.
- $I_B^{\ell} : \mathbb{Z}_{\ell} \otimes \mathcal{T}_B \rightarrow \mathcal{T}_{\ell}$ is defined by $1 \otimes (2\pi i)^{-1} \mapsto \delta$.
- $I_{dR}^{\ell} : \mathcal{T}_{dR} \otimes_{\mathbb{Z}} \mathbb{Z}_{\ell} \rightarrow \mathcal{T}_{\ell\text{-crys}}$ is given by $\iota \mapsto \iota$.
- $I^{\ell} : B_{dR, \ell} \otimes \mathbb{Q}[-1] \rightarrow B_{dR, \ell} \otimes_{\mathbb{Q}_{\ell}} \mathbb{Q}(-1)_{\ell}$ is defined by $1 \otimes \iota \mapsto t \otimes \delta$ where $t = \log[\varepsilon]$ (see e.g., [41, I.2.1.3]).
- For $\ell > 2$, $\mathcal{T}_{\ell\text{-crys}}$ is an object of \mathcal{MF}^0 and I^{ℓ} is induced by an isomorphism $\mathbb{V}(\mathcal{T}_{\ell\text{-crys}}) \cong \mathcal{T}_{\ell}$.

The above data defines objects in $\mathbf{PM}_{\mathbb{Q}}$ and $\mathcal{PM}_{\mathbb{Q}}^{\{2\}}$ which we denote by $\mathbb{Q}(-1)$ and \mathcal{T} . These could be described equivalently by $H^2(\mathbb{P}^1)$, or indeed $H^2(X)$ for any smooth, proper, geometrically connected curve X over \mathbb{Q} .

The Tate premotivic structure $\mathbb{Q}(1)$ is the object of $\mathbf{PM}_{\mathbb{Q}}$ defined by $\text{Hom}_{\mathbb{Q}}(\mathbb{Q}(-1), \mathbb{Q})$ and $\mathbb{Q}(n)$ is defined by $\mathbb{Q}(1)^{\otimes n}$ for integer n . We have $L(\mathbb{Q}(n), s) = \zeta(s+n)$ where ζ is the Riemann ζ -function. More generally, for any object M in \mathbf{PM}_K and integer n , $M(n)$ is defined as $M \otimes_{\mathbb{Q}} \mathbb{Q}(1)^{\otimes n}$. For any integer $n \geq 0$, $\mathcal{T}^{S, \otimes n}$ defines an object of $\mathcal{PM}_{\mathbb{Q}}^S$ where S is any set of primes containing those dividing $(n+1)!$; note also that $\mathbb{Q} \otimes \mathcal{T}^{S, \otimes n} \cong \mathbb{Q}(-n)^S$.

For any number field $F \subset \bar{\mathbb{Q}}$, let M_F denote the premotivic structure M_F of weight zero defined by $H^0(\text{Spec } F)$, called the Dedekind premotivic structure of F . To give an explicit description, let S denote the set of primes dividing $D = \text{Disc}(F/\mathbb{Q})$. We let

- $\mathcal{M}_{F, B} = \mathbb{Z}^{\mathbf{I}_F}$ with the natural action of $G_{\mathbb{R}}$, and $M_{F, B} = \mathbb{Q} \otimes \mathcal{M}_{F, B} = \mathbb{Q}^{\mathbf{I}_F}$. (Recall that we identified \mathbf{I}_F with the set of embeddings $F \rightarrow \mathbb{C}$ via the chosen embedding $\bar{\mathbb{Q}} \rightarrow \mathbb{C}$, so for $\alpha : \mathbf{I}_F \rightarrow \mathbb{Z}$ and $\sigma \in G_{\mathbb{R}}$, we define $\sigma\alpha$ by $\tau \mapsto \alpha(\sigma^{-1} \circ \tau)$);
- $\mathcal{M}_{F, dR} = \mathcal{O}_F[1/D]$ with $\text{Fil}^0 \mathcal{M}_{F, dR} = \mathcal{M}_{F, dR}$ and $\text{Fil}^1 \mathcal{M}_{F, dR} = 0$, and $M_{F, dR} = \mathbb{Q} \otimes \mathcal{M}_{F, dR} = F$;
- $\mathcal{M}_{F, \ell} = \mathbb{Z}_{\ell} \otimes \mathcal{M}_{F, B} = \mathbb{Z}_{\ell}^{\mathbf{I}_F}$ with the natural action of $G_{\mathbb{Q}}$, and $M_{F, \ell} = \mathbb{Q} \otimes \mathcal{M}_{F, \ell} = \mathbb{Q}_{\ell}^{\mathbf{I}_F}$ (so for $\alpha : \mathbf{I}_F \rightarrow \mathbb{Z}_{\ell}$, $\sigma \in G_{\mathbb{Q}}$ and $\tau : F \rightarrow \mathbb{Q}$, we have $(\sigma\alpha)(\tau) = \alpha(\sigma^{-1} \circ \tau)$);
- $\mathcal{M}_{F, \ell\text{-crys}} = \mathbb{Z}_{\ell} \otimes \mathcal{M}_{F, dR} = \mathbb{Z}_{\ell} \otimes \mathcal{O}_F$ for $\ell \notin S$, with the same filtration as $\mathcal{M}_{F, dR}$ and with $\phi^0 = \phi_{\ell}$.

The comparisons I_B^{ℓ} and I_{dR}^{ℓ} are identity maps, I^{∞} is defined by $I^{\infty}(1 \otimes x)(\tau) = \tau(x)$ after identifying $\mathbb{C} \otimes M_{F, B}$ with $\mathbb{C}^{\mathbf{I}_F}$, and I^{ℓ} is defined similarly. We thus obtain objects \mathcal{M}_F of $\mathcal{PM}_{\mathbb{Q}}^S$ and M_F of $\mathbf{PM}_{\mathbb{Q}}$, and $L(M_F, s)$ is the ζ -function of F . If F is Galois over \mathbb{Q} , then there is a natural action of $G = \text{Gal}(F/\mathbb{Q})$ on \mathcal{M}_F , where for $\alpha \in M_{F, B}$, $g \in G$ and $\tau \in \mathbf{I}_F$, we have $g\alpha$ by $\tau \mapsto \alpha(\tau \circ g)$.

Let $\psi : \hat{\mathbb{Z}}^{\times} \rightarrow K^{\times}$ be a character, regarded also as a character of \mathbb{A}^{\times} and $G_{\mathbb{Q}}$ via the isomorphisms $\hat{\mathbb{Z}}^{\times} \cong \mathbb{A}^{\times}/\mathbb{R}_{>0}^{\times} \mathbb{Q}^{\times} \cong G_{\mathbb{Q}}^{\text{ab}}$, where the first isomorphism is induced by the natural inclusion $\hat{\mathbb{Z}}^{\times} \rightarrow \mathbb{A}^{\times}$ and the second is given by class field theory. (Our convention is that a uniformizer in \mathbb{Q}_p^{\times} maps to Frob_p in the Galois group of any abelian extension of \mathbb{Q} unramified at p .) If F is a Galois extension of \mathbb{Q} such that ψ is trivial on the image of G_F in $G_{\mathbb{Q}}$, then we can regard ψ as a character of $G = \text{Gal}(F/\mathbb{Q})$ and define the Dirichlet premotivic structure M_{ψ} as $(V \otimes M_F)^G$ where $V = K$ with G acting by ψ . The construction is independent of the choice of F and embedding of F in $\bar{\mathbb{Q}}$. To describe M_{ψ} explicitly, we choose $F = \mathbb{Q}(e^{2\pi i/N}) \subset \mathbb{C}$ where

ψ has conductor N . We let $\tau_0 : F \rightarrow \bar{\mathbb{Q}}$ denote the embedding compatible with our fixed $\bar{\mathbb{Q}} \rightarrow \mathbb{C}$, and we regard ψ as a Dirichlet character via the canonical isomorphism $(\mathbb{Z}/N\mathbb{Z})^\times \cong G$. Let S denote the set of primes in K lying over those dividing N and define an object \mathcal{M}_ψ of \mathcal{PM}_K^S by $(\mathcal{V} \otimes \mathcal{M}_F)^G$ where $\mathcal{V} = \mathcal{O}_K^S$ with G acting by ψ . We then have:

- $\mathcal{M}_{\psi,B}$ is the \mathcal{O}_K -submodule of $\mathcal{O}_K^{\mathbf{I}_F}$ spanned by the map b_B defined by $\tau_0 \circ g \mapsto \psi^{-1}(g)$, where τ_0 is the inclusion of F in $\bar{\mathbb{Q}}$;
- $\mathcal{M}_{\psi,dR}$ is the $\mathcal{O} = \mathcal{O}_K[1/N]$ -submodule of $\mathcal{O}_K \otimes \mathcal{O}_F[1/N]$ spanned by

$$b_{dR} = \sum_a \psi(a) \otimes e^{2\pi ia/N},$$

where a runs over $(\mathbb{Z}/N\mathbb{Z})^\times$ with $\text{Fil}^1 \mathcal{M}_{\psi,dR} = 0$ and $\text{Fil}^0 \mathcal{M}_{\psi,dR} = \mathcal{M}_{\psi,dR}$;

- $\mathcal{M}_{\psi,\lambda} = \mathcal{O}_\lambda \otimes_{\mathcal{O}_K} \mathcal{M}_{\psi,B}$ with $G_{\mathbb{Q}}$ acting via ψ ;
- for $\lambda \nmid N$, $\mathcal{M}_{\psi,\lambda\text{-crys}} = \mathcal{O}_\lambda \otimes_{\mathcal{O}_K} \mathcal{M}_{\psi,dR}$ with the same filtration as $\mathcal{M}_{\psi,dR}$ and $\phi^0 = \psi^{-1}(\ell)$.

The comparison isomorphisms are induced from those of \mathcal{M}_F . Similarly, we get the object M_ψ of \mathbf{PM}_K by setting $M_{\psi,?} = \mathbb{Q} \otimes \mathcal{M}_{\psi,?}$ with comparison isomorphisms induced from those of M_F . In particular, we have $I^\infty(1 \otimes b_{dR}) = G_\psi(1 \otimes b_B)$ where G_ψ is the Gauss sum $\sum_a e^{2\pi ia/N} \otimes \psi(a)$ in $\mathbb{C} \otimes K$.

We have that $\mathbb{Q} \otimes \mathcal{M}_\psi \cong M_\psi^S$, M_ψ has good reduction at all primes not dividing the conductor of ψ and is L -admissible everywhere, and $L(M_\psi, s)$ is the Dirichlet L -function $L(\psi^{-1}, s)$.

1.2. Premotivic structures for level N modular forms

In this section we review the construction of premotivic structures associated to the space of modular forms of weight k and level N . More precisely, if $k \geq 2$ and $N \geq 3$, we construct objects of $\mathcal{PM}_{\mathbb{Q}}^S$ whose de Rham realization contains the space of such forms, where $S = S_N$ is the set of primes dividing $Nk!$.

1.2.1. Level N modular curves

These premotivic structures are obtained from the cohomology of modular curves. Let k and N be integers with $k \geq 2$ and $N \geq 3$. Let $T = \text{Spec } \mathbb{Z}[1/Nk!]$, and consider the functor which associates to a T -scheme T' the set of isomorphism classes of generalized elliptic curves over T' with level N structure [18, IV.6.6]. By [18, IV.6.7], the functor is represented by a smooth, proper curve over T . We denote this curve by \bar{X} , and we let $\bar{s} : \bar{E} \rightarrow \bar{X}$ denote the universal generalized elliptic curve with level N structure. We let X denote the open subscheme of \bar{X} over which \bar{E} is smooth. Then X is the complement of a reduced divisor, called *the cuspidal divisor*, which we denote by X^∞ . We let $E = \bar{s}^{-1}X$, $s = \bar{s}|_E$ and $E^\infty = \bar{E} \times_{\bar{X}} X^\infty$. Using the arguments of [18, VII.2.4], one can check that \bar{E} is smooth over T and E^∞ is a reduced divisor with strict normal crossings (in the sense of [46, 1.8] as well as [1, XIII.2.1]).

Let us also recall the standard description of

$$\bar{s}^{\text{an}} : \bar{E}^{\text{an}} \rightarrow \bar{X}^{\text{an}},$$

where we use $^{\text{an}}$ to denote the associated complex analytic space. We let

$$X_N = \prod_{t \in (\mathbb{Z}/N\mathbb{Z})^\times} X_{N,t},$$

where for each t , $X_{N,t}$ denotes a copy of $\Gamma(N)\backslash\mathfrak{H}$, the quotient of the complex upper half-plane \mathfrak{H} , by the principal congruence subgroup $\Gamma(N)$ of $\mathrm{SL}_2(\mathbb{Z})$. Similarly we let

$$\bar{X}_N = \coprod_{t \in (\mathbb{Z}/N\mathbb{Z})^\times} \bar{X}_{N,t},$$

where $\bar{X}_{N,t}$ is the compactification of $X_{N,t}$ obtained by adjoining the cusps. We write \bar{X}_N^{alg} for the corresponding algebraic curve over \mathbb{C} .

For each $t \in (\mathbb{Z}/N\mathbb{Z})^\times$, we define the complex analytic surface $E_{N,t}$ to be a copy of the quotient

$$\Gamma(N)\backslash((\mathfrak{H} \times \mathbb{C})/(\mathbb{Z} \times \mathbb{Z})),$$

where $(m, n) \in \mathbb{Z} \times \mathbb{Z}$ acts on $\mathfrak{H} \times \mathbb{C}$ via $(\tau, z) \mapsto (\tau, z + m\tau + n)$, and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N)$ acts by sending the class of (τ, z) to that of $(\gamma(\tau), (c\tau + d)^{-1}z)$. We can regard $E_N = \coprod E_{N,t}$ as a complex analytic family of elliptic curves over X_N with level N -structure defined by the pair of sections $(\tau, \tau/N)$ and $(\tau, t/N)$ on $X_{N,t}$. We can then extend E_N to a family \bar{E}_N of generalized elliptic curves with level N -structure over \bar{X}_N using analytic Tate curves, as in [18, VII.4]. One checks that \bar{E}_N is algebraic, so $\bar{E}_N \rightarrow \bar{X}_N$ is the analytification of a generalized elliptic curve $\bar{E}_N^{\mathrm{alg}} \rightarrow \bar{X}_N^{\mathrm{alg}}$. The resulting morphism $\bar{X}_N^{\mathrm{alg}} \rightarrow \bar{X}_{\mathbb{C}}$ induces an isomorphism $X_N \rightarrow X^{\mathrm{an}}$. The analytification of the universal generalized elliptic curve with level N -structure is therefore isomorphic to $\bar{E}_N \rightarrow \bar{X}_N$ with the level N -structure defined above.

1.2.2. Betti realization

To construct the Betti realization, define \mathcal{F}_B as the locally constant sheaf $R^1s_*^{\mathrm{an}}\mathbb{Z}$ on X^{an} . Let $\mathcal{F}_B^k = \mathrm{Sym}_{\mathbb{Z}}^{k-2} \mathcal{F}_B$, where our convention for defining symmetric powers is to take coinvariants under the symmetric group. We then define $\mathcal{M}_B = H^1(X^{\mathrm{an}}, \mathcal{F}_B^k)$, and $\mathcal{M}_{c,B} = H_c^1(X^{\mathrm{an}}, \mathcal{F}_B^k)$. Identifying X^{an} with X_N as above, we find that \mathcal{F}_B^k is identified with the locally constant sheaf defined by

$$\Gamma(N)\backslash(\mathfrak{H} \times \mathrm{Sym}^{k-2} \mathbb{Z}^2)$$

where $\Gamma(N)$ acts on \mathbb{Z}^2 by left-multiplication. It follows that

$$(2) \quad H^i(X^{\mathrm{an}}, \mathcal{F}_B^k) \cong \bigoplus_{t \in (\mathbb{Z}/N\mathbb{Z})^\times} H^i(\Gamma(N), \mathrm{Sym}^{k-2} \mathbb{Z}^2).$$

In particular, it follows easily that \mathcal{M}_B has no ℓ -torsion if $k = 2$ or ℓ does not divide $N(k - 2)!$. The actions of complex conjugation in $G_{\mathbb{R}}$ on \mathcal{M}_B and $\mathcal{M}_{c,B}$ are induced by its action on E^{an} and X^{an} . We let $M_B = \mathbb{Q} \otimes \mathcal{M}_B$ and $M_{c,B} = \mathbb{Q} \otimes \mathcal{M}_{c,B}$.

1.2.3. ℓ -adic realizations

For any finite prime ℓ , we let \mathcal{F}_ℓ denote the ℓ -adic sheaf $R^1s_*\mathbb{Z}_\ell$ on X . Let $\mathcal{F}_\ell^k = \mathrm{Sym}_{\mathbb{Z}_\ell}^{k-2} \mathcal{F}_\ell$, $\mathcal{M}_\ell = H^1(X_{\mathbb{Q}}, \mathcal{F}_\ell^k)$ and $\mathcal{M}_{c,\ell} = H_c^1(X_{\mathbb{Q}}, \mathcal{F}_\ell^k)$. Then $G_{\mathbb{Q}}$ acts on \mathcal{M}_ℓ and $\mathcal{M}_{c,\ell}$ by transport of structure. We let $M_\ell = \mathbb{Q} \otimes \mathcal{M}_\ell$ and $M_{c,\ell} = \mathbb{Q} \otimes \mathcal{M}_{c,\ell}$. A standard construction using the comparison between classical and étale cohomology [2, XI.4.4, XVII.5.3] yields the isomorphisms $\mathbb{Z}_\ell \otimes \mathcal{M}_B \cong \mathcal{M}_\ell$ and $\mathbb{Z}_\ell \otimes \mathcal{M}_{c,B} \cong \mathcal{M}_{c,\ell}$.

1.2.4. de Rham realization

The construction of the de Rham realization is similar to the one given in [74] except we use the language of log schemes [54]. We let \mathcal{N}_E (resp. \mathcal{N}_X) denote the log structure on \bar{E} (resp.

\bar{X}) associated to E^∞ (resp. X^∞) [54, 1.5]. By [54, 3.5, 3.12], we have an exact sequence of coherent locally free $\mathcal{O}_{\bar{E}}$ -modules

$$(3) \quad 0 \rightarrow \bar{s}^* \omega_{\bar{X}/T}^1 \rightarrow \omega_{\bar{E}/T}^1 \rightarrow \omega_{\bar{E}/\bar{X}}^1 \rightarrow 0,$$

where ω^1 denotes the sheaf of logarithmic relative differentials defined in [54, 1.7]. The sheaves $\omega_{\bar{X}/T}^1$ and $\omega_{\bar{E}/\bar{X}}^1$ are invertible, and can be identified, respectively, with $\Omega_{\bar{X}/T}^1(X^\infty)$ and the sheaf of regular differentials for \bar{s} (denoted $\omega_{\bar{E}/\bar{X}}$ in [18, I.2.1]). Define \mathcal{F}_{dR} as the locally free sheaf $\mathbf{R}^1 \bar{s}_* \omega_{\bar{E}/\bar{X}}^\bullet$ of $\mathcal{O}_{\bar{X}}$ -modules on \bar{X} , where $\omega_{\bar{E}/\bar{X}}^\bullet$ is the complex $d: \mathcal{O}_{\bar{E}} \rightarrow \omega_{\bar{E}/\bar{X}}^1$. This has a decreasing filtration with $\text{Fil}^2 \mathcal{F}_{\text{dR}} = 0$, $\text{Fil}^1 \mathcal{F}_{\text{dR}} = \bar{s}_* \omega_{\bar{E}/\bar{X}}^1$, and $\text{Fil}^0 \mathcal{F}_{\text{dR}} = \mathcal{F}_{\text{dR}}$. We denote $\text{Fil}^1 \mathcal{F}_{\text{dR}}$ simply as ω . We define $\mathcal{F}_{\text{dR}}^k$ as the filtered sheaf of $\mathcal{O}_{\bar{X}}$ -modules $\text{Sym}_{\mathcal{O}_{\bar{X}}}^{k-2} \mathcal{F}_{\text{dR}}$, and we let $\mathcal{F}_{c,\text{dR}}^k = \mathcal{F}_{\text{dR}}^k(-X^\infty)$.

The (logarithmic) Gauss–Manin connection

$$\nabla: \mathcal{F}_{\text{dR}} \rightarrow \mathcal{F}_{\text{dR}} \otimes_{\mathcal{O}_{\bar{X}}} \omega_{\bar{X}/T}^1$$

induces logarithmic connections on $\mathcal{F}_{\text{dR}}^k$ and $\mathcal{F}_{c,\text{dR}}^k$ satisfying Griffiths transversality. We set $\mathcal{M}_{\text{dR}} = \mathbf{H}^1(\bar{X}, \omega^\bullet(\mathcal{F}_{\text{dR}}^k))$ and $\mathcal{M}_{c,\text{dR}} = \mathbf{H}^1(\bar{X}, \omega^\bullet(\mathcal{F}_{c,\text{dR}}^k))$, where we write $\omega^\bullet(\mathcal{G})$ for the complex associated to the module \mathcal{G} with its connection. The filtrations on \mathcal{M}_{dR} and $\mathcal{M}_{c,\text{dR}}$ are defined by those on $\mathcal{F}_{\text{dR}}^k$ and $\mathcal{F}_{c,\text{dR}}^k$. We let $M_{\text{dR}} = \mathbb{Q} \otimes \mathcal{M}_{\text{dR}}$ and $M_{c,\text{dR}} = \mathbb{Q} \otimes \mathcal{M}_{c,\text{dR}}$. Letting $\omega = \bar{s}_* \omega_{\bar{E}/\bar{X}}^1$, we have $\text{gr}^0 \mathcal{F}_{\text{dR}} \cong \omega^{-1}$ by Grothendieck–Serre duality and $\omega^2 \cong \omega_{\bar{X}/T}^1$ by the Kodaira–Spencer isomorphism [18, VI.4.5.2]. It follows that $\omega^2 \cong \omega_{\bar{X}/T}^1$, and one deduces that

$$\text{gr}^i \mathcal{M}_{\text{dR}} \cong \begin{cases} H^0(\bar{X}, \omega^{k-2} \otimes \omega_{\bar{X}/T}^1), & \text{if } i = k - 1; \\ H^1(\bar{X}, \omega^{2-k}), & \text{if } i = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Similarly one finds

$$\text{gr}^i \mathcal{M}_{c,\text{dR}} \cong \begin{cases} H^0(\bar{X}, \omega^{k-2} \otimes \Omega_{\bar{X}/T}^1), & \text{if } i = k - 1; \\ H^1(\bar{X}, \omega^{2-k}(-X^\infty)), & \text{if } i = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Pulling back to $\coprod_t \mathfrak{H}$ and trivializing by $(2\pi i)^{k-1} (dz)^{\otimes(k-2)} d\tau$ yields an isomorphism

$$(4) \quad \alpha: \mathbb{C} \otimes \text{Fil}^{k-1} \mathcal{M}_{\text{dR}} \rightarrow \bigoplus_{t \in (\mathbb{Z}/N\mathbb{Z})^\times} M_k(\Gamma(N))$$

where $M_k(\Gamma(N))$ is the space of modular forms of weight k with respect to $\Gamma(N)$. By the q -expansion principle [18, VII], the map

$$\bigoplus_{t \in (\mathbb{Z}/N\mathbb{Z})^\times} M_k(\Gamma(N)) \rightarrow \bigoplus_{t \in (\mathbb{Z}/N\mathbb{Z})^\times} \mathbb{C}[[q^{1/N}]]$$

that sends $f(\tau) = g(e^{2\pi i \tau/N})$ to $g(q^{1/N})$ identifies $\text{Fil}^{k-1} \mathcal{M}_{\text{dR}}$ as the subset of

$$\bigoplus_{t \in (\mathbb{Z}/N\mathbb{Z})^\times} M_k(\Gamma(N))$$

whose q -expansion at ∞ has coefficients in $\mathbb{Z}[1/Nk!, \mu_N]$, which we view as a subring of $\prod_t \mathbb{C}$ via the embedding defined by $(e^{2\pi it/N})_t$. The same assertions hold with \mathcal{M}_{dR} replaced by $\mathcal{M}_{c,\text{dR}}$ and $M_k(\Gamma(N))$ by $S_k(\Gamma(N))$, the subspace of cusp forms.

To construct the comparison isomorphisms I^∞ , apply GAGA [76] and the Poincaré Lemma to conclude that the pull-back of $\omega^\bullet(\mathcal{F}_{\text{dR}})$ to X^{an} defines a resolution of $\mathbb{C} \otimes \mathcal{F}_B$. Taking symmetric powers then provides resolutions of $\mathbb{C} \otimes \mathcal{F}_B^k$ and $\mathbb{C} \otimes j_! \mathcal{F}_B^k$ where $j: X^{\text{an}} \rightarrow \bar{X}^{\text{an}}$, and taking cohomology yields the desired comparisons. We refer the reader to [25] for more details.

1.2.5. Crystalline realization

We define the crystalline realization using the language of logarithmic crystals as in [31]. Suppose that \bar{Y} is a smooth, proper scheme over $\text{Spec } \mathbb{Z}_\ell$ with a relative divisor D with strict normal crossings, and let $Y = \bar{Y} - D$. For each integer a with $0 \leq a \leq \ell - 2$, Faltings [31, §2i)] defines a category $\mathcal{MF}_{[0,a]}^\nabla(Y)$. In the case of $Y = \bar{Y} = \text{Spec } \mathbb{Z}_\ell$, Faltings’ category can be identified with the full subcategory of $\mathcal{MF}_{\text{tor}}^0$ whose objects A satisfy $\text{Fil}^{a+1} A = 0$. Assuming ℓ does not divide $2N$, let $\mathcal{F}_{\ell\text{-crys}}$ denote the inverse system in $\mathcal{MF}_{[0,1]}^\nabla(X_{\mathbb{Z}_\ell})$ defined by reduction mod ℓ^n of \mathcal{F}_{dR} with its filtration, logarithmic Gauss–Manin connection and locally defined Frobenius maps. Assuming further that $\ell > k - 1$, we let $\mathcal{F}_{\ell\text{-crys}}^k$ denote the inverse system in $\mathcal{MF}_{[0,k-2]}^\nabla(X_{\mathbb{Z}_\ell})$ defined by $\text{Sym}_{\mathcal{O}_{\bar{Y}}}^{k-2} \mathcal{F}_{\ell\text{-crys}}$. If $\ell > k$, we obtain an object

$$\mathcal{M}_{\ell\text{-crys}} = H_{\text{crys}}^1(X_{\mathbb{Z}_\ell}, \mathcal{F}_{\ell\text{-crys}}^k)$$

of \mathcal{MF}^0 whose underlying filtered module can be identified with $\mathbb{Z}_\ell \otimes \mathcal{M}_{\text{dR}}$ (see [31, §4c)). Similarly, taking cohomology with compact support (in the sense of [31]) yields an object $\mathcal{M}_{c,\ell\text{-crys}}$ whose underlying filtered module is $\mathbb{Z}_\ell \otimes \mathcal{M}_{c,\text{dR}}$.

The above identifications provide the comparison isomorphisms I_{dR}^ℓ . The construction of the comparisons I^ℓ relies on Faltings’ comparison theorem between ℓ -adic and crystalline cohomology. Faltings [31, Theorem 2.6] defines a functor \mathbb{D} from $\mathcal{MF}_{[0,a]}^\nabla(Y)$ to the category of finite locally constant étale sheaves on $Y_{\mathbb{Q}_\ell}$ so that $\mathbb{V} = \text{Hom}(\mathbb{D}(\cdot), \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ coincides with that of [39] for $Y = \text{Spec } \mathbb{Z}_\ell$. If $\ell \nmid Nk!$, then we have $\mathbb{V}(\mathcal{F}_{\ell\text{-crys}}) \cong \mathcal{F}_\ell$ by [31, Theorem 6.2], so $\mathbb{V}(\mathcal{F}_{\ell\text{-crys}}^k) \cong \mathcal{F}_\ell^k$ by [31, IIIh], giving $\mathbb{V}(\mathcal{M}_{\ell\text{-crys}}) \cong \mathcal{M}_\ell$ and $\mathbb{V}(\mathcal{M}_{c,\ell\text{-crys}}) \cong \mathcal{M}_{c,\ell}$ by [31, Theorem 5.3].

1.2.6. Weight filtration

There is a natural map $\mathcal{M}_{c,?} \rightarrow \mathcal{M}_?$ for each realization respecting all of the data and comparison isomorphisms. Setting

$$W_i \mathcal{M}_? = \begin{cases} 0, & \text{if } i < k - 1; \\ \text{im}(\mathcal{M}_{c,?} \rightarrow \mathcal{M}_?), & \text{if } k - 1 \leq i < 2(k - 1); \\ \mathcal{M}_?, & \text{if } 2(k - 1) \leq i; \end{cases}$$

$$W_i \mathcal{M}_{c,?} = \begin{cases} 0, & \text{if } i < 0; \\ \ker(\mathcal{M}_{c,?} \rightarrow \mathcal{M}_?), & \text{if } 0 \leq i < k - 1; \\ \mathcal{M}_{c,?}, & \text{if } k - 1 \leq i \end{cases}$$

defines weight filtrations. So we can regard \mathcal{M} and \mathcal{M}_c as objects of $\mathcal{PM}_{\mathbb{Q}}^S$, and $M = \mathbb{Q} \otimes \mathcal{M}$ and $M_c = \mathbb{Q} \otimes \mathcal{M}_c$ as objects of $\mathbf{PM}_{\mathbb{Q}}^S$, where S contains the set of primes dividing $Nk!$. The integer $k \geq 2$ will always be fixed in the discussion and suppressed from the notation; when it is necessary to specify N , we denote the objects $\mathcal{M}(N)$ and $\mathcal{M}(N)_c$.

We let \mathcal{M}_{tf} denote the maximal torsion-free quotient of \mathcal{M} (i.e., $\mathcal{M}/\mathcal{M}[r]$ where $r \in \mathbb{Z}_{>0}$ is chosen to annihilate the torsion in \mathcal{M}_B and $\mathcal{M}[r]$ denotes the kernel of multiplication by r on

\mathcal{M} .) We then let $\mathcal{M}_!$ denote the premotivic structure $\text{im}(\mathcal{M}_c \rightarrow \mathcal{M}_{\text{tf}})$ in $\mathcal{PM}_{\mathbb{Q}}^S$, pure of weight $k - 1$. We let $M_! = \mathbb{Q} \otimes \mathcal{M}_!$.

1.3. The action of $\text{GL}_2(\mathbb{A}_f)$

In this section we define the adelic action on premotivic structures associated to modular forms.

1.3.1. Action on modular forms

We first recall the adelic definition of modular curves and forms. Suppose that U is an open compact subgroup of $\text{GL}_2(\mathbb{A}_f)$ where \mathbb{A}_f denotes the finite adeles. Let U_∞ denote the stabilizer of i in $\text{GL}_2(\mathbb{R})$, so $U_\infty = \mathbb{R}^\times \text{SO}_2(\mathbb{R})$. The analytic modular curve X_U of level U is defined as the quotient

$$\text{GL}_2(\mathbb{Q}) \backslash \text{GL}_2(\mathbb{A}) / UU_\infty.$$

The analytic structure is characterized by requiring that if g is in $\text{GL}_2(\mathbb{A}_f)$, then the map $\mathfrak{H} \rightarrow X_U$ defined by $\gamma(i) \rightarrow \text{GL}_2(\mathbb{Q})g\gamma UU_\infty$, $\gamma \in \text{GL}_2^+(\mathbb{R})$, is holomorphic.

If $\phi: \text{GL}_2(\mathbb{A}) \rightarrow \mathbb{C}$ is such that $\phi(\delta xuv) = \det v (ci + d)^{-k} \phi(x)$ for all $\delta \in \text{GL}_2(\mathbb{Q})$, $x \in \text{GL}_2(\mathbb{A})$, $u \in U$ and $v = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_\infty$, then we define $\phi_g: \mathfrak{H} \rightarrow \mathbb{C}$

$$\phi_g(\gamma(i)) = (\det \gamma)^{-1} (ci + d)^k \phi(g\gamma) \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2^+(\mathbb{R}).$$

We say that such a function ϕ is a *modular form of level U* if for all $g \in \text{GL}_2(\mathbb{A}_f)$, ϕ_g is a modular form of weight k with respect to $gUg^{-1} \cap \text{GL}_2^+(\mathbb{Q})$. We denote this space by $M_k(U)$, and similarly define $S_k(U)$, the space of cusp forms of level U .

Suppose now that U and U' are open compact subgroups of $\text{GL}_2(\mathbb{A}_f)$, and g is an element of $\text{GL}_2(\mathbb{A}_f)$ such that $g^{-1}U'g \subset U$. Note that right multiplication by g induces a holomorphic map $X_{U'} \rightarrow X_U$, and inclusions $M_k(U) \rightarrow M_k(U')$ and $S_k(U) \rightarrow S_k(U')$. We thus obtain an action of $\text{GL}_2(\mathbb{A}_f)$ on

$$\mathcal{A}_k = \varinjlim_U M_k(U) \quad \text{and} \quad \mathcal{A}_k^0 = \varinjlim_U S_k(U).$$

Suppose now that $U = U_N$ for some $N \geq 3$, where $U_N \subset \text{GL}_2(\mathbb{A}_f)$ is the kernel of the reduction map $\text{GL}_2(\hat{\mathbb{Z}}) \rightarrow \text{GL}_2(\mathbb{Z}/N\mathbb{Z})$. For each class $t \in (\mathbb{Z}/N\mathbb{Z})^\times$, we choose an element $g_t \in \text{GL}_2(\hat{\mathbb{Z}})$ whose image in $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})$ is $\begin{pmatrix} 1 & 0 \\ 0 & t^{-1} \end{pmatrix}$. We identify X_N with X_U via the maps $\eta_t: X_{N,t} \rightarrow X_U$ defined by

$$\Gamma(N) \cdot \gamma(i) \mapsto \text{GL}_2(\mathbb{Q}) \cdot g_t \gamma \cdot UU_\infty$$

for $\gamma \in \text{GL}_2^+(\mathbb{R})$. We identify $M_k(U)$ with $\bigoplus_t M_k(\Gamma(N))$ via the isomorphism β defined by $\beta(\phi)_t = \phi_{g_t}$. (Note that η and β are independent of the choices of the g_t .) We thus obtain isomorphisms

$$(5) \quad \beta^{-1} \circ \alpha : \mathbb{C} \otimes \text{Fil}^{k-1} \mathcal{M}_{\text{dR}} \cong M_k(U) \quad \text{and} \quad \mathcal{A}_k \cong \mathbb{C} \otimes \varinjlim_N \text{Fil}^{k-1} \mathcal{M}(N)_{\text{dR}},$$

where α was defined in (4).

1.3.2. Action on premotivic structures

For $h \in \text{GL}_2(\mathbb{A}_f)$ and integers $N, N' \geq 3$, we call (h, N, N') an *admissible triple* if both h and $N'N^{-1}h^{-1} \in M_2(\hat{\mathbb{Z}})$. If (h, N, N') is an admissible triple, then $N|N'$ and $h^{-1}U_{N'}h \subset U_N$. Let

\bar{E}/\bar{X} (resp. \bar{E}'/\bar{X}') denote the universal generalized elliptic curve with level N (respectively N') structure. Note that right multiplication by $N'h^{-1} \in M_2(\hat{\mathbb{Z}})$ defines an endomorphism of $\bar{E}'[N'] = (\mathbb{Z}/N'\mathbb{Z})^2_{/X'}$. We define G to be its image which is a finite flat subgroup of E' . Right multiplication by $N^{-1}N'h^{-1}$ defines an injective map

$$(\mathbb{Z}/N\mathbb{Z})^2 \rightarrow (\mathbb{Z}/N'\mathbb{Z})^2 / ((\mathbb{Z}/N'\mathbb{Z})^2(N'h^{-1}))$$

which gives rise to a level N -structure on E'/G extending to $(\bar{E}'/G)_{\text{cont}}$, the contraction of \bar{E}'/G whose cuspidal fibers are N -gons [18, IV.1]. By the universal property of \bar{E}/\bar{X} , this defines a map $\bar{X}' \rightarrow \bar{X}$ such that $(\bar{E}'/G)_{\text{cont}} \rightarrow \bar{E} \times_{\bar{X}} \bar{X}'$ as generalized elliptic curves with level N -structures. Composing with the natural map $\bar{E}' \rightarrow (\bar{E}'/G)_{\text{cont}}$, we get a commutative diagram

$$(6) \quad \begin{array}{ccc} \bar{E}' & \longrightarrow & \bar{E} \\ \downarrow & & \downarrow \\ \bar{X}' & \longrightarrow & \bar{X}. \end{array}$$

Suppose that $g \in M_2(\hat{\mathbb{Z}}) \cap \text{GL}_2(\mathbb{A}_f)$ with $g^{-1}U_{N'}g \subseteq U_N$. One can then factor $g = rh$ so that $r \in \mathbb{Z}$ and (h, N, N') is admissible. Suppose that S is a set of primes containing those dividing $N'k!$. For $\sharp = \emptyset, c$ and $!$, we write $\mathcal{M}_{\sharp} = \mathcal{M}(N)_{\sharp}^S$ and $\mathcal{M}'_{\sharp} = \mathcal{M}(N')_{\sharp}^S$ for the objects of $\mathcal{PM}_{\mathbb{Q}}^S$ defined in Section 1.2.6. For $? = B, \text{dR}, \ell$ and ℓ -crys with $\ell \nmid N'k!$, we use the top row of (6) to define compatible maps from $\mathcal{F}'_{?}$ to the pullback of $\mathcal{F}_{?}$ along the bottom row, take symmetric products and then take cohomology, yielding morphisms $[h]_{\sharp} : \mathcal{M}_{\sharp} \rightarrow \mathcal{M}'_{\sharp}$. We then obtain morphisms $[g]_{\sharp} : \mathcal{M}_{\sharp} \rightarrow \mathcal{M}'_{\sharp}$ by defining $[g]_{\sharp} = r^{k-2}[h]_{\sharp}$, and this is independent of the factorization $g = rh$. Furthermore, if $N'' \geq 3$ is an integer, $g' \in M_2(\hat{\mathbb{Z}}) \cap \text{GL}_2(\mathbb{A}_f)$ is such that $g'^{-1}U_{N''}g' \subseteq U_N$ and S contains the set of primes dividing $N''k!$, then $[g']_{\sharp} \circ [g]_{\sharp} = [g'g]_{\sharp}$ in $\mathcal{PM}_{\mathbb{Q}}^{S''}$ for $\sharp = \emptyset, c$ and $!$, where S'' is the set of primes dividing $N''k!$. In particular, we obtain an action of $\text{GL}_2(\hat{\mathbb{Z}})/U_N \cong \text{GL}_2(\mathbb{Z}/N\mathbb{Z})$ on \mathcal{M}_{\sharp} . We note the following:

LEMMA 1.1. – *If $g \in M_2(\hat{\mathbb{Z}}) \cap \text{GL}_2(\mathbb{A}_f)$ and $U_{N'} \subset gU_Ng^{-1} \subseteq \text{GL}_2(\hat{\mathbb{Z}})$, then the injective morphism $[g]_c : \mathcal{M}_c \rightarrow (\mathcal{M}'_c)^{gU_Ng^{-1}}$ has cokernel killed by $\|\det g\|^{-1}$.*

Suppose now that $g \in \text{GL}_2(\mathbb{A}_f)$ with $g^{-1}U_{N'}g \subseteq U_N$. We can then write $g = rh$ for some $r \in \mathbb{Q}$ so that (h, N, N') is admissible and obtain morphisms in $\mathcal{PM}_{\mathbb{Q}}^S$ which we also denote $[g]_{\sharp}$. These behave naturally under composition and the resulting action of $\text{GL}_2(\mathbb{A}_f)$ is compatible with the isomorphisms in (5).

1.4. The premotivic structure for forms of level N and character ψ

1.4.1. σ -constructions

Suppose U is any open compact subgroup of $\text{GL}_2(\hat{\mathbb{Z}})$. Let K be a number field with ring of integers \mathcal{O}_K . Let V be a finite dimensional vector space over K and let $\sigma : U \rightarrow \text{Aut}_K(V)$ be a continuous representation of U . Define

$$S_{\sigma} = \{ \ell \in S_f(\mathbb{Q}) \mid \ell \mid k! \text{ or } \text{GL}_2(\mathbb{Z}_{\ell}) \not\subset \ker \sigma \},$$

and suppose that $S \subset S_f(K)$ with $S_{\sigma}^K \subset S$.

Choose $N \geq 3$ such that $U_N \subset \ker \sigma$ and N is divisible only by primes in S , and let $\mathcal{M} = \mathcal{M}(N)^S$. Since U_N is normal in U , by Section 1.3.2, we have a group action of U on

\mathcal{M} . Let \mathcal{V} be an \mathcal{O}_K -lattice in V that is stable under the action of U . We then have an object $\mathcal{M} \otimes \mathcal{V}$ of \mathcal{PM}_K^S defined by $(\mathcal{M} \otimes \mathcal{V})_? = \mathcal{M}_? \otimes \mathcal{V}$ where all additional structures on \mathcal{V} are trivial. Letting U act diagonally on $\mathcal{M} \otimes \mathcal{V}$, we obtain an object

$$\mathcal{M}(\sigma) = (\mathcal{M} \otimes \mathcal{V})^U$$

of \mathcal{PM}_K^S as in Section 1.1.1. We also define objects $\mathcal{M}(\sigma)_\# = (\mathcal{M}_\# \otimes \mathcal{V})^U$ for $\# = \text{tf}$ or c and define

$$\mathcal{M}(\sigma)_! = \text{im}(\mathcal{M}(\sigma)_c \rightarrow \mathcal{M}(\sigma)_{\text{tf}}).$$

We remark that $\mathcal{M}(\sigma)_!$ may lie properly in $(\mathcal{M}_! \otimes \mathcal{V})^U$ and that $\mathcal{M}(\sigma)$ and $\mathcal{M}(\sigma)_{\text{tf}}$ may rely on the choice of N as above. However using the fact that if N' is another choice with $N|N'$ and \mathcal{M}'_c denotes $\mathcal{M}(N')_c^S$, then the natural map $\mathcal{M}_c \rightarrow (\mathcal{M}'_c)^{U_N}$ is an isomorphism (by Lemma 1.1), we conclude that $\mathcal{M}(\sigma)_c$ and $\mathcal{M}(\sigma)_!$ are independent of N . We also define $\mathcal{M}(\sigma)_\# = \mathcal{M}(\sigma)_\# \otimes \mathbb{Q}$ in \mathbf{PM}_K^S for $\# = \emptyset, c$ and $!$; these are also independent of N .

Let U and U' be two open compact subgroups contained in $\text{GL}_2(\hat{\mathbb{Z}})$. Let $\sigma : U \rightarrow \text{Aut}_K(V)$ and $\sigma' : U' \rightarrow \text{Aut}_K(V')$ be two representations with stable \mathcal{O}_K -lattices \mathcal{V} and \mathcal{V}' . Suppose we are given a $g \in \text{GL}_2(\mathbb{A}_f) \cap M_2(\hat{\mathbb{Z}})$ and a K -linear homomorphism $\tau : V \rightarrow V'$ such that $\tau(\sigma(g^{-1}ug)v) = \sigma'(u)\tau(v)$ for all $v \in V$ and $u \in U'_1 = U' \cap gUg^{-1}$. Let S be a subset of $S_f(K)$ containing $S_\sigma^K \cup S_{\sigma'}^K \cup S_g^K$ where S_g is the set of ℓ such that $g_\ell \notin \text{GL}_2(\mathbb{Z}_\ell)$. Choosing suitable N and N' and a coset decomposition $U' = \coprod_i g_i U_1$, the formula

$$x \otimes v \mapsto \sum_i [g_i g]_\# x \otimes \sigma'(g_i)\tau(v)$$

defines maps $[U'gU]_\# : \mathcal{M}(\sigma)_\# \rightarrow \mathcal{M}(\sigma')_\#$. The map is independent of the choices for $\# = c$ and $!$, and for $\# = \emptyset$ after tensoring with \mathbb{Q} ,

1.4.2. Premotivic structure of level N and character ψ

Suppose that $k \geq 2$ and $N \geq 1$. Let ψ be a character $\hat{\mathbb{Z}}^\times \rightarrow K^\times$ of conductor dividing N . Let $U = U_0(N)$ denote the set of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\hat{\mathbb{Z}})$ with $c \in N\hat{\mathbb{Z}}$. Define $\sigma = \sigma(N, \psi)$ by the character $\psi : U_0(N) \rightarrow K^\times$ sending $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to $\psi^{-1}(a_N)$, where a_N denotes the image of a in $\prod_{p|N} \mathbb{Z}_p$. Define $V = V(N, \psi)$ to be the vector space K with an action of U by σ . Let $\mathcal{V} = \mathcal{O}_K \subset V$. Note that $S_\sigma = S_N^K$. We let $\mathcal{M}(N, \psi)_\#$ denote the premotivic structure $\mathcal{M}(\sigma)_\#$ for $\# = c$ or $!$, and let $M(N, \psi)_\# = M(\sigma)_\#$ for $\# = \emptyset, c$ or $!$.

Recall that the isomorphism $\mathbb{C} \otimes \text{Fil}^{k-1} \mathcal{M}(M)_{\text{dR}} \cong M_k(U_M)$ in (5) respects the action of $\text{GL}_2(\hat{\mathbb{Z}})$. It follows that for any embedding $K \rightarrow \mathbb{C}$, the isomorphism identifies $\mathbb{C} \otimes_K \text{Fil}^{k-1} M(N, \psi)_{\text{dR}}$ with $M_k(N, \psi)$, the space of classical modular forms of weight k , level N and character ψ . Under this isomorphism, $\text{Fil}^{k-1} \mathcal{M}(N, \psi)_{\text{dR}}$ corresponds to the set of forms whose q -expansion at ∞ has coefficients in $\tau(\mathcal{O}_S)$. Replacing $M(N, \psi)$ by $\mathcal{M}(N, \psi)_c$ or $\mathcal{M}(N, \psi)_!$ gives the same identifications, but for the space of cusp forms $S_k(N, \psi)$.

1.5. Duality

We now define duality morphisms arising from pairings on the realizations of the premotivic structures associated to modular forms.

1.5.1. Duality at level N

For $N \geq 3$, we let $\mathcal{H} = H_N$ denote the premotivic structure $H_c^2(X_N) = H^2(\bar{X}_N)$. More precisely, we let $\mathcal{H}_B = H^2(\bar{X}^{\text{an}}, \mathbb{Z})$, $\mathcal{H}_\ell = H^2(\bar{X}_{\bar{\mathbb{Q}}}, \mathbb{Z}_\ell)$, $\mathcal{H}_{\text{dR}} = H^1(\bar{X}, \Omega_{\bar{X}/T}^1)$ and

$\mathcal{H}_{\ell\text{-crys}} = H_{\text{crys}}^2(\bar{X}_{\mathbb{Z}\ell}, \mathcal{O}_{\bar{X}_{\mathbb{Z}\ell}, \text{crys}})$ for $\ell \notin S_N$. These come equipped with additional structure and comparison isomorphisms making \mathcal{H} an object of $\mathcal{PM}_{\mathbb{Q}}^{S_N}$, and we let $H = \mathbb{Q} \otimes \mathcal{H}$.

Let $F = \mathbb{Q}(\mu_N)$. The Weil pairing on $(\mathbb{Z}/N\mathbb{Z})_X^2 \cong E[N]$ defines an isomorphism between $(\mathbb{Z}/N\mathbb{Z})_X$ and $\mu_{N,X}$, hence a morphism $X \rightarrow \text{Spec } \mathcal{O}_F[1/N]$. The fibers being geometrically connected, this induces an isomorphism

$$(7) \quad \mathcal{M}_F(-1) \rightarrow \mathcal{H}$$

in $\mathcal{PM}_{\mathbb{Q}}^{S_N}$ whose realizations are given by Poincaré and Serre dualities. Furthermore, the action of $u \in \text{GL}_2(\hat{\mathbb{Z}})$ on \mathcal{H} corresponds to that of $\det u$ on \mathcal{M}_F , where the action on \mathcal{H} arises from the action on \bar{X} (see Section 1.3.2) and that of $\hat{\mathbb{Z}}^\times$ on \mathcal{M}_F is via the isomorphism of class field theory (see Section 1.1.3).

Recall from Section 1.2.2 that \mathcal{F}_B denotes the sheaf $R^1 s_*^{\text{an}} \mathbb{Z}$ on X^{an} . The cup product defines a morphism $\mathcal{F}_B \otimes \mathcal{F}_B \rightarrow (2\pi i)^{-1} \mathbb{Z}$ of locally constant sheaves on X^{an} , inducing a morphism $\mathcal{F}_B^k \otimes \mathcal{F}_B^k \rightarrow (2\pi i)^{2-k} \mathbb{Z}$ defined on sections by

$$(8) \quad x_1 \otimes \cdots \otimes x_{k-2} \otimes y_1 \otimes \cdots \otimes y_{k-2} \mapsto \sum_{\sigma \in \Sigma_{k-2}} \prod_{i=1}^{k-2} x_i \cup y_{\sigma(i)}.$$

Taking cohomology and composing this with the cup product yields a morphism

$$(\ , \)_B : \mathcal{M}_{c,B} \otimes \mathcal{M}_B \rightarrow \mathcal{H}_B(2 - k) \cong \mathcal{M}_F(1 - k)_B,$$

which induces $\mathcal{M}_{c,B} \rightarrow \text{Hom}_{\mathbb{Z}}(\mathcal{M}_B, \mathcal{M}_F(1 - k)_B)$. Defining pairings $(\ , \)_{\text{dR}}$ and $(\ , \)_{\ell}$, one finds that they respect the comparison isomorphisms and weight filtrations yielding morphisms

$$(9) \quad \delta : \mathcal{M}_c \rightarrow \text{Hom}(\mathcal{M}, \mathcal{M}_F(1 - k)) \quad \text{and} \quad \delta_{\ell} : \mathcal{M}_{\ell} \rightarrow \text{Hom}(\mathcal{M}_{\ell}, \mathcal{M}_F(1 - k))$$

in $\mathcal{PM}_{\mathbb{Q}}^S$ if $S_N \subset S$.

The pairing $(\ , \)_{\text{dR}}$ is compatible with the Petersson inner product

$$(g, h)_{\Gamma(N)} = (-2i)^{-1} \int_{\Gamma(N) \backslash \mathfrak{H}} g(\tau) \overline{h(\tau)} (\text{Im } \tau)^{k-2} d\tau \wedge d\bar{\tau}$$

$$\text{for } g \in S_k(\Gamma(N)), h \in M_k(\Gamma(N))$$

as follows. For $g \in \mathbb{C} \otimes \text{Fil}^{k-1} M_{c, \text{dR}}$ and $h \in \mathbb{C} \otimes \text{Fil}^{k-1} M_{\text{dR}}$, write

$$\alpha(g) = (g_t)_t \in \bigoplus_{t \in (\mathbb{Z}/N\mathbb{Z})^\times} S_k(\Gamma(N)) \quad \text{and} \quad \alpha(h) = (h_t)_t \in \bigoplus_{t \in (\mathbb{Z}/N\mathbb{Z})^\times} M_k(\Gamma(N)).$$

After extending scalars to \mathbb{C} for the pairing $(\ , \)_{\text{dR}}$, we have

$$(10) \quad \pi_t(g, (I^\infty)^{-1}(F_\infty \otimes 1)I^\infty h)_{\text{dR}} = (k - 2)!(4\pi)^{k-1} \sum_{t \in (\mathbb{Z}/N\mathbb{Z})^\times} (g_t, h_t)_{\Gamma(N)} \otimes t^{k-1}$$

where $\pi_t : \mathbb{C} \otimes M_{F, \text{dR}} = \mathbb{C} \otimes F \rightarrow \mathbb{C}$ is defined by $e^{2\pi it/N} \in \mu_N(\mathbb{C})$.

1.5.2. Duality for σ -constructions

For the rest of the section, we assume U is an open compact subgroup of $GL_2(\hat{\mathbb{Z}})$ satisfying $\det U = \hat{\mathbb{Z}}^\times$, and let $\psi: \hat{\mathbb{Z}}^\times \rightarrow K^\times$ be a continuous character. For a continuous representation $\sigma: U \rightarrow \text{Aut}_{\mathcal{O}_K} \mathcal{V}$, let $\hat{\sigma}$ denote the representation defined by $\text{Hom}_{\mathcal{O}_K}(\mathcal{V}, \mathcal{O}_K)$. Suppose $N \geq 3$ is such that $U_N \subset \ker \sigma$, the conductor of ψ divides N and $S_N^K \subset S$. Restricting the pairings $(\cdot, \cdot)_?$ to U -invariants, we get a morphism

$$(11) \quad \delta_N: \mathcal{M}_c(\hat{\sigma} \otimes (\psi^{-1} \circ \det)) \rightarrow \text{Hom}_{\mathcal{O}_K}(\mathcal{M}(\sigma), \mathcal{M}_\psi(1 - k))$$

which depends on the choice of N . Tensoring with \mathbb{Q} and normalizing by dividing by $[U : U_N]$, we get an isomorphism

$$\bar{\delta}: M_c(\hat{\sigma} \otimes (\psi^{-1} \circ \det)) \rightarrow \text{Hom}_K(M(\sigma), M_\psi(1 - k))$$

which is independent of N , and we similarly define

$$(12) \quad \bar{\delta}_!: M_!(\hat{\sigma} \otimes (\psi^{-1} \circ \det)) \rightarrow \text{Hom}_K(M_!(\sigma), M_\psi(1 - k)).$$

We say that U is *sufficiently small* if U acts freely on $GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}) / U_\infty$. In particular U is sufficiently small if $U \subset U_1(d)$ for some $d \geq 4$, where $U_1(d)$ denotes the preimage in $GL_2(\hat{\mathbb{Z}})$ of the subgroup of $GL_2(\mathbb{Z}/d\mathbb{Z})$ consisting of matrices of the form $\begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$. One then has a description of $\mathcal{M}(\sigma)_{!,B}$ in terms of the cohomology of the curve X_U with coefficients in a sheaf depending on σ . Poincaré duality on X_U then shows that the isomorphism $\bar{\delta}$ arises from an injective morphism

$$\mathcal{M}(\hat{\sigma} \otimes (\psi^{-1} \circ \det))_! \rightarrow \text{Hom}_{\mathcal{O}_K}(\mathcal{M}(\sigma)_!, \mathcal{M}_\psi(1 - k))$$

whose cokernel \mathcal{C} satisfies $\mathcal{C}_\ell = 0$ for $\ell \nmid N(k - 2)!$. We deduce the following:

LEMMA 1.2. – *Suppose that U has a sufficiently small open compact normal subgroup U' such that $\det U' = \hat{\mathbb{Z}}^\times$ and $\ell \nmid [U : U']$. If $\ell > k - 2$ and $\ker \sigma \subset U_N$ for some N not divisible by ℓ , then $\bar{\delta}_\lambda$ arises from an isomorphism*

$$\mathcal{M}(\hat{\sigma} \otimes (\psi^{-1} \circ \det))_{!,\lambda} \rightarrow \text{Hom}_{\mathcal{O}_{K,\lambda}}(\mathcal{M}(\sigma)_{!,\lambda}, \mathcal{M}_\psi(1 - k)_\lambda)$$

for every λ dividing ℓ .

Suppose now that σ, σ', g, τ and S are as in Section 1.4, so we have morphisms

$$[U'gU]_{\tau,\#}: M(\sigma)_\# \rightarrow M(\sigma')_\#$$

for $\# = \emptyset, c$ and $!$. We then also have morphisms

$$[U(\| \det g \| g)^{-1} U']_{\tau^t \otimes \psi(\det(g)),\#}: M(\sigma' \otimes (\psi^{-1} \circ \det))_\# \rightarrow M(\sigma \otimes (\psi^{-1} \circ \det))_\#$$

which we denote by $[U'gU]_{\tau,\#}^T$. One finds then that $[U'gU]_{\tau,c}^T$ (respectively, $[U'gU]_{\tau,!}^T$) is the adjoint of $[U'gU]_\tau$ (respectively, $[U'gU]_{\tau,!}$) with respect to the pairing $\bar{\delta}$, (respectively, $\bar{\delta}_!$).

1.5.3. Duality for level N , character ψ

We now define duality morphisms for premotivic structures defined in Section 1.4.2. Suppose now that $N \geq 1$, ψ has conductor dividing N and $S_N^K \subset S$. Let $U = U_0(N)$ and $\sigma = \sigma(N, \psi)$ be the representation on $V = V(N, \psi)$ as in Section 1.4.2. Let V' be the one dimensional representation $\text{Hom}_K(V, K) \otimes K_{\psi^{-1} \circ \det}$ of U with natural lattice \mathcal{V}' . We denote by $\sigma' = \hat{\sigma} \otimes (\psi^{-1} \circ \det)$ the representation on V' . Define $\omega: \mathcal{V} \rightarrow \mathcal{V}'$ by sending v_0 to $\hat{v}_0 \otimes 1$, where $\hat{v}_0 \in \text{Hom}(\mathcal{V}, \mathcal{O})$ is such that $\hat{v}_0(v_0) = 1$.

Let w denote $\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}_N$ in $\prod_{p|N} \text{GL}_2(\mathbb{Z}_p)$. The operator $[UwU]_{\omega, \#}$ defines an isomorphism

$$(13) \quad M(N, \psi)_{\#} \rightarrow M(\sigma')_{\#}$$

in \mathcal{PM}_K^S for $\# = \emptyset, c$ and $!$, restricting to an isomorphism on integral structures for $\# = c$ and $!$. One finds that $[UwU]_{\omega}^{-1} = [Uw^{-1}U]_{\omega^t \otimes \psi(\det w)}$ and $[UwU]_{\omega}$ is adjoint to $N^{k-2}[Uw^{-1}U]_{\omega^t \otimes \psi(\det w), c}$ and coincides with $\psi(-1_{\infty})N^{k-2}[Uw^{-1}U]_{\omega}$. Composing the operator $[UwU]_{\omega}$ with the duality morphism $\bar{\delta}$, we obtain a duality isomorphism

$$(14) \quad \hat{\delta}: M(N, \psi) \rightarrow \text{Hom}_K(M(N, \psi)_c, M_{\psi}(1 - k)).$$

Similar assertions hold for $M(N, \psi)!$ yielding an isomorphism $\hat{\delta}_!$. Since $M(N, \psi) = 0$ unless $\psi(-1_{\infty}) = (-1)^{k-2}$, we find that the corresponding perfect pairing is alternating.

1.6. Premotivic structure of a newform

We keep the notation of Section 1.4.2. In particular, we assume $N \geq 1$, ψ is a K -valued Dirichlet character of conductor dividing N , S is a set of primes containing S_N^K and $M(N, \psi)_{\#}$ and $M(N, \psi)_{\#}$ are premotivic structures associated to modular forms of weight k , level N and character ψ . We describe the premotivic structures associated to Hecke eigenforms.

1.6.1. Hecke action

We now define the action of Hecke operators on the premotivic structures defined in Section 1.4.2. For each rational prime p , we have an action of the classical Hecke operator T_p on the spaces $M_k(N, \psi)$ and $S_k(N, \psi)$. Let $\tilde{\mathbb{T}}$ denote the polynomial algebra over \mathcal{O}_K generated by the variables t_p for all primes p . The operators T_p commute on $M_k(N, \psi)$ and $S_k(N, \psi)$ making them $\tilde{\mathbb{T}}$ -modules with t_p acting as T_p . Denote their annihilators $\mathfrak{a}' \subset \mathfrak{a}$, let $\mathbb{T}' = \tilde{\mathbb{T}}/\mathfrak{a}'$ and $\mathbb{T} = \tilde{\mathbb{T}}/\mathfrak{a}$.

PROPOSITION 1.3. – *There is a natural action of \mathbb{T} on $M(N, \psi)!$ and of \mathbb{T}' on $M(N, \psi)_c$ and $M(N, \psi)$ compatible with the isomorphisms*

$$\text{Fil}^{k-1} M(N, \psi)_{!, \text{dR}} \otimes_K \mathbb{C} \cong S_k(N, \psi), \quad \text{Fil}^{k-1} M(N, \psi)_{\text{dR}} \otimes_K \mathbb{C} \cong M_k(N, \psi),$$

the natural morphisms

$$M(N, \psi)_c \rightarrow M(N, \psi)! \rightarrow M(N, \psi)$$

and the duality morphisms $\hat{\delta}$ and $\hat{\delta}_!$ of (14).

Proof. – For $S' = S \cup S_{\{p\}}^K$, the double coset operator

$$[U_0(N) \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}_p U_0(N)]_{\psi(p_p)^{-1}}$$

defines endomorphisms of $\mathcal{M}(N, \psi)_c^{S'}$, $\mathcal{M}(N, \psi)_!^{S'}$ and $M(N, \psi)^{S'}$. It is straightforward to check the compatibility with T_p on $S_k(N, \psi)$ and $M_k(N, \psi)$ and with the indicated morphisms of objects of $\mathbf{PM}_K^{S'}$. These double coset operators commute, yielding an action of $\tilde{\mathbb{T}}$ on $M(N, \psi)_{\sharp, ?}$ for $\sharp = c, !$ and \emptyset and $? = B, \text{dR}$ and λ , restricting to an action on $\mathcal{M}(N, \psi)_{\sharp, ?}$ for $\sharp = c$ and $!$ and $? = B$ and λ .

If $T \in \mathfrak{a}$, then T annihilates $\text{Fil}^{k-1} M(N, \psi)_{!, \text{dR}}$ and is compatible with I_∞ , so it also annihilates

$$\overline{\text{Fil}}^{k-1} M(N, \psi)_{!, \text{dR}} = (I^\infty)^{-1} \circ (\text{id} \otimes c) \circ I^\infty (\text{Fil}^{k-1} M(N, \psi)_{!, \text{dR}}).$$

From the opposition of filtrations in the Hodge structure, we deduce that the action of $\tilde{\mathbb{T}}$ on $M(N, \psi)_{!, \text{dR}}$ and $M(N, \psi)_{!, B}$ factors through \mathbb{T} . From the compatibility with I_B^λ , we deduce the same for $M(N, \psi)_{!, \lambda}$. The same argument shows that the action of $\tilde{\mathbb{T}}$ on $M(N, \psi)_?$ factors through \mathbb{T}' for $? = B, \text{dR}$ and λ . It follows then from the compatibility with $\hat{\delta}$ that the action on $M(N, \psi)_{c, ?}$ factors through \mathbb{T}' for these realizations.

Suppose now that λ is not in S . There is then a unique action of \mathbb{T}' on $\mathcal{M}(N, \psi)_{c, \lambda\text{-crys}}$ compatible with its action on $\mathcal{M}(N, \psi)_\lambda$ and the comparison isomorphism I^λ . For p not divisible by λ , the action of T_p is given by the above double coset operator, hence is compatible with I_{dR}^λ as well. Since such T_p generate the K -algebra $\mathbb{T}' \otimes \mathbb{Q}$, it follows that the action of \mathbb{T}' on $M(N, \psi)_{c, \text{dR}}$ preserves the localization of $\mathcal{M}(N, \psi)_{\text{dR}}$ at λ and is compatible with I_{dR}^λ . We thus obtain the desired action of \mathbb{T}' on the object $\mathcal{M}(N, \psi)_c$ of \mathcal{PM}_K^S . Similarly we conclude that \mathbb{T}' acts on $M(N, \psi)$ and \mathbb{T} acts on $\mathcal{M}(N, \psi)_!$ as desired. \square

1.6.2. Premotivic structure for an eigenform

Now suppose that f is an eigenform in $\text{Fil}^{k-1} \mathcal{M}(N, \psi)_{!, \text{dR}}$ for the action of \mathbb{T} . So for $T \in \mathbb{T}$ we have $T(f) = \theta_f(T)f$ for some \mathcal{O}_K -linear homomorphism $\mathbb{T} \rightarrow \mathcal{O}_K$. We assume f is normalized so that its q -expansion $\sum a_n(f)q^n$ at ∞ has leading term q . We then have $a_p(f) = \theta_f(T_p) \in \mathcal{O}_K$ for all primes p . Let $I_f = \ker \theta_f$ and $\mathcal{M}_f = \mathcal{M}(N, \psi)_![I_f]$ in the notation of Section 1.1.2; thus \mathcal{M}_f is an object of \mathcal{PM}_K^S and $M_f = \mathcal{M}_f \otimes_{\mathcal{O}_K} K$ is in \mathbf{PM}_K^S . Then

$$(15) \quad \text{Fil}^{k-1} \mathcal{M}_{f, \text{dR}} = \mathcal{O}_{K, S} f$$

and M_f is a premotivic structure of rank 2 over K . The $G_{\mathbb{Q}}$ -module $M_{f, \lambda}$ is irreducible. We write $\mathcal{M}_{f, \lambda}$ for the residual representation $\mathcal{M}_{f, \lambda} / \lambda \mathcal{M}_{f, \lambda}$.

For each embedding $\tau: K \rightarrow \mathbb{C}$, we obtain a (classical) normalized eigenform $\tau(f) = \sum \tau(a_n(f))q^n$ in $S_k(N, \psi)$. Conversely, if f is a normalized eigenform in $S_k(N, \psi)$, its q -expansion coefficients $a_n(f)$ generate a number field $K_f \subset \mathbb{C}$, and taking $K \supset K_f$, we can regard f as an eigenform in $\text{Fil}^{k-1} \mathcal{M}(N, \psi)_{!, \text{dR}}$ and consider the associated premotivic structures \mathcal{M}_f and M_f .

We say that f is a newform of level N if each (equivalently, some) $\tau(f)$ is a classical newform of level N . If g is a normalized eigenform in $\text{Fil}^{k-1} \mathcal{M}(N, \psi)_{!, \text{dR}}$, then there is a unique newform f of some level N_f dividing N such that $a_p(f) = a_p(g)$ for all p not dividing N/N_f . In that case, a straightforward construction using double-coset operators defines an isomorphism $M_f \cong M_g$ in \mathbf{PM}_K^S (see Proposition 1.4 below for the cases we need).

If f is a newform of level N , then the pairing $\hat{\delta}_!$ on $M(N, \psi)_!$ restricts to a perfect alternating pairing on M_f , i.e., an isomorphism

$$(16) \quad \wedge_K^2 M_f \cong M_\psi(1 - k).$$

With our normalization of T_p , the Eichler–Shimura relation on $\mathcal{M}(N, \psi)_{!,\lambda}$ due to Deligne [15] takes the form

$$(17) \quad \text{Frob}_p^2 - \psi(p)^{-1} T_p \text{Frob}_p + \psi(p)^{-1} p^{k-1} = 0$$

for all p not dividing $N\ell$, where $\psi(p) = \psi(p_N) = \psi(p_p)^{-1}$. It follows that Frob_p on $M_{f,\lambda}$ has characteristic polynomial

$$(18) \quad X^2 - \psi(p)^{-1} a_p(f) X + \psi(p)^{-1} p^{k-1}.$$

1.6.3. The L -function

Suppose that $f = \sum a_n q^n \in S_k(N, \psi)$ is a newform of weight k , conductor N_f and character ψ_f . Associated to f is the L -function with Euler product factorization:

$$L(f, s) = \sum_{n \geq 1} a_n n^{-s} = \prod_{p \nmid N_f} (1 - a_p p^{-s} + \psi_f(p) p^{k-1-2s})^{-1} \prod_{p \mid N_f} (1 - a_p p^{-s})^{-1}.$$

There is also an irreducible $\text{GL}_2(\mathbb{A}_f)$ -subrepresentation $\pi(f)$ of \mathcal{A}_k^0 with central character $\psi_f \parallel \parallel^{2-k}$ such that f spans the image of $\pi(f)^{U_1(N_f)}$ under the isomorphism

$$(\mathcal{A}_k^0)^{U_1(N_f)} \cong S_k(\Gamma_1(N_f)),$$

where we view ψ_f as a character on $\mathbb{A}_f^\times \subset \mathbb{A}^\times$. (Recall from Section 1.3.1 that

$$\mathcal{A}_k^0 = \varinjlim_{\vec{N}} S_k(U_N) \cong \varinjlim_{\vec{N}} \text{Fil}^{k-1} \mathcal{M}(N)_{!,\text{dR}} \otimes \mathbb{C}.)$$

Moreover, we have the decomposition $\mathcal{A}_k^0 = \bigoplus_f \pi(f)$ where f runs over newforms of weight k of any conductor and character. For each f we have a factorization $\pi(f) \cong \otimes'_p \pi_p(f)$ where $\pi_p(f)$ is an irreducible admissible representation of $\text{GL}_2(\mathbb{Q}_p)$ and \otimes' is a restricted tensor product.

Suppose now that f is as above with $K_f \subset K \subset \mathbb{C}$. For every prime p of \mathbb{Q} and $\lambda \notin S$, the representation $D_{\text{pst}}(M_{f,\lambda} | G_p)^{\text{ss}}$ of the Weil–Deligne group of \mathbb{Q}_p is K -rational and corresponds via local Langlands to $\pi_p(f)$ (where we extend scalars to \mathbb{C} via τ and normalize the local Langlands correspondence as in [9]). For λ not dividing p and p not dividing N , this is the Eichler–Shimura relation (18); for λ dividing p and p dividing N , this is due to Deligne, Langlands and Carayol [9]; for $\lambda|p$, $p \notin S$, this is due to Scholl [75]. It follows that M_f is L -admissible everywhere¹ and that its L -function is related to that of f by the formula

$$(19) \quad L(M_f \otimes_K M_{\psi_f^{-1}}, s) = L(f, s),$$

where the Euler factors defining the first L -function are viewed as \mathbb{C} -valued via the inclusion $K \subset \mathbb{C}$. More generally, for a newform f with coefficients in a number field K , we have

$$L(M_g \otimes_K M_{\psi_g^{-1}}, \tau, s) = L(\tau(g), s)$$

for each embedding $\tau : K \rightarrow \mathbb{C}$, so (19) holds as an identity of $K \otimes \mathbb{C}$ -valued functions.

¹ In fact, the main theorem of [71] shows that $M_f = M^S$ for an object M of \mathbf{PM}_K which is L -admissible everywhere.

1.7. The adjoint premotivic structure

1.7.1. Realizations of the adjoint premotivic structure

Suppose now that f is a newform of weight k , character ψ and level N , with coefficients in K . We define $A_f = \text{ad}^0 M_f$ to be the kernel of the trace morphism

$$\text{Hom}_K(M_f, M_f) \rightarrow K.$$

It is a premotivic structure in \mathbf{PM}_K^S for $S \supseteq S_N^K$.

For $? = B, \text{dR}$ or λ , $A_{f,?}$ has an integral structure given by

$$\mathcal{A}_{f,?} = \{a \in \text{End}(\mathcal{M}_{f,?}) \mid \text{tr}(a) = 0\}.$$

The extra structures on the realizations of A_f are obtained by restrictions from those of $\text{End}(\mathcal{M}_f)$. For example the filtration on $\mathcal{A}_{f,\text{dR}}$ is given by

$$\begin{aligned} \text{Fil}^n \mathcal{A}_{f,\text{dR}} &= \{a \in \mathcal{A}_{\text{dR}} \subseteq \text{End}(\mathcal{M}_{f,\text{dR}}) \mid a(\text{Fil}^i \mathcal{M}_{f,\text{dR}}) \subseteq \text{Fil}^{n+i} \mathcal{M}_{f,\text{dR}}, \forall j\} \\ &= \begin{cases} \mathcal{A}_{f,\text{dR}}, & n \leq 1 - k, \\ \{a \in \mathcal{A}_{f,\text{dR}} \mid a(\text{Fil}^0 \mathcal{M}_{f,\text{dR}}) \subseteq \text{Fil}^0 \mathcal{M}_{f,\text{dR}}\}, & 1 - k < n \leq 0, \\ \{a \in \mathcal{A}_{f,\text{dR}} \mid a(\mathcal{M}_{f,\text{dR}}) \subseteq \text{Fil}^0 \mathcal{M}_{f,\text{dR}}, a(\text{Fil}^0 \mathcal{M}_{f,\text{dR}}) = 0\}, & 0 < n \leq k - 1, \\ 0, & n > k - 1. \end{cases} \end{aligned}$$

Note that defining $\mathcal{A}_{f,\lambda\text{-crys}}$ as above does not yield an object of \mathcal{MF}^0 since the non-trivial graded pieces are in degree $1 - k, 0$ and $k - 1$ (though one can obtain such an object by suitably twisting if $\ell - 1 > 2(k - 1)$).

There is a canonical isomorphism $\det_K A_f \cong K$ in \mathbf{PM}_K^S which restricts to an isomorphism

$$(20) \quad \det_{\mathcal{O}_K} \mathcal{A}_{f,?} \cong \mathcal{O}_{K,?}$$

for $? \in \{B, \text{dR}, \lambda\}$, where $\mathcal{O}_{K,B} = \mathcal{O}_K$ and $\mathcal{O}_{K,\text{dR}} = \mathcal{O}_{K,S}$. (To see this, note that

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \wedge \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \wedge \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is independent of the choice of basis used to represent an endomorphism.) We note also that A_f and $\text{Hom}_K(A_f, K)$ are canonically isomorphic, the isomorphism being defined by the pairing

$$(21) \quad \alpha \otimes \beta \mapsto \text{tr}(\alpha \circ \beta)$$

on each realization of A_f .

Suppose that ψ' is a character $\mathbb{A}^\times \rightarrow K^\times$ of conductor D and that S contains S_K^D as well. Let $f \otimes \psi'$ denote the newform (of weight k , level dividing ND^2 and character $\psi(\psi')^2$) associated to the normalized eigenform $g = \sum_{(n,D)=1} \psi'(n_D) a_n q^n$. Then $M_{f \otimes \psi'}$ is an object of \mathbf{PM}_K^S and one checks that the double coset operator

$$[U_0(ND^2) \begin{pmatrix} 1 & 1/D \\ 0 & 1 \end{pmatrix} U_0(N)]_1$$

induces an isomorphism $M_f \otimes_K M_{\psi'} \cong M_g$. It follows that $M_f \otimes_K M_{\psi'} \cong M_{f \otimes \psi'}$, so

$$(22) \quad A_{f \otimes \psi'} \cong A_f$$

in \mathbf{PM}_K^S . We may therefore assume f has minimal conductor among its twists when considering A_f . We also note that if we replace K by $K' \supset K$ and S by a subset S' of the primes over those in S , then A_f is replaced by $(A_f \otimes_K K')^{S'}$.

1.7.2. Euler factors and functional equation

For each prime p , we let $c_p = v_p(N)$ and let δ_p denote the dimension of $M_{f,\lambda}^{I_p}$ for any λ not dividing p , so

$$\delta_p = \begin{cases} 2, & \text{if } p \nmid N, \\ 1, & \text{if } p \mid N \text{ and } a_p \neq 0, \\ 0, & \text{if } p \mid N \text{ and } a_p = 0. \end{cases}$$

We set $L_p^{\text{nv}}(A_f, s) = L_p(A_f, s)$ if $\delta_p > 0$, and $L_p^{\text{nv}}(A_f, s) = 1$ if $\delta_p = 0$. We let $\Sigma_e = \Sigma_e(f)$ denote the set of primes p such that $\delta_p = 0$ and $L_p(A_f, s) \neq 1$, and set

$$L^{\text{nv}}(A_f, s) = \prod_p L_p^{\text{nv}}(A_f, s) = \prod_{p \notin \Sigma_e(f)} L_p(A_f, s).$$

We call the primes in Σ_e *exceptional* for f .

Recall that if $\delta_p = 2$, then writing

$$L_p(f, s) = (1 - a_p p^{-s} + \psi(p)p^{k-1-2s})^{-1} = (1 - \alpha_p p^{-s})^{-1}(1 - \beta_p p^{-s})^{-1},$$

we have

$$L_p(A_f, s) = (1 - \alpha_p \beta_p^{-1} p^{-s})^{-1}(1 - p^{-s})^{-1}(1 - \alpha_p^{-1} \beta_p p^{-s})^{-1}.$$

If $\delta_p = 1$, then

$$(23) \quad L_p(A_f, s) = \begin{cases} (1 - p^{-1-s})^{-1} & \text{if } \pi_p(f) \text{ is special;} \\ (1 - p^{-s})^{-1} & \text{if } \pi_p(f) \text{ is principal series.} \end{cases}$$

Shimura [80] proved that $L(A_f, s)$ extends to an entire function on the complex plane. Recall that we regard $L(M, s)$ as taking values in $K \otimes \mathbb{C}$. Each embedding $\tau : K \rightarrow \mathbb{C}$ gives a map $K \otimes \mathbb{C} \rightarrow \mathbb{C}$ and we write $L(M, \tau, s)$ for the composite with $L(M, s)$. Moreover, the work of Gelbart and Jacquet [43] and others (see [73]) shows that

$$\begin{aligned} \Lambda(A_f, s) &= L(A_f, s) \Gamma_{\mathbb{R}}(s + 1) \Gamma_{\mathbb{C}}(s + k - 1) \\ &= 2^{2-k-s} \pi^{(1-2k-3s)/2} L(A_f, s) \Gamma\left(\frac{s+1}{2}\right) \Gamma(s + k - 1) \end{aligned}$$

satisfies the functional equation

$$(24) \quad \Lambda(A_f, s) = \epsilon(A_f, s) \Lambda(A_f, 1 - s),$$

where $\epsilon(A_f, s)$ is as defined by Deligne [16]. Here we have used that A_f and $\text{Hom}_K(A_f, K)$ are isomorphic (using (21)).

1.7.3. Variation of integral structures

We maintain the above notation, but now we fix a prime λ of K not in S_N^K and let $S = S_f(K) \setminus \{\lambda\}$. For each finite set of primes $\Sigma \subset S$, we shall define integral structures on M_f and $\wedge_K^2 M_f$. We then compare these as Σ varies, showing that under certain hypotheses, the integral

structure on M_f is invariant, but the variation on $\wedge_K^2 M_f$ is controlled by Euler factors of the adjoint L -function.

Let $N^\Sigma = N \prod_{p \in \Sigma} p^{\delta_p}$. Setting $a'_n = 0$ if n is divisible by a prime in Σ and $a'_n = a_n$ otherwise, we have that $f^\Sigma = \sum a'_n q^n$ is an eigenform of level N^Σ with associated newform f . The construction of Section 1.6.2 thus yields a premotivic structure \mathcal{M}_{f^Σ} in \mathcal{PM}_K^S contained in $\mathcal{M}(N^\Sigma, \psi)_!$. The pairing $\hat{\delta}_!$ defined in Section 1.5.3 restricts to an alternating pairing on M_{f^Σ} , which Proposition 1.4 below shows is non-degenerate, hence induces an isomorphism $\wedge_K^2 M_{f^\Sigma} \rightarrow M_\psi(1-k)$. The image of $\wedge_{\mathcal{O}_K}^2 \mathcal{M}_{f^\Sigma}$ therefore defines an integral structure for $M_\psi(1-k)$, necessarily of the form $\eta_f^\Sigma \cdot \mathcal{M}_\psi(1-k)$ for some fractional ideal $\eta_f^\Sigma \subset K$. We call η_f^Σ the (naive, Σ -finite) congruence \mathcal{O}_K -ideal of f . Note that η_f^Σ is well-behaved under extension of scalars $\mathcal{O}_{K'} \otimes_{\mathcal{O}_K} \cdot$ if $K \subset K'$.

For positive integers m dividing $N^\Sigma/N = \prod_{p \in \Sigma} p^{\delta_p}$, we let ϵ_m denote the morphism $M(N, \psi)_! \rightarrow M(N^\Sigma, \psi)_!$ defined by the operator

$$m^{-1} [U_0(N^\Sigma) \begin{pmatrix} m^{-1} & 0 \\ 0 & 1 \end{pmatrix} U_0(N)]_1 = m^{1-k} [U_0(N^\Sigma) \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix} U_0(N)]_1.$$

We also define the endomorphism ϕ_m of $M(N, \psi)_!$ by

- $\phi_1 = 1, \phi_p = -T_p, \phi_{p^2} = \psi(p)p^{k-1};$
- $\phi_{m_1 m_2} = \phi_{m_1} \phi_{m_2}$ if $(m_1, m_2) = 1.$

We also define

$$\gamma = \sum_m \epsilon_m \phi_m : M(N, \psi)_! \rightarrow M(N^\Sigma, \psi)_!$$

and let γ^t denote its adjoint with respect to the pairings defined in Section 1.5.3.

PROPOSITION 1.4. –

- (a) *The morphism γ restricts to an isomorphism $M_f \rightarrow M_{f^\Sigma}$ in \mathbf{PM}^S with $\gamma_{\text{dR}}(f) = f^\Sigma.$*
- (b) *We have*

$$\gamma^t \circ \gamma = \phi_{N^\Sigma/N_f} \prod_{p \in \Sigma} L_p^{\text{nv}}(A_f, 1)^{-1}$$

on M_f , so $\hat{\delta}_!$ is non-degenerate on $M_{f^\Sigma}.$

- (c) *If $\bar{\mathcal{M}}_{f,\lambda}$ is an irreducible $(\mathcal{O}_K/\lambda)[G_\mathbb{Q}]$ -module, then γ induces an isomorphism $\mathcal{M}_{f,\lambda} \rightarrow \mathcal{M}_{f^\Sigma,\lambda}^\Sigma$ in \mathcal{PM}^S and*

$$\eta_{f,\lambda}^\Sigma = \eta_{f,\lambda}^\emptyset \prod_{p \in \Sigma} L_p^{\text{nv}}(A_f, 1)^{-1}.$$

Proof. – Part (a) and the formula in (b) follow from straightforward double-coset computations similar to those in Chapter 2 of [88] (see also p. 121 of [14]). The non-degeneracy of the pairing follows from ϕ_{N^Σ/N_f} being non-zero on M_f ; in fact it is invertible on $\mathcal{M}_{f,\lambda}$. If $\bar{\mathcal{M}}_{f,\lambda}$ is irreducible, then the image of $\mathcal{M}_{f,\lambda}$ must be of the form $\lambda^n \mathcal{M}_{f^\Sigma,\lambda}$ for some $n \geq 0$; since $\gamma_{\text{dR}}(f) = f^\Sigma$, we see that $n = 0$. The formula for $\eta_{f,\lambda}^\Sigma$ in part (c) then follows from part (b). \square

1.8. Refined integral structures

We now modify some of the constructions of the preceding sections in order to obtain perfect pairings on integral structures and to account for the congruences corresponding to Euler factors missing from $L^{\text{nv}}(A_f, s)$. We also prove some technical results needed for Section 3.2.

We maintain the notation of Section 1.7.3. In particular $S = S_f(K) \setminus \{\lambda\}$ for some $\lambda \notin S_N^K$. We assume also that f has minimal conductor among its twists. We further assume that the representation of $G_{\mathbb{Q}}$ on $M_{f,\lambda}$ is irreducible; moreover if $3 \in \lambda$, then we require its restriction to $G_{\mathbb{Q}(\mu_3)}$ to be absolutely irreducible.

1.8.1. Σ -level structure

Recall that Σ_e denotes the set of exceptional primes defined in Section 1.7.2. Since we assume f has minimal conductor among its twists, we have $p \in \Sigma_e$ if and only if $L_p(A_f, s) = (1 + p^{-s})^{-1}$, which is equivalent to $M_{f,\lambda}|_{G_p}$ being an absolutely irreducible representation induced from a character of G_F where F is the unramified quadratic extension of \mathbb{Q}_p . In particular, its conductor exponent $c_p = v_p(N)$ is even and $\delta_p = 0$. We let

$$\Sigma_1 = \{p \in \Sigma_e \mid p \equiv -1 \pmod{\lambda}\}.$$

The irreducibility hypotheses allow us to choose an auxiliary prime $r > 3$ not dividing $N\ell$ such that $L_r(A_f, 1)^{-1} \in \mathcal{O}_{K,S}^\times$ by Lemma 3 of [27]. We then define

$$N_1^\Sigma = \prod_{p \notin \Sigma_1 \cup \Sigma} p^{c_p} \prod_{p \in \Sigma \cup \{r\}} p^{c_p + \delta_p}.$$

We then define $U^\Sigma = U_0(N_1^\Sigma)$. Note that $U^\Sigma = \prod_p U_p^\Sigma$, where the subgroup U_p^Σ of $\mathrm{GL}_2(\mathbb{Z}_p)$ is determined by whether $p \in \Sigma$. Next we shall define a representation of U^Σ as a tensor product of certain representations of the U_p^Σ for $p|N$.

If $p \in \Sigma$ or $p \notin \Sigma_1$, we let $\mathcal{V}_p = \mathcal{O}_K$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_p^\Sigma$ acting via $\psi(a)$ (which is trivial if $c_p = 0$). For $p \in \Sigma_1$, we let $U_p = U_0(p^{c_p}) \cap \mathrm{GL}_2(\mathbb{Z}_p)$ and $g_p = \begin{pmatrix} 1 & 0 \\ 0 & p^{c_p/2} \end{pmatrix}$. We then define a representation \mathcal{V}'_p of $\mathrm{GL}_2(\mathbb{Z}_p)$ by the following lemma:

LEMMA 1.5. – *There is a finite extension K' of K , a prime λ' of $\mathcal{O}' = \mathcal{O}_{K'}$ over λ and a finite flat \mathcal{O}' -module \mathcal{V}'_p with a continuous action of $\mathrm{GL}_2(\mathbb{Z}_p)$ such that the following hold:*

- (a) $(\mathcal{V}'_p \otimes_{\mathcal{O}', \tau} \pi_p(\tau(f)))^{\mathrm{GL}_2(\mathbb{Z}_p)}$ is one-dimensional for any embedding $\tau: K' \rightarrow \mathbb{C}$;
- (b) $\mathcal{V}'_p/\lambda'\mathcal{V}'_p$ is an absolutely irreducible $(\mathcal{O}'/\lambda')[\mathrm{GL}_2(\mathbb{Z}_p)]$ -module;
- (c) there is a homomorphism of $\mathcal{O}'[\mathrm{GL}_2(\mathbb{Z}_p)]$ -modules

$$\omega_p: \mathcal{V}'_p \cong \mathrm{Hom}'_{\mathcal{O}'}(\mathcal{V}'_p, \mathcal{O}'(\psi^{-1} \circ \det))$$

such that $\omega_{p,\lambda'}$ is an isomorphism;

- (d) there is a homomorphism of $\mathcal{O}'[U_p]$ -modules

$$\tau_p: \mathrm{res}_{g_p \mathrm{GL}_2(\mathbb{Z}_p) g_p^{-1}}^{U_p} g_p \mathcal{V}'_p \rightarrow \mathcal{V}_p$$

such that $\tau_{p,\lambda'}$ is surjective.

Proof. – Let $\varepsilon: \mathcal{O}_F^\times \rightarrow \bar{K}_\lambda^\times$ be the restriction of a character of F^\times corresponding via class field theory to one from which $M_{f,\lambda}|_{G_p}$ is induced. The minimality of the conductor of f among twists implies that $\varepsilon/(\varepsilon \circ \mathrm{Frob}_p)$ has conductor $p^{c_p/2} \mathcal{O}_F$. We let K' be a finite extension of K over which the K -rational representation denoted $\Theta(\varepsilon)$ in Section 3 [13] is defined. Then part (a) follows from compatibility with the local Langlands correspondence and its explicit description in Section 3 of [44]. Part (b) is contained in Lemma 3.2.1 of [13]. Part (c), after tensoring with K' , follows from the first paragraph of Section 3.3 of [13]. Rescaling and applying (b) gives

the desired homomorphism ω_p . Part (d), after tensoring with \mathbb{C} via τ , follows from the fact that $(\mathcal{V}_p \otimes_{\mathcal{O}', \tau} \pi_p(\tau(f)))^{U_p}$ is one-dimensional. It therefore holds after tensoring with K' , and rescaling again gives the desired homomorphism. \square

Replacing K by a larger K' (and λ by λ') if necessary to define the representations \mathcal{V}'_p for $p \in \Sigma_1$, we let $\mathcal{V}^\Sigma_p = \mathcal{V}'_p$ or \mathcal{V}_p according to whether $p \in \Sigma_1 \setminus \Sigma$. We then let $\mathcal{V}^\Sigma = \bigotimes_p \mathcal{V}^\Sigma_p$, $\sigma^\Sigma : U^\Sigma \rightarrow \text{Aut}_{\mathcal{O}_K}(\mathcal{V}^\Sigma)$ and consider the Σ -level premotivic structures $M(\sigma^\Sigma)$ and $\mathcal{M}(\sigma^\Sigma)_\#$ for $\# = c$ and $!$. Note that if $\Sigma_1 \cup \{r\} \subset \Sigma$, then $N_1^\Sigma = N^\Sigma$, $\sigma^\Sigma = \sigma(N^\Sigma, \psi)$ as in Section 1.7.3.

We now define a perfect pairing on $M(\sigma^\Sigma)_!$ giving rise to a perfect pairing on $\mathcal{M}(\sigma^\Sigma)_{!, \lambda}$. Define $w = w_1^\Sigma = \begin{pmatrix} 0 & -1 \\ N_1^\Sigma & 0 \end{pmatrix}_{N_1^\Sigma} \in \text{GL}_2(\mathbb{A}_f)$. Let $\sigma = \sigma^\Sigma$, $\mathcal{V} = \mathcal{V}^\Sigma$ and define

$$\omega : \mathcal{V} \rightarrow \text{Hom}_{\mathcal{O}_K}(\mathcal{V}, \mathcal{O}_K(\psi^{-1} \circ \det)) \cong \otimes \text{Hom}_{\mathcal{O}_K}(\mathcal{V}_p^\Sigma, \mathcal{O}_K(\psi^{-1} \circ \det))$$

as the tensor product of the maps ω_p , where ω_p is defined in Lemma 1.5 if $p \in \Sigma_1 - \Sigma$, and by sending a generator v_0 to the map $v_0 \mapsto 1$ otherwise. We then have that

$$\omega(\sigma(w^{-1}uw)v) = \sigma'(u)\omega(v)$$

for all $u \in U^\Sigma$ and $v \in \mathcal{V}$, so the operator $[UwU]_\omega$ is well-defined and induces a morphism

$$\mathcal{M}(\sigma)_! \rightarrow \mathcal{M}(\hat{\sigma} \otimes (\psi^{-1} \circ \det))_{!, 1}$$

Composing with the isomorphism of (12), we obtain an isomorphism

$$(25) \quad \hat{\delta}_1^\Sigma : M(\sigma)_! \rightarrow \text{Hom}_K(M(\sigma)_!, M_\psi(1 - k))$$

arising from a perfect alternating pairing on $M(\sigma)_!$. Moreover Lemma 1.2 with $U' = U^\Sigma \cap U_1(r)$ yields the following:

COROLLARY 1.6. – *The pairing $\hat{\delta}_{!, \lambda}$ of (25) restricts to an isomorphism*

$$\mathcal{M}(\sigma)_{!, \lambda} \rightarrow \text{Hom}_{\mathcal{O}_{K, \lambda}}(\mathcal{M}(\sigma)_{!, \lambda}, \mathcal{M}_\psi(1 - k)_\lambda).$$

1.8.2. Hecke action and localization

We now define an action of Hecke operators on the Σ -level premotivic structures. For a finite set of primes Ψ , we write $\tilde{\mathbb{T}}^\Psi$ for the \mathcal{O}_K -subalgebra of $\tilde{\mathbb{T}}$ generated by the variables t_p for $p \notin \Psi$. Let Ψ_Σ denote the finite set of primes $p \notin \Sigma$ such that $\delta_p = 1$ or $p \in \Sigma_1$. For primes $p \notin \Psi_\Sigma$, we write T_p for the double coset operator

$$T_p = [U^\Sigma \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}_p U^\Sigma]_{\psi^{-1}(p_p)}.$$

As in Section 1.6.1, we obtain an action of $\tilde{\mathbb{T}}^{\Psi_\Sigma}$ on $M(\sigma^\Sigma)$, $\mathcal{M}(\sigma^\Sigma)_c$ and $\mathcal{M}(\sigma^\Sigma)_!$ factoring through the quotient of $\tilde{\mathbb{T}}^{\Psi_\Sigma}$ by the annihilator of $\text{Fil}^{k-1} M(\sigma^\Sigma)_{\text{dR}}$. Moreover the Hecke operators are self-adjoint with respect to the pairing of (25).

Recall that we are assuming irreducibility of the representation of $G_{\mathbb{Q}}$ on $\bar{M}_{f, \lambda}$. One way we use this hypothesis is to relate cohomology groups with different supports after localizing at maximal ideals of Hecke algebras. For the rest of the section, Σ and Ψ denote finite subsets of S with $\Psi_\Sigma \subset \Psi$ and \mathfrak{m} is the maximal ideal of the maximal ideal $\tilde{\mathbb{T}}^\Psi$ generated by λ and the elements $t_p - a_p(f^{\Sigma \cup \{r\}})$ for all primes $p \notin \Psi$.

We shall need to consider a slightly more general setting than that of the $M(\sigma^\Sigma)_\#$, but can then restrict attention to Betti and λ -adic realizations. Suppose that $N' = N_1^\Sigma D$ for some positive integer D not divisible by any primes in $\Sigma_1 \cup \{\ell\} \setminus \Sigma$, and that U is an open compact subgroup of $\mathrm{GL}_2(\mathbb{A}_f)$ satisfying $U_1(N') \subset U \subset U_0(N')$. Setting $\sigma = \sigma^\Sigma|U$, we define an action of \mathbb{T}^Ψ on the $M(\sigma)_{\#,\?}$ for $\? \in \{B, \lambda\}$ and $\Psi \supset \Psi_\Sigma$ by letting t_p act as

$$T_p = p^{k-2} [U \begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix} U]_1.$$

One checks that the action respects the comparison isomorphism I_B^λ , that $M(\sigma)_{\#,\?}$ is stable for $\# \in \{c, !\}$, and that the resulting action coincides with the ones defined above if $U = U^\Sigma$.

Let $\Sigma' = \Sigma \cup \Sigma_1$, $U' = U^{\Sigma'} \cap U$ and $\sigma' = \sigma^{\Sigma'}|U'$. Define $g \in \mathrm{GL}_2(\mathbb{A}_f)$ with g_p as in Lemma 1.5 for $p \in \Sigma_1 \setminus \Sigma$ and $g_p = 1$ otherwise. Define $\alpha: \mathcal{V}^\Sigma \rightarrow \mathcal{V}^{\Sigma'}$ by $\otimes_p \alpha_p$ with $\alpha_p = \tau_p g_p$ as in Lemma 1.5 for $p \in \Sigma_1 \setminus \Sigma$ and the identity otherwise. The operator $[U' g U]_{\alpha,c,B}$ then defines a \mathbb{T}^Ψ -linear homomorphism $\mathcal{M}(\sigma)_{c,B} \rightarrow \mathcal{M}(\sigma')_{c,\?}$.

LEMMA 1.7. – *The map $[U' g U]_{\alpha,c,B}$ is injective.*

Proof. – Let $d = \prod_{p \in \Sigma_1 \setminus \Sigma} p^{c_p/2}$, $\mathcal{M}_c = \mathcal{M}_c(N'd)$, $\mathcal{M}'_c = \mathcal{M}_c(N'd^2)$, $\mathcal{V} = \mathcal{V}^\Sigma$ and $\mathcal{V}' = \mathcal{V}^{\Sigma'}$. Writing $g^{-1}\alpha$ as a composite $\mathcal{V} \rightarrow \mathrm{Ind}_{g^{-1}U'g}^U g^{-1}\mathcal{V} \rightarrow g^{-1}\mathcal{V}'$, we can write $[U' g U]_{\alpha,c}$ as

$$(\mathcal{M}_{c,B} \otimes \mathcal{V})^U \rightarrow (\mathcal{M}_{c,B} \otimes \mathrm{Ind}_{g^{-1}U'g}^U g^{-1}\mathcal{V}')^U \rightarrow (\mathcal{M}_{c,B} \otimes g^{-1}\mathcal{V}')^{g^{-1}U'g} \rightarrow (\mathcal{M}'_{c,B} \otimes \mathcal{V}')^{U'}$$

where the last map is defined by $[g]_{c,B} \otimes g$. The first map is injective since \mathcal{V} is irreducible, the second is an isomorphism by Shapiro’s Lemma, and the last is injective by Lemma 1.1. \square

Suppose for the moment that we also have $U \subset U_1(r)$. Letting $\Gamma = \mathrm{SL}_2(\mathbb{Z}) \cap U$, we have that Γ acts freely on \mathfrak{H} and X_U can be identified with $\Gamma \backslash \mathfrak{H}$. We write \mathcal{F}_B^k for the locally constant sheaf $\mathrm{Sym}_{\mathbb{Z}}^{k-2} R^1 s_* \mathbb{Z}$, where s is the natural projection $E_U \rightarrow X_U$ with $E_U = \Gamma \backslash (\mathfrak{H} \times \mathbb{C}) / (\mathbb{Z} \times \mathbb{Z})$ defined as in Section 1.2.1. The representation σ defines an action of Γ on \mathcal{V} , and we let \mathcal{F}_σ denote the locally constant sheaf on X_U defined by $\Gamma \backslash (\mathfrak{H} \times \mathcal{V})$. We can then identify $\mathcal{M}(\sigma)_{c,B}$ with $H_c^1(X_U, \mathcal{F}_B^k \otimes \mathcal{F}_\sigma)$ and $M(\sigma)_B$ with $K \otimes_{\mathcal{O}_K} H^1(X_U, \mathcal{F}_B^k \otimes \mathcal{F}_\sigma)$. If σ is the trivial representation of U on \mathcal{O}_K , then the usual action of the Hecke operator T_p on $H_c^1(X_U, \mathcal{F}_B^k \otimes \mathcal{O}_K)$ and $H^1(X_U, \mathcal{F}_B^k \otimes \mathcal{O}_K)$ is compatible with the ones we defined on $\mathcal{M}(\sigma)_c$ and $M(\sigma)$.

LEMMA 1.8. – *If $U \subset U_1(r)$ and σ is trivial, then the natural map*

$$H_c^1(X_U, \mathcal{F}_B^k \otimes \mathcal{O}_K)_\mathfrak{m} \rightarrow H^1(X_U, \mathcal{F}_B^k \otimes \mathcal{O}_K)_\mathfrak{m}$$

is an isomorphism.

We recall the idea of the proof, which is standard. The kernel of the map is torsion-free, and the cokernel has no λ -torsion since $N'(k-2)! \notin \lambda$. After tensoring with K , one has $T_p = p^{k-1} + 1$ on the kernel and cokernel for all $p \equiv 1 \pmod{N'}$. Thus if \mathfrak{m} is in the support of the kernel or cokernel, then $T_p - 2 \in \mathfrak{m}$ for all $p \equiv 1 \pmod{N'\ell}$, $p \notin \Psi$. Arguing as in Proposition 2 of [26], one obtains a contradiction to the hypothesis that $\mathcal{M}_{f,\lambda}$ is absolutely irreducible.

Let us now return to the case of arbitrary U with $U_1(N') \subset U \subset U_0(N')$. We let $U'' = U' \cap U_1(r)$ and consider the commutative diagram of $\tilde{\mathbb{T}}^\Psi$ -modules

$$\begin{CD} \mathcal{M}(\sigma)_{c,B} @>>> \mathcal{M}(\sigma')_{c,B} @>>> H_c^1(X_{U''}, \mathcal{F}_B^k \otimes \mathcal{O}_K) \\ @VVV @VVV @VVV \\ \mathcal{M}(\sigma)_{!,B} @>>> \mathcal{M}(\sigma')_{!,B} @>>> H^1(X_{U''}, \mathcal{F}_B^k \otimes \mathcal{O}_K)_{\text{tf}}. \end{CD}$$

The horizontal maps in the top row are injective (the first by Lemma 1.7) and the right most vertical map is an isomorphism after localizing at \mathfrak{m} by Lemma 1.8. We thus have:

COROLLARY 1.9. – *The localization at \mathfrak{m} of the natural map $\mathcal{M}(\sigma)_{c,?} \rightarrow \mathcal{M}(\sigma)_{!,?}$ is an isomorphism for $? \in \{B, \lambda\}$.*

Note that the case of $? = \lambda$ follows from that of $? = B$. In fact, $\mathcal{M}(\sigma)_{\#, \lambda}$ is isomorphic as a $\tilde{\mathbb{T}}^\Psi$ -module to the completion of $\mathcal{M}(\sigma)_{\#, B}$ at λ . Note also that $\mathcal{M}(\sigma)_{\#, \lambda}$ is isomorphic to the direct sum of its localizations at the maximal ideals over λ in its support as a $\tilde{\mathbb{T}}^\Psi$ -module.

Finally we shall need the following generalization of a lemma of De Shalit in Section 3.2.

LEMMA 1.10. – *Suppose that U has ℓ -power index in $U_0(N')$ and let $\Delta = U_0(N')/U$. Then $\mathcal{M}(\sigma)_{!, \lambda, \mathfrak{m}}^-$ is a free $\mathcal{O}_{K, \lambda}[\Delta]$ -module.*

Proof. – Let $U'' = U \cap U_1(r)$ and $\sigma'' = \sigma|_{U''}$. Note that $(\mathbb{Z}/r\mathbb{Z})^\times$ has order not divisible by ℓ , and that $U_0(N')/U'' \cong \Delta \times (\mathbb{Z}/r\mathbb{Z})^\times$ acts on $\mathcal{M}(\sigma'')_{c, \lambda, \mathfrak{m}}$. It follows that $\mathcal{M}(\sigma)_{c, \lambda, \mathfrak{m}} \cong \mathcal{M}(\sigma'')_{c, \lambda, \mathfrak{m}}^{(\mathbb{Z}/r\mathbb{Z})^\times}$ is an $\mathcal{O}_{K, \lambda}[\Delta]$ -module summand of $\mathcal{M}(\sigma'')_{c, \lambda, \mathfrak{m}}$ and $\mathcal{M}(\sigma)_{c, \lambda, \mathfrak{m}}^-$ is an $\mathcal{O}_{K, \lambda}[\Delta]$ -module summand of $\mathcal{M}(\sigma'')_{c, \lambda, \mathfrak{m}}^-$. Note also that the ring $\mathcal{O}_{K, \lambda}[\Delta]$ is local.

Suppose first that $k = 2$, ψ is trivial and $\Sigma_1 \subset \Sigma$. The argument of Proposition 1 of [86] shows that $H^1(X_{U''}, \mathcal{F}_B^k \otimes \mathcal{O}_{K, \lambda})^-$ is free over $\mathcal{O}_{K, \lambda}[\Delta]$, hence so is its summand

$$\mathcal{M}(\sigma'')_{c, \lambda, \mathfrak{m}}^- \cong H_c^1(X_{U''}, \mathcal{F}_B^k \otimes \mathcal{O}_{K, \lambda})_{\mathfrak{m}}^- \cong H^1(X_{U''}, \mathcal{F}_B^k \otimes \mathcal{O}_{K, \lambda})_{\mathfrak{m}}^-,$$

where the first isomorphism is gotten from I_B^λ and the second from Lemma 1.8. It follows that its summand $\mathcal{M}(\sigma)_{c, \lambda, \mathfrak{m}}^-$ is free, hence so is $\mathcal{M}(\sigma)_{!, \lambda, \mathfrak{m}}^-$ by Corollary 1.9.

Suppose next that $k > 2$, ψ is non-trivial or $\Sigma_1 \not\subset \Sigma$. Denoting $\mathcal{O}_{K, \lambda} \otimes_{\mathcal{O}_K} \mathcal{F}_\sigma$ by $\mathcal{F}_{\sigma'', \lambda}$, we have

$$H_c^1(X_{U''}, \mathcal{F}_B^k \otimes \mathcal{F}_{\sigma'', \lambda}) \cong \begin{cases} \mathcal{M}(\sigma'')_{c, \lambda}, & \text{if } i = 1, \\ 0, & \text{otherwise} \end{cases}$$

(the case $i = 2$ following from the vanishing of $H^0(X_{U''}, \mathcal{F}_B^k \otimes \mathcal{F}_{\sigma''}/\lambda)$). Note that this holds in the case $U = U_0(N')$ as well, and the Serre–Hochschild spectral sequence with respect to the cover $X_{U''} \rightarrow X_{U_0(N') \cap U_1(r)}$ gives $H^i(\Delta, \mathcal{M}(\sigma'')_{c, \lambda}) = 0$ for all $i > 0$. By [6, VI.8.10], it follows that $\mathcal{M}(\sigma'')_{c, \lambda}$ is free, hence so is its summand $\mathcal{M}(\sigma)_{!, \lambda, \mathfrak{m}}^- \cong \mathcal{M}(\sigma)_{c, \lambda, \mathfrak{m}}^-$. \square

1.8.3. Ihara’s Lemma

For finite subsets $\Sigma \subset \Sigma'$ of $S = S_f(K) \setminus \{\lambda\}$, we shall define a morphism

$$\gamma_{\Sigma}^{\Sigma'} : M(\sigma^\Sigma)_! \rightarrow M(\sigma^{\Sigma'})_!$$

generalizing the one in Section 1.7.3. We shall prove a result needed in Section 3.2—that this map is injective with torsion-free cokernel on certain localizations of the integral Betti realization. The

result stems from a lemma of Ihara which has been generalized in various ways for applications to congruences between modular forms (see [68,20] and Chapter 2 of [88]).

For positive integers m dividing $N^{\Sigma' \cup \{r\}}/N^{\Sigma \cup \{r\}}$, we define

$$\epsilon_{\Sigma, m}^{\Sigma'} = m^{1-k} [U^{\Sigma'} \begin{pmatrix} 1 & 0 \\ 0 & md \end{pmatrix} U^{\Sigma}]_{\alpha, !} : M(\sigma^{\Sigma})_! \rightarrow M(\sigma^{\Sigma'})_!,$$

where d and α are as in Lemma 1.7. We then define

$$\gamma_{\Sigma}^{\Sigma'} = \sum_m \epsilon_{\Sigma, m}^{\Sigma'} \phi_m$$

where the sum is over positive divisors of $N^{\Sigma' \cup \{r\}}/N^{\Sigma \cup \{r\}}$ and ϕ_m is defined in Section 1.7.3. Note that $\gamma_{\Sigma}^{\Sigma'}$ is $\tilde{\mathbb{T}}^{\Psi}$ -linear where Ψ is the union of Ψ_{Σ} and the set of primes dividing $N_1^{\Sigma'}/N_1^{\Sigma}$. One checks also that if $\Sigma \subset \Sigma' \subset \Sigma''$, then $\gamma_{\Sigma}^{\Sigma''} = \gamma_{\Sigma'}^{\Sigma''} \circ \gamma_{\Sigma}^{\Sigma'}$.

We let \mathfrak{m} denote the maximal ideal of $\tilde{\mathbb{T}}^{\Psi_{\Sigma}}$ defined as in Section 1.8.2, and similarly define \mathfrak{m}' using Σ' . Note that \mathfrak{m}' might not lie over \mathfrak{m} , but that they lie over the same maximal ideal \mathfrak{m}'' of $\tilde{\mathbb{T}}^{\Psi}$.

The argument in the first part of the proof of the lemma on p. 491 of [88] shows that $\tilde{\mathbb{T}}^{\Psi}$ and $\tilde{\mathbb{T}}^{\Psi_{\Sigma}}$ have the same image in $\text{End}_K M(\sigma^{\Sigma})_!$. Since \mathfrak{m} is in the support of $\mathcal{M}(\sigma^{\Sigma})_{!, B}$, it follows that the localization map $\mathcal{M}(\sigma^{\Sigma})_{!, B, \mathfrak{m}''} \rightarrow \mathcal{M}(\sigma^{\Sigma})_{!, B, \mathfrak{m}}$ is an isomorphism. Composing its inverse with the map

$$\mathcal{M}(\sigma^{\Sigma})_{!, B, \mathfrak{m}''} \rightarrow \mathcal{M}(\sigma^{\Sigma'})_{!, B, \mathfrak{m}''} \rightarrow \mathcal{M}(\sigma^{\Sigma'})_{!, B, \mathfrak{m}'}$$

induced by $\gamma_{\Sigma}^{\Sigma'}$, we obtain a morphism

$$(26) \quad \mathcal{M}(\sigma^{\Sigma})_{!, B, \mathfrak{m}} \rightarrow \mathcal{M}(\sigma^{\Sigma'})_{!, B, \mathfrak{m}'}$$

which we denote $\gamma_{\mathfrak{m}} = \gamma_{\Sigma, \mathfrak{m}}^{\Sigma'}$. Similarly, we have a morphism

$$(27) \quad \hat{\gamma}_{\mathfrak{m}} = \hat{\gamma}_{\Sigma, \mathfrak{m}}^{\Sigma'} : \mathcal{M}(\sigma^{\Sigma})_{!, \lambda, \mathfrak{m}} \rightarrow \mathcal{M}(\sigma^{\Sigma'})_{!, \lambda, \mathfrak{m}'}$$

LEMMA 1.11. – *The $\mathcal{O}_{K, S}$ -linear (respectively, $\mathcal{O}_{K, \lambda}$ -linear) map $\gamma_{\mathfrak{m}}$ (respectively, $\hat{\gamma}_{\mathfrak{m}}$) is injective with torsion-free cokernel.*

Proof. – First note the lemma is equivalent to the injectivity of $\gamma_{\mathfrak{m}}$ mod λ , and by the formula $\gamma_{\Sigma, \mathfrak{m}}^{\Sigma''} = \gamma_{\Sigma', \mathfrak{m}}^{\Sigma''} \circ \gamma_{\Sigma, \mathfrak{m}}^{\Sigma'}$, we can assume $\Sigma' = \Sigma \cup \{p\}$ for some $p \notin \Sigma$. Note that the case $p = r$ is clear, and that if p divides $N_1^{\Sigma'}/N_1^{\Sigma}$, then $T_p \circ \gamma = 0$ and $\mathfrak{m}' = (\mathfrak{m}'', t_p)$. Thus by Corollary 1.9, it suffices to prove that if $p \neq r$, then

$$(28) \quad \mathcal{M}(\sigma^{\Sigma})_{c, B, \mathfrak{m}''}/\lambda \rightarrow \mathcal{M}(\sigma^{\Sigma'})_{c, B, \mathfrak{m}''}/\lambda$$

is injective.

First we consider the case $\delta_p = 0$. If $p \notin \Sigma_1$ then γ is the identity, so we assume $p \in \Sigma_1$. Using part (b) of Lemma 1.5, the argument in the proof of Lemma 1.7 carries over mod λ , giving the injectivity of

$$(\mathcal{M}_c \otimes \mathcal{V}/\lambda)^U \rightarrow (\mathcal{M}'_c \otimes \mathcal{V}'/\lambda)^{U'}$$

hence that of (28).

Having proved the lemma for $p \in \Sigma_1 \cup \{r\}$, we may assume for the remaining cases ($\delta_p = 1$ or 2 , $p \neq r$), that $\Sigma_1 \cup \{r\} \subset \Sigma$. Setting $g = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$, $\mathcal{V}_0 = \mathcal{V}(N, \psi)/\lambda$, $N' = N^\Sigma$, $N'' = N^{\Sigma p^{\delta_p}}$, $U = U_0(N')$, $U' = U_0(N'')$, $A = \mathcal{M}(N')_{c,B} \otimes \mathcal{V}_0$ and $A' = \mathcal{M}(N'')_{c,B} \otimes \mathcal{V}_0$, it suffices to show that the map

$$(29) \quad \bigoplus_{i=0}^{\delta_p} A_{\mathfrak{m}''}^U \rightarrow (A')_{\mathfrak{m}''}^{U'}$$

defined by $([1]_c, [g]_c, \dots, [g]_c^{\delta_p})$ is injective.

Suppose now that $\delta_p = 1$. Then Lemma 1.1 yields an isomorphism $(A')^{U'} \rightarrow (A')^{g^{-1}U'g}$ induced by $[g]_c$ (each module being identified with the U' -invariants in $\mathcal{M}(N^{\Sigma p^2})_{c,B} \otimes \mathcal{V}_0$). Furthermore one finds that $[1]_c$ (respectively, $[g]_c$) maps A^U isomorphically $(A')^{g^{-1}U'g}$ (respectively, $(A')^{U'}$). Therefore it suffices to prove these have trivial intersection. Suppose then that $x \otimes v \in A'$ with $v \neq 0$ is invariant under U and $g^{-1}U'g$. Since the p -part of the conductor of ψ is p^{c_p} , we may choose $a \in 1 + p^{c_p-1}\mathbb{Z}_p$ so that $\psi(a) \neq 1$. One checks that $h = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ is in the subgroup of $\text{GL}_2(\hat{\mathbb{Z}})$ generated by U and $g^{-1}U'g$. Therefore $x \otimes v = h(x \otimes v) = x \otimes \psi(a)v$ implies that $x = 0$.

Finally consider the case $\delta_p = 2$. Let $A'' = \mathcal{M}(N'p)_{c,B} \otimes \mathcal{V}_0$ and $U'' = U_0(N'p)$. Since U is generated by U'' and $g^{-1}U''g$, the argument in the case $\delta_p = 1$ applied to U'' instead of U now yields an exact sequence

$$(30) \quad A^U \rightarrow (A'')^{U''} \times (A'')^{U''} \rightarrow (A')^{U'}$$

where the maps are given by $\begin{pmatrix} -[g]_c \\ [1]_c \end{pmatrix}$ and $([1]_c, [g]_c)$. We combine this with Lemma 3.2 of [20], whose proof shows that the map

$$H_p^1(X_1(N'), \mathcal{F}_B^k/\lambda)^2 \rightarrow H_p^1(X_1(N', p), \mathcal{F}_B^k/\lambda)$$

induced by $([1], [g])$ is injective, where $X_1(N', p)$ is the modular curve associated to $U_1(N') \cap U_0(p)$. Lemma 1.8 then gives the injectivity of

$$H_c^1(X_1(N'), \mathcal{F}_B^k/\lambda)_{\mathfrak{m}''}^2 \rightarrow H_c^1(X_1(N', p), \mathcal{F}_B^k/\lambda)_{\mathfrak{m}''},$$

whence the injectivity of $([1]_c, [g]_c): (A^U)_{\mathfrak{m}''}^2 \rightarrow (A'')_{\mathfrak{m}''}^{U''}$. Combining this with the exactness of the localization at \mathfrak{m}'' of (30) we deduce the injectivity of (29). \square

1.8.4. Comparison of integral structures

We now generalize Proposition 1.4 to the setting of the refined integral structures defined in Section 1.8.1. Define $\mathcal{M}_{f,1}^\Sigma = \mathcal{M}(\sigma^\Sigma)_! [\tilde{I}_f^\Sigma]$ where \tilde{I}_f^Σ is the preimage of $I_{f^{\Sigma \cup \{r\}}}$ in $\tilde{\mathbb{T}}^{\Psi^\Sigma}$. (Recall that $f^{\Sigma \cup \{r\}}$ is the eigenform of level N^Σ defined in Section 1.7.3 and I_g was defined in Section 1.6.2.) Using strong multiplicity one and Lemma 1.5(a), one sees that $\dim_K \text{Fil}^{k-1} M_{f,1,\text{dR}}^\Sigma = 1$ and therefore that $M_{f,1,\text{dR}}^\Sigma$ has rank two over K , where $M_{f,1}^\Sigma = K \otimes_{\mathcal{O}_K} \mathcal{M}_{f,1}^\Sigma$. Note that if $\Sigma_1 \cup \{r\} \subset \Sigma$, then $\mathcal{M}_{f,1}^\Sigma = \mathcal{M}_{f^\Sigma}$.

If $\Sigma \subset \Sigma'$, then the restriction of $\gamma_{\Sigma'}^\Psi$ (defined in Section 1.8.3) defines a morphism $M_{f,1}^{\Sigma'} \rightarrow M_{f,1}^{\Sigma}$. (This follows from $\tilde{\mathbb{T}}^\Psi$ -linearity with Ψ as in Section 1.8.3 and the fact that $T_p \gamma_{\Sigma'}^\Psi = 0$ for $p | N_1^{\Sigma'} / N_1^\Sigma$.) Note that the maximal ideal of $\tilde{\mathbb{T}}^{\Psi^\Sigma}$ defined in Section 1.8.2 is simply $(\tilde{I}_f^\Sigma, \lambda)$, so the natural map $\mathcal{M}_{f,1,B}^\Sigma \rightarrow \mathcal{M}(\sigma^\Sigma)_!, B, \mathfrak{m}$ is injective. It therefore follows from Lemma 1.11 that

$\gamma_{\Sigma'}^{\Sigma}$ is injective on $M_{f,1}^{\Sigma}$, hence induces an isomorphism $M_{f,1}^{\Sigma} \xrightarrow{\sim} M_{f,1}^{\Sigma'}$. Moreover it restricts to isomorphisms $\mathcal{M}_{f,1,?}^{\Sigma} \xrightarrow{\sim} \mathcal{M}_{f,1,?}^{\Sigma'}$ for $? = \lambda$ and dR.

Now let $\gamma_{\Sigma'}^{\Sigma}$ denote the transpose of $\gamma_{\Sigma'}^{\Sigma}$ with respect to the pairings δ_1^{Σ} and $\delta_1^{\Sigma'}$ defined in Section 1.8.1. Using $\tilde{\Psi}$ -linearity again, we see that $\gamma_{\Sigma'}^{\Sigma}$ maps $M_{f,1}^{\Sigma'}$ to $M_{f,1}^{\Sigma}$. Recall that if $\Sigma_1 \cup \{r\} \subset \Sigma'$, then the pairing on $M_{f,1}^{\Sigma'} = M_{f,\Sigma'}$ is alternating and non-degenerate (Proposition 1.4). It follows that the same is true for arbitrary Σ and that for any $\Sigma \subset \Sigma'$, the restriction of $\gamma_{\Sigma'}^{\Sigma}$ is an isomorphism. We thus obtain an isomorphism

$$\wedge_K^2 M_{f,1}^{\Sigma} \rightarrow M_{\psi}(1 - k).$$

We let $\eta_{f,1}^{\Sigma}$ be the ideal in $\mathcal{O}_{K,\lambda}$ such that $\wedge_{\mathcal{O}_{K,\lambda}}^2 \mathcal{M}_{f,1,\lambda}^{\Sigma}$ maps isomorphically to $\eta_{f,1}^{\Sigma} \mathcal{M}_{\psi}(1 - k)_{\lambda}$. Note that if $\Sigma_1 \cup \{r\} \subset \Sigma$, then $\eta_{f,1}^{\Sigma} = \eta_{f,\lambda}^{\Sigma}$.

We now state the generalization of Proposition 1.4.

PROPOSITION 1.12. – *Suppose that $\Sigma \subset \Sigma'$ are finite subsets of $S = S_f(K) \setminus \{\lambda\}$.*

- (a) *The morphism $\gamma_{\Sigma'}^{\Sigma}$ restricts to an isomorphism $M_{f,1}^{\Sigma} \rightarrow M_{f,1}^{\Sigma'}$ in \mathbf{PM}^S with $\mathcal{M}_{f,1,?}^{\Sigma} \xrightarrow{\sim} \mathcal{M}_{f,1,?}^{\Sigma'}$ for $? = \lambda$ and dR.*
- (b) *We have*

$$\gamma_{\Sigma'}^{\Sigma} \circ \gamma_{\Sigma}^{\Sigma'} = \beta_{f,\Sigma}^{\Sigma'} \prod_{p \in \Sigma} L_p(A_f, 1)^{-1}$$

on $M_{f,1}^{\Sigma}$ for some non-zero $\beta_{f,\Sigma}^{\Sigma'}$ in $\mathcal{O}_{K,S}$. Moreover $\beta_{f,\Sigma}^{\Sigma'} = \varphi_{N^{\Sigma'} / N^{\Sigma}}$ if $\Sigma_1 \cap \Sigma' \setminus \Sigma = \emptyset$ (cf. Proposition 1.4).

- (c) *The pairing δ_1^{Σ} is non-degenerate and S -integral on $M_{f,1}^{\Sigma}$, and*

$$\eta_{f,1}^{\Sigma} \subset \eta_{f,1}^{\emptyset} \prod_{p \in \Sigma} L_p(A_f, 1)^{-1}.$$

Proof. – Part (a) and the first part of (c) have already been shown, and the formula in (c) follows from the one in (b). Part (b) reduces to the case $\Sigma' = \Sigma \cup \{p\}$ for some $p \notin \Sigma$. If $p = r$, the result is clear since $L_r(A_f, 1) \in \mathcal{O}_{K,S}^{\times}$. If $p \notin \Sigma_1 \cup \{r\}$, the computation is the same as in Proposition 1.4. Finally, for $p \in \Sigma_1$, we factor $\gamma_{\Sigma'}^{\Sigma} = [U^{\Sigma'} g U^{\Sigma}]_{\alpha} = \gamma_2 \circ \gamma_1$ where

$$\gamma_1 = [U1U^{\Sigma}]_{1,!} : M(\sigma^{\Sigma})! \rightarrow M(\sigma)! \quad \text{and} \quad \gamma_2 = [U^{\Sigma'} g U]_{\alpha,!} : M(\sigma)! \rightarrow M(\sigma^{\Sigma'})!,$$

where $U = U_0(N_1^{\Sigma} p)$ and $\sigma = \sigma^{\Sigma}|U$. Defining a pairing on $M(\sigma)!$ exactly as for $M(\sigma^{\Sigma})!$, using the same w and ω , we find that $\gamma_1^t \gamma_1 = p + 1$ and $\gamma_2^t \gamma_2 = \beta$ for some $\mathcal{O}_K[U]$ -linear endomorphism β of \mathcal{V}^{Σ} , necessarily a scalar by Lemma 3.2.1 of [13]. The desired formula follows with $\beta_{f,\Sigma}^{\Sigma'} = p\beta$. \square

2. The Bloch–Kato conjecture for A_f and $A_f(1)$

In this section we shall deduce the Bloch–Kato conjecture from the main result, Theorem 3.7, of Section 3 below. More precisely, we prove the λ -part of the Bloch–Kato conjecture [4] for A_f and $B_f := A_f(1)$, where f is a newform of weight $k \geq 2$, conductor $N \geq 1$, with coefficients in

the number field K , and λ is a prime of K not contained in the set

$$(31) S_f = \left\{ \begin{array}{l} \lambda \mid Nk\ell, \text{ or the } (\mathcal{O}_K/\lambda)[G_F]\text{-module } \bar{\mathcal{M}}_{f,\lambda} \text{ is not absolutely irreducible,} \\ \text{where } F = \mathbb{Q}(\sqrt{(-1)^{(\ell-1)/2}\ell}) \text{ and } \lambda \mid \ell \end{array} \right\}.$$

By (22) we can assume that f has minimal conductor among its twists and we shall do so in this section. Our formulation of the conjecture follows Fontaine and Perrin-Riou [41], generalized to motives with coefficients in K . For a more systematic discussion of the Bloch–Kato conjecture for motives with coefficients we refer to [7]. If A_f is the scalar extension of a premotivic structure with coefficients in a subfield $K' \subseteq K$ then Theorem 4.1 and Lemma 11b) of [7] show that the conjecture over K implies the one over K' (in the context of Deligne’s conjecture this was already noted in [17, Rem 2.10]). So we need not be concerned with finding the smallest coefficient field for A_f .

2.1. Galois cohomology

For any field F and continuous G_F -module M we write $H^i(F, M)$ for $H^i_{\text{cont}}(G_F, M)$. Let V be a continuous finite-dimensional representation of $G_{\mathbb{Q}}$ over \mathbb{Q}_{ℓ} , unramified at all but finitely many primes, and let $T \subseteq V$ be a $G_{\mathbb{Q}}$ -stable \mathbb{Z}_{ℓ} -lattice. We set $W := V/T$. For each place p of \mathbb{Q} , Bloch and Kato (see [4] or [41]) define a subspace $H^1_{\mathbf{f}}(\mathbb{Q}_p, V) \subseteq H^1(\mathbb{Q}_p, V)$ by

$$H^1_{\mathbf{f}}(\mathbb{Q}_p, V) := \begin{cases} H^1_{\text{ur}}(\mathbb{Q}_p, V) & p \neq \ell, \infty, \\ \ker(H^1(\mathbb{Q}_p, V) \rightarrow H^1(\mathbb{Q}_p, B_{\text{crys}} \otimes V)) & p = \ell, \\ 0 & p = \infty, \end{cases}$$

where

$$H^1_{\text{ur}}(\mathbb{Q}_p, M) := H^1(\mathbb{F}_p, M^{I_p}) = \ker(H^1(\mathbb{Q}_p, M) \rightarrow H^1(I_p, M))$$

for any G_p -module M . They then define groups

$$H^1_{\mathbf{f}}(\mathbb{Q}_p, W) := \text{im}(H^1_{\mathbf{f}}(\mathbb{Q}_p, V) \rightarrow H^1(\mathbb{Q}_p, W))$$

and a Selmer group

$$H^1_{\mathbf{f}}(\mathbb{Q}, M) := \ker\left(H^1(\mathbb{Q}, M) \rightarrow \bigoplus_p \frac{H^1(\mathbb{Q}_p, M)}{H^1_{\mathbf{f}}(\mathbb{Q}_p, M)}\right),$$

where M is either V or W and the sum is over all places p of \mathbb{Q} . For $p \notin \{\ell, \infty\}$, note that $H^1_{\mathbf{f}}(\mathbb{Q}_p, W)$ is the maximal divisible subgroup of $H^1_{\text{ur}}(\mathbb{Q}_p, W)$ and that the two groups coincide if W^{I_p} is divisible, e.g. when W is unramified. For any finite set Σ of prime numbers *not* containing ℓ we define a larger Selmer group

$$H^1_{\Sigma}(\mathbb{Q}, W) := \ker\left(H^1(\mathbb{Q}, W) \rightarrow \bigoplus_{p \notin \Sigma \cup \{\ell, \infty\}} \frac{H^1(\mathbb{Q}_p, W)}{H^1_{\text{ur}}(\mathbb{Q}_p, W)} \oplus \frac{H^1(\mathbb{Q}_{\ell}, W)}{H^1_{\mathbf{f}}(\mathbb{Q}_{\ell}, W)}\right)$$

without local conditions at $p \in \Sigma \cup \{\infty\}$ and slightly relaxed local conditions at primes p where $H^1_{\text{ur}}(\mathbb{Q}_p, W)$ is not divisible. The group $H^1_{\mathbf{f}}(\mathbb{Q}, W)$ appears in the Bloch–Kato conjecture whereas $H^1_{\Sigma}(\mathbb{Q}, W)$ can be analyzed using the Taylor–Wiles method in our situation.

LEMMA 2.1. – Put $T^D = \text{Hom}_{\mathbb{Z}_\ell}(T, \mathbb{Z}_\ell(1))$, $V^D = T^D \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ and $W^D = V^D/T^D$, and denote by M^* the Pontryagin dual of a locally compact abelian group M . Set

$$H_{\mathbf{f}}^1(\mathbb{Q}_p, T^D) := \iota^{-1} H_{\mathbf{f}}^1(\mathbb{Q}_p, V^D)$$

where $\iota: H^1(\mathbb{Q}_p, T^D) \rightarrow H^1(\mathbb{Q}_p, V^D)$ is the natural map. If Σ is nonempty and contains all primes where $H_{\mathbf{f}}^1(\mathbb{Q}_p, W) \neq H_{\text{ur}}^1(\mathbb{Q}_p, W)$, and if moreover $H^0(\mathbb{Q}, V^D) = H_{\mathbf{f}}^1(\mathbb{Q}, V^D) = 0$ then there is an exact sequence

$$0 \rightarrow H_{\mathbf{f}}^1(\mathbb{Q}, W) \rightarrow H_{\Sigma}^1(\mathbb{Q}, W) \rightarrow \bigoplus_{p \in \Sigma \cup \{\infty\}} H_{\mathbf{f}}^1(\mathbb{Q}_p, T^D)^* \rightarrow H^0(\mathbb{Q}, W^D)^* \rightarrow 0.$$

Proof. – By [32, Proposition 1.4] there is a long exact sequence

$$\begin{aligned} 0 \rightarrow H_{\mathbf{f}}^1(\mathbb{Q}, W) \rightarrow H^1(G_S, W) \xrightarrow{\rho} \bigoplus_{p \in S} \frac{H^1(\mathbb{Q}_p, W)}{H_{\mathbf{f}}^1(\mathbb{Q}_p, W)} \xrightarrow{\rho^{D,*}} H_{\mathbf{f}}^1(\mathbb{Q}, T^D)^* \\ \rightarrow H^2(G_S, W) \rightarrow \bigoplus_{p \in S} H^2(\mathbb{Q}_p, W) \rightarrow H^0(\mathbb{Q}, T^D)^* \rightarrow 0 \end{aligned}$$

where G_S is the Galois group of the maximal extension of \mathbb{Q} unramified outside $S := \{\ell, \infty\} \cup \Sigma$ and $H_{\mathbf{f}}^1(\mathbb{Q}, T^D) = \iota^{-1} H_{\mathbf{f}}^1(\mathbb{Q}, V^D)$. By our assumption

$$H^0(\mathbb{Q}, V^D) = H_{\mathbf{f}}^1(\mathbb{Q}, V^D) = 0,$$

the natural (boundary) map $H^0(\mathbb{Q}, W^D) \rightarrow H_{\mathbf{f}}^1(\mathbb{Q}, T^D)$ is an isomorphism. The map $\rho^{D,*}$ is Pontryagin dual to the restriction map

$$H^0(\mathbb{Q}, W^D) = H_{\mathbf{f}}^1(\mathbb{Q}, T^D) \xrightarrow{\rho^D} \bigoplus_{p \in S} H_{\mathbf{f}}^1(\mathbb{Q}_p, T^D).$$

Clearly, ρ^D is injective as $H^0(\mathbb{Q}, W^D)$ injects into $H^0(\mathbb{Q}_p, W^D) \cong H_{\mathbf{f}}^1(\mathbb{Q}_p, T^D)_{\text{tor}}$ for any $p \in S \setminus \{\infty\} \neq \emptyset$. This argument also shows that $\rho^{D,*}$ restricted to

$$L := \bigoplus_{p \in \Sigma \cup \{\infty\}} \frac{H^1(\mathbb{Q}_p, W)}{H_{\mathbf{f}}^1(\mathbb{Q}_p, W)} \cong \bigoplus_{p \in \Sigma \cup \{\infty\}} H_{\mathbf{f}}^1(\mathbb{Q}_p, T^D)^*$$

is still surjective since the dual map is still injective. On the other hand we have $\rho^{-1}(L) = H_{\Sigma}^1(\mathbb{Q}, W)$ which yields the lemma. \square

Suppose now that K_λ is a finite extension of \mathbb{Q}_ℓ with ring of integers \mathcal{O}_λ and uniformizer λ . For $i = 1, 2$, let V_i be representations of $G_{\mathbb{Q}}$ over K_λ which are pseudo-geometric and short as defined in Sections 1.1.1 and 1.1.2 respectively. Suppose that L_i is a $G_{\mathbb{Q}}$ -stable \mathcal{O}_λ -lattice in V_i and set

$$V = \text{Hom}_{K_\lambda}(V_1, V_2), \quad T = \text{Hom}_{\mathcal{O}_\lambda}(L_1, L_2), \quad W = V/T.$$

For $n \geq 1$, put

$$W_n = \{x \in W \mid \lambda^n x = 0\} \cong T/\lambda^n T$$

and note that we have a natural isomorphism

$$H^1(F, W_n) = \text{Ext}_{\mathcal{O}_\lambda/\lambda^n[G_F]}^1(L_1/\lambda^n L_1, \lambda^{-n} L_2/L_2)$$

since $L_1/\lambda^n L_1$ is free over $\mathcal{O}_\lambda/\lambda^n$ (here $F = \mathbb{Q}$ or $F = \mathbb{Q}_p$). Since V_i is short the G_ℓ -modules $L_i/\lambda^n L_i$ are in the essential image of the functor

$$\mathbb{V} : \mathcal{MF}_{\text{tor}}^0 \rightarrow \mathcal{O}_\lambda[G_\ell] - \text{Mod}$$

of Section 1.1.2. Let

$$H_{\mathbf{f}}^1(\mathbb{Q}_\ell, W_n) \subseteq H^1(\mathbb{Q}_\ell, W_n)$$

be the subset of extensions of $\mathcal{O}_\lambda/\lambda^n[G_\ell]$ -modules

$$(32) \quad 0 \rightarrow \lambda^{-n} L_2/L_2 \rightarrow \mathcal{E} \rightarrow L_1/\lambda^n L_1 \rightarrow 0$$

so that \mathcal{E} is in the essential image of \mathbb{V} . Using the stability of this essential image under direct sums, subobjects and quotients, one checks that $H_{\mathbf{f}}^1(\mathbb{Q}_\ell, W_n)$ is a \mathcal{O}_λ -submodule, and that $H_{\mathbf{f}}^1(\mathbb{Q}_\ell, W_n)$ is the preimage of $H_{\mathbf{f}}^1(\mathbb{Q}_\ell, W_{n+1})$ under the natural map $H^1(\mathbb{Q}_\ell, W_n) \rightarrow H^1(\mathbb{Q}_\ell, W_{n+1})$.

PROPOSITION 2.2. – *The group $H_{\mathbf{f}}^1(\mathbb{Q}_\ell, W)$ is divisible of \mathcal{O}_λ -corank*

$$d = \dim_{K_\lambda} H^0(\mathbb{Q}_\ell, V) + \dim_{K_\lambda} V - \dim_{K_\lambda} \text{Fil}^0 D_{\text{crys}}(V).$$

Moreover, the natural isomorphism

$$\varinjlim_n H^1(\mathbb{Q}_p, W_n) \cong H^1(\mathbb{Q}_p, W)$$

induces isomorphisms

$$\varinjlim_n H_{\text{ur}}^1(\mathbb{Q}_p, W_n) \cong H_{\text{ur}}^1(\mathbb{Q}_p, W), \quad \varinjlim_n H_{\mathbf{f}}^1(\mathbb{Q}_\ell, W_n) \cong H_{\mathbf{f}}^1(\mathbb{Q}_\ell, W).$$

Proof. – The divisibility of $H_{\mathbf{f}}^1(\mathbb{Q}_\ell, W)$ follows from its definition, as does the fact that its corank coincides with $\dim_{K_\lambda} H_{\mathbf{f}}^1(\mathbb{Q}_\ell, V) = d$ (see [4] for this last identity). The statement concerning H_{ur}^1 follows from the fact that continuous group cohomology commutes with direct limits. For $H_{\mathbf{f}}^1$ we first note that

$$\iota_n^{-1} H_{\mathbf{f}}^1(\mathbb{Q}_\ell, W) \subseteq H_{\mathbf{f}}^1(\mathbb{Q}_\ell, W_n)$$

where $\iota_n : H^1(\mathbb{Q}_\ell, W_n) \rightarrow H^1(\mathbb{Q}_\ell, W)$ is the natural map. Indeed, on the level of extensions the map ι_n is given by pushout via $\lambda^{-n} L_2/L_2 \rightarrow V_2/L_2$, pullback via $L_1 \rightarrow L_1/\lambda^n L_1$, and the isomorphism $H^1(\mathbb{Q}_\ell, W) \cong \text{Ext}_{\mathcal{O}_\lambda[G_\ell]}^1(L_1, V_2/L_2)$. Similarly, the map

$$\pi : H^1(\mathbb{Q}_\ell, V) \rightarrow H^1(\mathbb{Q}_\ell, W)$$

is given by pushout via $V_2 \rightarrow V_2/L_2$ and pullback via $L_1 \rightarrow V_1$. So for $E \in H_{\mathbf{f}}^1(\mathbb{Q}_\ell, V)$ all finite subquotients of the locally compact continuous $\mathcal{O}_\lambda[G_\ell]$ -module $\pi(E)$ are in the essential image

of \mathbb{V} . Hence, if \mathcal{E} is as in (32) and $\iota_n(\mathcal{E}) = \pi(E)$ for $E \in H_{\mathbf{f}}^1(\mathbb{Q}_\ell, V)$ then \mathcal{E} lies in the essential image of \mathbb{V} .

So we obtain an inclusion of torsion groups

$$H_{\mathbf{f}}^1(\mathbb{Q}_\ell, W) \simeq \varinjlim_n \iota_n^{-1} H_{\mathbf{f}}^1(\mathbb{Q}_\ell, W) \subseteq \varinjlim_n H_{\mathbf{f}}^1(\mathbb{Q}_\ell, W_n)$$

which is an isomorphism if and only if the induced inclusion on the λ -torsion submodules is an isomorphism (as $H_{\mathbf{f}}^1(\mathbb{Q}_\ell, W)$ is divisible). There is an exact sequence

$$(33) \quad 0 \rightarrow H^0(\mathbb{Q}_\ell, W)/\lambda \rightarrow H_{\mathbf{f}}^1(\mathbb{Q}_\ell, W_1) \rightarrow \left(\varinjlim_n H_{\mathbf{f}}^1(\mathbb{Q}_\ell, W_n)\right)[\lambda] \rightarrow 0,$$

and we need to prove that the right hand term has $\kappa := \mathcal{O}_\lambda/\lambda$ -dimension d . Pick objects \mathcal{D}_i of \mathcal{MF}^0 so that $\mathbb{V}(\mathcal{D}_i) \cong L_i$. Then $\mathcal{D} := \text{Hom}_{\mathcal{O}_\lambda}(\mathcal{D}_1, \mathcal{D}_2)$ is also an object of \mathcal{MF} and $\mathcal{D} \otimes_{\mathcal{O}_\lambda} K_\lambda \cong D_{\text{crys}}(V)$ (see Eq. (1) in Section 1.1.2). Put $\bar{\mathcal{D}}_i = \mathcal{D}_i/\lambda$, $\bar{\mathcal{D}} = \mathcal{D}/\lambda$ so that $\mathbb{V}(\bar{\mathcal{D}}_i) \cong L_i/\lambda L_i$ and $\mathbb{V}(\bar{\mathcal{D}}) \cong W_1$. For all $j \in \mathbb{Z}$ we have

$$(34) \quad \dim_{K_\lambda} \text{Fil}^j D_{\text{crys}}(V) = \dim_{\mathcal{O}_\lambda} \text{Fil}^j \mathcal{D} = \dim_\kappa \text{Fil}^j \bar{\mathcal{D}}.$$

Denote by $\kappa\text{-}\mathcal{MF}$ the category of κ -modules in \mathcal{MF} . Then

$$(35) \quad \dim_\kappa H_{\mathbf{f}}^1(\mathbb{Q}_\ell, W_1) = \dim_\kappa \text{Ext}_{\kappa\text{-}\mathcal{MF}}^1(\bar{\mathcal{D}}_1, \bar{\mathcal{D}}_2)$$

and

$$(36) \quad \begin{aligned} \dim_\kappa H^0(\mathbb{Q}_\ell, W)/\lambda &= \dim_\kappa H^0(\mathbb{Q}_\ell, W_1) - \dim_{K_\lambda} H^0(\mathbb{Q}_\ell, V) \\ &= \dim_\kappa \text{Hom}_{\kappa\text{-}\mathcal{MF}}(\bar{\mathcal{D}}_1, \bar{\mathcal{D}}_2) - \dim_{K_\lambda} H^0(\mathbb{Q}_\ell, V). \end{aligned}$$

There is an exact sequence

$$(37) \quad \begin{aligned} 0 \rightarrow \text{Hom}_{\kappa\text{-}\mathcal{MF}}(\bar{\mathcal{D}}_1, \bar{\mathcal{D}}_2) &\rightarrow \text{Hom}_{\kappa, \text{Fil}}(\bar{\mathcal{D}}_1, \bar{\mathcal{D}}_2) = \text{Fil}^0 \bar{\mathcal{D}} \\ &\xrightarrow{1-\phi^0} \text{Hom}_\kappa(\bar{\mathcal{D}}_1, \bar{\mathcal{D}}_2) = \bar{\mathcal{D}} \rightarrow \text{Ext}_{\kappa\text{-}\mathcal{MF}}^1(\bar{\mathcal{D}}_1, \bar{\mathcal{D}}_2) \rightarrow 0 \end{aligned}$$

(see diagram (61) below for a similar computation) and the combination of (33)–(38) then shows that indeed

$$\dim_\kappa \left(\varinjlim_n H_{\mathbf{f}}^1(\mathbb{Q}_\ell, W_n)\right)[\lambda] = d. \quad \square$$

COROLLARY 2.3. – *Suppose that \bar{L} is a two-dimensional $G_{\mathbb{Q}}$ -representation over the finite field κ of characteristic $\ell > 2$ so that $L|_{G_\ell} \cong \mathbb{V}(\bar{\mathcal{D}}')$ for some object $\bar{\mathcal{D}}'$ of $\kappa\text{-}\mathcal{MF}^0$ with $\dim_\kappa \text{Fil}^1 \bar{\mathcal{D}}' = 1$. Let $\text{ad}_\kappa^0 \bar{L} \subset \text{ad}_\kappa \bar{L} := \text{Hom}_\kappa(\bar{L}, \bar{L})$ be the endomorphisms of trace zero. Then*

$$\dim_\kappa H_{\mathbf{f}}^1(\mathbb{Q}_\ell, \text{ad}_\kappa^0 \bar{L}) = 1 + \dim_\kappa H^0(\mathbb{Q}_\ell, \text{ad}_\kappa^0 \bar{L}).$$

Proof. – From (38) applied to $\bar{\mathcal{D}}_1 = \bar{\mathcal{D}}_2 = \bar{\mathcal{D}}'$ we have

$$\dim_\kappa H_{\mathbf{f}}^1(\mathbb{Q}_\ell, \text{ad}_\kappa \bar{L}) = 2 \cdot 2 - 3 + \dim_\kappa H^0(\mathbb{Q}_\ell, \text{ad}_\kappa \bar{L})$$

and (38) applied to $\bar{\mathcal{D}}_1 = \bar{\mathcal{D}}_2 = \kappa[0]$ (the unit object of $\kappa\text{-}\mathcal{MF}$) shows that $\dim_\kappa H_{\mathbf{f}}^1(\mathbb{Q}_\ell, \kappa) = \dim_\kappa H^0(\mathbb{Q}_\ell, \kappa)$. Since $\ell > 2$ we have $\text{ad}_\kappa \bar{L} = \kappa \oplus \text{ad}_\kappa^0 \bar{L}$ which gives the lemma. \square

Finally we record the following fact in this subsection.

LEMMA 2.4. – *The set S_f defined in (31) is finite.*

Proof. – Suppose that λ does not divide $Nk!$ and $\bar{\mathcal{M}}_{f,\lambda}$ is reducible. Its semisimplification is of the form $\psi_1 \oplus \psi_2$ where ψ_1 and ψ_2 are characters of $\text{Gal}(\mathbb{Q}(\mu_{N\ell})/\mathbb{Q})$. The representation is necessarily ordinary at ℓ (see [30]), so one of the characters is unramified at ℓ and the other has restriction χ_ℓ^{1-k} on I_ℓ where $\chi_\ell: G_\mathbb{Q} \rightarrow \text{Aut}(\mu_\ell)$ is the cyclotomic character. It follows that $a_p \equiv p^{k-1} + 1 \pmod{\lambda}$ for all $p \equiv 1 \pmod{N}$. If this holds for infinitely many λ , then we get $a_p = p^{k-1} + 1$ for all such p , violating the Ramanujan conjecture (a theorem of Deligne [15]). Having established irreducibility of $\mathcal{M}_{f,\lambda}$ for all but finitely many λ , the proof is then finished by the following lemma. \square

LEMMA 2.5. – *Suppose that λ does not divide $N(2k - 1)(2k - 3)k!$. If $\bar{\mathcal{M}}_{f,\lambda}$ is irreducible, then its restriction to G_F is absolutely irreducible.*

Proof. – Consider the restriction of $\bar{\mathcal{M}}_{f,\lambda}$ to I_ℓ . By results of Deligne and Fontaine (see [30]), this restriction has semisimplification of the form $\chi_\ell^{1-k} \oplus 1$ or $\psi_\ell^{1-k} \oplus \psi_\ell^{\ell(1-k)}$ (after extending scalars if necessary), where ψ_ℓ is a fundamental character of level 2, according to whether or not a_ℓ is a unit mod λ .

Suppose that $\bar{\mathcal{M}}_{f,\lambda}$ is irreducible but its restriction to G_F is not absolutely irreducible. Then (after extending scalars) $\bar{\mathcal{M}}_{f,\lambda}$ is induced from a character of G_F , and its restriction to I_ℓ is induced from a character of its subgroup of index 2. It follows that the ratio of the characters into which this restriction decomposes is quadratic. Since ψ_ℓ has order $\ell^2 - 1$, this forces either $(\ell - 1) | 2(k - 1)$ or $(\ell + 1) | 2(k - 1)$ and we arrive at a contradiction. \square

2.2. Order of vanishing

Suppose that M is an L -admissible object of \mathbf{PM}_K and let $M^D = \text{Hom}_K(M, K(1))$. We recall the conjectured order of vanishing of $L(M, s)$ at $s = 0$ [41, III. 4.2.2].

CONJECTURE 2.6. – *Let $\tau: K \rightarrow \mathbb{C}$ be an embedding and λ any finite prime of K . Then*

$$\text{ord}_{s=0} L(M, \tau, s) = \dim_{K_\lambda} H_{\mathbf{f}}^1(\mathbb{Q}, M_\lambda^D) - \dim_{K_\lambda} H^0(\mathbb{Q}, M_\lambda^D).$$

THEOREM 2.7. – *Conjecture 2.6 holds for both $M = A_f$ and $M = B_f$ if λ is not in S_f . More precisely, we have $\text{ord}_{s=0} L(A_f, \tau, s) = \text{ord}_{s=0} L(B_f, \tau, s) = 0$ and*

$$H^0(\mathbb{Q}, A_{f,\lambda}) \cong H_{\mathbf{f}}^1(\mathbb{Q}, A_{f,\lambda}) \cong H_{\mathbf{f}}^1(\mathbb{Q}, B_{f,\lambda}) \cong H^0(\mathbb{Q}, B_{f,\lambda}) \cong 0$$

if $\lambda \notin S_f$.

Proof. – Lemma 2.12 below shows that

$$L(A_f, \tau, 1) = L^{\text{nv}}(A_f, \tau, 1) \prod_{p \in \Sigma_e(f)} L_p(A_f, \tau, 1)$$

is a nonzero multiple of the Petersson inner product of f with itself and hence it follows that $L(B_f, \tau, 0) = L(A_f, \tau, 1) \neq 0$ for each τ . It follows from the functional equation (24) that

$$(38) \quad L(A_f, \tau, 0) = \frac{(k - 1)\epsilon(A_f)}{2\pi^2} L(A_f, \tau, 1) \neq 0$$

for each τ as well. The absolute irreducibility of $M_{f,\lambda}$ for each λ implies that

$$\text{End}_{K_\lambda[G_\mathbb{Q}]}(M_{f,\lambda}) = K_\lambda,$$

so $H^0(\mathbb{Q}, A_{f,\lambda}) = 0$, and since $M_{f,\lambda}$ is not isomorphic to $M_{f,\lambda}(1)$, we also have $H^0(\mathbb{Q}, B_{f,\lambda}) = 0$. It follows then from [41, II.2.2.2] (see also [32, Corollary 1.5]) that

$$\dim_{K_\lambda} H_{\mathfrak{f}}^1(\mathbb{Q}, A_{f,\lambda}) = \dim_{K_\lambda} H_{\mathfrak{f}}^1(\mathbb{Q}, B_{f,\lambda})$$

for all λ and hence that Theorem 2.7 is implied by the vanishing of $H_{\mathfrak{f}}^1(\mathbb{Q}, A_{f,\lambda})$. Theorem 3.7 shows that

$$H_{\mathfrak{f}}^1(\mathbb{Q}, A_{f,\lambda} / \text{ad}_{\mathcal{O}_\lambda}^0 \mathcal{M}_{f,\lambda}) \subset H_{\Sigma}^1(\mathbb{Q}, A_{f,\lambda} / \text{ad}_{\mathcal{O}_\lambda}^0 \mathcal{M}_{f,\lambda})$$

is finite for λ in S_f . Since the kernel of

$$H_{\mathfrak{f}}^1(\mathbb{Q}, A_{f,\lambda}) \rightarrow H_{\mathfrak{f}}^1(\mathbb{Q}, A_{f,\lambda} / \text{ad}_{\mathcal{O}_\lambda}^0 \mathcal{M}_{f,\lambda})$$

is finitely generated over \mathcal{O}_λ we deduce $H_{\mathfrak{f}}^1(\mathbb{Q}, A_{f,\lambda}) = 0$ and Theorem 2.7 follows. \square

2.3. Deligne’s period

We now recall the formulation in [41] of Deligne’s conjecture [17] for the “transcendental part” of $L(M, 0)$ for $M = A_f$ or B_f . The authors there actually discuss the more general conjecture of Beilinson concerning the leading coefficient $L^*(M, 0)$ for premotivic structures arising from motives, but their formulation relies on the conjectural existence of a category of mixed motives with certain properties. We restrict our attention to those M , such as A_f and B_f , for which $L(M, 0) \neq 0$ and which are critical in the sense of Deligne. In that case Beilinson’s conjecture reduces (conjecturally) to Deligne’s, which can be stated without reference to the category of mixed motives.

Under these hypotheses, the *fundamental line* for M is the K -line defined by

$$\Delta_{\mathfrak{f}}(M) = \text{Hom}_K(\det_K M_B^+, \det_K t_M)$$

where $+$ indicates the subspace fixed by F_∞ and $t_M = M_{\text{dR}} / \text{Fil}^0 M_{\text{dR}}$. Furthermore the composite

$$\mathbb{R} \otimes M_B^+ \rightarrow (\mathbb{C} \otimes M_B)^+ \xrightarrow{(I^\infty)^{-1}} \mathbb{R} \otimes M_{\text{dR}} \rightarrow \mathbb{R} \otimes t_M$$

is an $\mathbb{R} \otimes K$ -linear isomorphism. Its determinant over $\mathbb{R} \otimes K$ defines a basis for $\mathbb{R} \otimes \Delta_{\mathfrak{f}}(M)$ called the Deligne period, denoted $c^+(M)$.

CONJECTURE 2.8. – *There exists a basis $b(M)$ for $\Delta_{\mathfrak{f}}(M)$ such that*

$$L(M, 0)(1 \otimes b(M)) = c^+(M).$$

There are various rationality results for $L(A_f, 0)$ and $L(B_f, 0)$ in the literature (see for example [73, Theorem 2.3]) although the precise relationship with Conjecture 2.8 for $M = A_f$ or B_f is not always clear. In this section we recall the proof of Conjecture 2.8 for $M = A_f$ and B_f and give convenient natural descriptions for $b(A_f)$ and $b(B_f)$.

We begin by observing that $A_{f,B}^+$ and t_{A_f} are one-dimensional over K . Furthermore, complex conjugation

$$F_\infty : M_{f,B} \rightarrow M_{f,B}$$

has trace zero and commutes with F_∞ , so it is a basis for $A_{f,B}^+$. Note also that the natural map

$$A_{f,dR} \rightarrow \text{Hom}_K(\text{Fil}^{k-1} M_{f,dR}, M_{f,dR}/\text{Fil}^{k-1} M_{f,dR})$$

factors through an isomorphism

$$(39) \quad t_{A_f} \rightarrow \text{Hom}_K(\text{Fil}^{k-1} M_{f,dR}, M_{f,dR}/\text{Fil}^{k-1} M_{f,dR}).$$

The fundamental line $\Delta_f(A_f)$ can therefore be identified with

$$\text{Hom}_K(\text{Fil}^{k-1} M_{f,dR} \otimes \mathbb{Q} \cdot F_\infty, M_{f,dR}/\text{Fil}^{k-1} M_{f,dR}).$$

We shall describe $b(A_f)$ by specifying the image of the canonical basis $f \otimes F_\infty$ for $\text{Fil}^{k-1} M_{f,dR} \otimes K \cdot F_\infty$ where we view f as an element of $M_{f,dR}$ by (15). Recall that we defined in (16) a perfect alternating pairing

$$\langle \cdot, \cdot \rangle : M_f \otimes_K M_f \rightarrow M_\psi(1-k),$$

and this induces an isomorphism

$$M_{f,dR}/\text{Fil}^{k-1} M_{f,dR} \rightarrow \text{Hom}_K(\text{Fil}^{k-1} M_{f,dR}, M_\psi(1-k)_{dR}).$$

We shall eventually define $b(A_f)$ by specifying the element $\langle f, b(A_f)(f \otimes F_\infty) \rangle$ of $M_\psi(1-k)_{dR}$.

We can make a similar analysis of the fundamental line $\Delta_f(B_f)$. One finds that $B_{f,B}^+$ and t_{B_f} are two-dimensional over K . Note that $B_{f,B}^+$ can be identified with $A_{f,B}^- \otimes \mathbb{Q}(1)_B$ and that the natural map

$$A_{f,B}^- \rightarrow \text{Hom}_K(M_{f,B}^+, M_{f,B}^-) \oplus \text{Hom}_K(M_{f,B}^-, M_{f,B}^+)$$

defined by restrictions is an isomorphism. We therefore have an isomorphism

$$\det_K B_{f,B}^+ \rightarrow K(2)_B$$

which is canonical up to sign. To fix the choice of sign, we use $\alpha \wedge \alpha^{-1}$ as a basis for $\det_K A_{f,B}^-$ where $\alpha : M_{f,B}^+ \rightarrow M_{f,B}^-$ is any K -linear isomorphism. Next note that the natural map

$$B_{f,dR} \rightarrow \text{Hom}_K(M_{f,dR}, M_f(1)_{dR}) \rightarrow \text{Hom}_K(\text{Fil}^{k-1} M_{f,dR}, M_f(1)_{dR})$$

factors through an isomorphism

$$t_{B_f} \rightarrow \text{Hom}_K(\text{Fil}^{k-1} M_{f,dR}, M_f(1)_{dR}).$$

Using the isomorphism

$$\det_K M_{f,dR} \rightarrow \text{Fil}^{k-1} M_{f,dR} \otimes_K (M_{f,dR}/\text{Fil}^{k-1} M_{f,dR})$$

(with choice of sign again indicated by the ordering), we find that $\det_K t_{B_f}$ is naturally isomorphic to

$$(40) \quad \text{Hom}_K(\text{Fil}^{k-1} M_{f,\text{dR}}, M_{f,\text{dR}}/\text{Fil}^{k-1} M_{f,\text{dR}}) \otimes \mathbb{Q}(2)_{\text{dR}}.$$

We can therefore identify $\Delta_f(B_f)$ with

$$(41) \quad \text{Hom}_K(\text{Fil}^{k-1} M_{f,\text{dR}} \otimes \mathbb{Q}(2)_B, (M_{f,\text{dR}}/\text{Fil}^{k-1} M_{f,\text{dR}}) \otimes \mathbb{Q}(2)_{\text{dR}}),$$

and we arrive at a canonical isomorphism

$$\Delta_f(A_f) \otimes \Delta_f(\mathbb{Q}(2)) \otimes A_{f,B}^+ \xrightarrow{\sim} \Delta_f(B_f).$$

Fixing the basis F_∞ of $A_{f,B}^+$ and the basis β of $\Delta_f(\mathbb{Q}(2))$ which sends $(2\pi i)^2$ to ι^{-2} , this defines an isomorphism of K -lines

$$(42) \quad \text{tw} : \Delta_f(A_f) \rightarrow \Delta_f(B_f)$$

so that $\text{tw}(\phi)(x \otimes y) = \phi(x \otimes F_\infty) \otimes \beta(y)$.

LEMMA 2.9. – We have

$$(\mathbb{R} \otimes \text{tw})(c^+(A_f)) = -\frac{1}{2\pi^2} c^+(B_f).$$

Proof. – Let $I_M^\infty : \mathbb{C} \otimes M_{f,\text{dR}} \cong \mathbb{C} \otimes M_{f,B}$ be the comparison isomorphism for M_f . Via the natural isomorphism $\mathbb{C} \otimes \text{End}_K(M_f)_? \cong \text{End}_{\mathbb{C} \otimes K}(\mathbb{C} \otimes M_{f,?})$ where $? = B$ or $? = \text{dR}$, I_M^∞ induces the comparison isomorphism I^∞ for both $\text{End}(M_f)$ and $A_f : I^\infty(\phi) = I_M^\infty \circ \phi \circ (I_M^\infty)^{-1}$. A similar formula holds for $c^+(A_f)$.

Suppose now that x is a $\mathbb{R} \otimes K$ -basis of $\mathbb{R} \otimes \text{Fil}^{k-1} M_{f,\text{dR}}$ and write $I_M^\infty(x) = y^+ + y^-$ with $y^\pm \in \mathbb{C} \otimes M_{f,B}^\pm$. Then

$$c^+(A_f)(x \otimes F_\infty) = (I_M^\infty)^{-1}(1 \otimes F_\infty) I_M^\infty(x) = (I_M^\infty)^{-1}(y^+ - y^-) \text{ mod } \mathbb{R} \otimes \text{Fil}^{k-1} M_{f,\text{dR}}.$$

On the other hand we have $\alpha(y^+) = \lambda y^-$ for some $\lambda \in (\mathbb{C} \otimes K)^\times$ and therefore $\alpha^{-1}(y^-) = \lambda^{-1} y^+$. Hence

$$\begin{aligned} (I^\infty)^{-1}(\alpha)(x) \wedge (I^\infty)^{-1}(\alpha^{-1})(x) &= (I_M^\infty)^{-1} \alpha(y^+) \wedge (I_M^\infty)^{-1} \alpha^{-1}(y^-) \\ &= (I_M^\infty)^{-1} \lambda y^- \wedge (I_M^\infty)^{-1} \lambda^{-1} y^+ \\ &= \frac{1}{2} (I_M^\infty)^{-1} (y^+ + y^-) \wedge (I_M^\infty)^{-1} (y^+ - y^-) \\ &= \frac{1}{2} x \wedge (I_M^\infty)^{-1} (y^+ - y^-) \end{aligned}$$

and in the description (41) of $\mathbb{R} \otimes \Delta_f(B_f)$ the element $c^+(B_f)$ is given by

$$\begin{aligned} c^+(B_f)(x \otimes (2\pi i)^2) \otimes \iota^2 &= (2\pi i)^2 \frac{1}{2} (I_M^\infty)^{-1} (y^+ - y^-) \text{ mod } \mathbb{R} \otimes \text{Fil}^{k-1} M_{f,\text{dR}} \\ &= -2\pi^2 c^+(A_f)(x \otimes F_\infty). \end{aligned}$$

In view of the definition of tw in (42) this gives the lemma. \square

Recall that $\Sigma_e(f)$ is the set of primes p such that $L_p^{\text{nv}}(A_f, s) = 1$ but $L_p(A_f, s) = (1 + p^{-s})^{-1}$. We write b_{dR} for the basis of $M_{\psi,\text{dR}}$ defined in Section 1.1.3, and pick $\eta \in \{0, 1\}$ so that

$\eta \equiv k \pmod 2$. Note that by Proposition 5.5 of [17], we have $\epsilon(M_f \otimes M_{\psi^{-1}})/\epsilon(M_{\psi^{-1}}) \in K^\times$. The same proposition together with (20) gives $\epsilon(A_f) \in K^\times$.

THEOREM 2.10. – *Let $b(A_f) \in \Delta_f(A_f)$ be defined by the formula*

$$\langle f, b(A_f)(f \otimes F_\infty) \rangle = \frac{i^{k-\eta}((k-2)!)^2 \epsilon(M_f \otimes M_{\psi^{-1}})}{2\epsilon(M_{\psi^{-1}})\epsilon(A_f)} \prod_{p \in \Sigma_e(f)} (1+p^{-1}) \cdot (b_{dR} \otimes \iota^{k-1}),$$

and $b(B_f) \in \Delta_f(B_f)$ by the formula

$$(43) \quad b(B_f) = (1-k)\epsilon(A_f) \text{tw}(b(A_f)).$$

Then $L(A_f, 0)(1 \otimes b(A_f)) = c^+(A_f)$ and $L(B_f, 0)(1 \otimes b(B_f)) = c^+(B_f)$.

Proof. – If we show

$$\langle f, c^+(A_f)(f \otimes F_\infty) \rangle = \frac{i^{k-\eta}(k-1)!(k-2)!\epsilon(M_f \otimes M_{\psi^{-1}})L^{nv}(A_f, 1)}{4\pi^2\epsilon(M_{\psi^{-1}})} \cdot (b_{dR} \otimes \iota^{k-1})$$

in $\mathbb{C} \otimes M_\psi(1-k)_{dR}$, then the statement concerning $b(A_f)$ is an immediate consequence of the functional equation (38). The identity $L(B_f, 0)(1 \otimes b(B_f)) = c^+(B_f)$ then follows by applying $(\mathbb{R} \otimes \text{tw})$ to the identity $L(A_f, 0)(1 \otimes b(A_f)) = c^+(A_f)$ and using (38) and Lemma 2.9.

As in Section 1.4.2 put $U = U_0(N)$, let $\sigma : U \rightarrow K^\times$ be the representation $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \psi^{-1}(a_N)$ and set $M(N, \psi) = M(\sigma) = M(N')(\sigma)$ for some $N' \geq 3$ so that $U_{N'} \subseteq U$. Put $w = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} \in \text{GL}_2(\mathbb{A}_f)$ and denote by $W = [UwU]_\omega : M(N')(\sigma) \rightarrow M(N')(\hat{\sigma} \otimes (\psi^{-1} \circ \det))$ the isomorphism in (13). Note that $w w_N^{-1} \in U$ so that we can work with w instead of w_N . For any one-dimensional K -representation σ of U whose kernel contains $U_{N'}$ we shall view $M(N')(\sigma)$ as a sub- PM_K -structure of $K \otimes M(N')$. With I^∞ denoting the comparison isomorphism for both M_f and $K \otimes M(N')$ we have

$$(44) \quad \begin{aligned} \langle f, c^+(A_f)(f \otimes F_\infty) \rangle &= \langle f, (I^\infty)^{-1}(1 \otimes F_\infty)I^\infty f \rangle \\ &= [U : U_{N'}]^{-1} \langle f, (I^\infty)^{-1}(1 \otimes F_\infty)I^\infty Wf \rangle_{N'} \end{aligned}$$

where this last pairing is the one defined in (12).

We proceed with the computation of $Wf \in \text{Fil}^{k-1} M(N')(\hat{\sigma} \otimes (\psi^{-1} \circ \det))_{!,dR}$. Note that the field K_f generated by the Fourier coefficients of the newform f is either totally real or a CM field and hence has a well defined automorphism ρ induced by complex conjugation. It is known that the Fourier expansion $f^\rho(z) = \sum_{n=1}^\infty a_n^\rho e^{2\pi izn}$ is a newform of conductor N and character ψ^{-1} [63, 4.6.15(2)], hence represents an element of $\text{Fil}^{k-1} M(N')(\hat{\sigma})_{!,dR}$.

Let P_1, P_2 be the canonical N' -torsion sections on the moduli scheme X of level N' introduced in Section 1.2.1, denote by $\zeta = \langle P_1, P_2 \rangle \in \Gamma(X, \mathcal{O}_X)$ their Weil pairing and consider the resulting morphism $X \rightarrow \text{Spec}(\mathcal{O}_F)$ where $F = \mathbb{Q}(\zeta)$. This induces isomorphisms $M_F \cong H^0(X)$ and

$$(45) \quad M_\psi \cong (H^0(X) \otimes K_{\psi^{-1} \circ \det})^U$$

where in the definition of M_ψ in Section 1.1.3 we have to replace $e^{2\pi iN'}$ by ζ . Then $M_\psi \otimes_K M(N')(\hat{\sigma})_!$ has a natural map into $M(N')(\hat{\sigma} \otimes (\psi^{-1} \circ \det))_!$ via the isomorphism (45) followed by cup product on X .

LEMMA 2.11. – We have

$$(46) \quad Wf = \psi(-1) \frac{i^{k-\eta} \epsilon(M_f \otimes M_{\psi^{-1}})}{N \epsilon(M_{\psi^{-1}})} b_{\text{dR}} \cup f^\rho$$

where b_{dR} is the basis of $M_{\psi, \text{dR}}$ defined in Section 1.1.3.

Proof. – We fix an embedding $\tau : K \rightarrow \mathbb{C}$ and compute the images of both sides in $S_k(U_{N'})$ (we shall suppress τ in the notation and view all elements of K as complex numbers via τ). Let $\phi \in (S_k(U_{N'}) \otimes_{\mathbb{C}} \mathbb{C}_\sigma)^U$ denote the element corresponding to f under the isomorphism (5). Recall that the isomorphism

$$\beta : S_k(U_{N'}) \cong \bigoplus_{t \in (\mathbb{Z}/N'\mathbb{Z})^\times} S_k(\Gamma(N'))$$

was defined before (5) by

$$\beta(F)_t(\gamma(i)) := (\det \gamma)^{-1} j(\gamma, i)^k F(g_t \gamma)$$

for $\gamma \in \text{GL}_2(\mathbb{R})^+$, $j\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z\right) = cz + d$ and $g_t \equiv \begin{pmatrix} 1 & 0 \\ 0 & t^{-1} \end{pmatrix} \pmod{N'}$. We have $\phi(xu) = \sigma^{-1}(u)\phi(x)$ for all $u \in U$ and $\beta(\phi)_t(z) = f(z)$ for all $t \in (\mathbb{Z}/N'\mathbb{Z})^\times$ since $g_t \in U$ and $\sigma(g_t) = 1$.

Recall the analytic description $X_{N'} = \prod_{t \in (\mathbb{Z}/N'\mathbb{Z})^\times} X_{N', t}$ of X and of P_1, P_2 from Section 1.2.1. One checks that $\langle (\tau, \frac{\tau}{N'}), (\tau, \frac{t}{N'}) \rangle = e^{-2\pi i t / N'}$. Hence

$$b_{\text{dR}} = \sum_{a \in (\mathbb{Z}/N\mathbb{Z})^\times} \psi(a) \otimes \zeta^{N'/N} \in \mathbb{C} \otimes F,$$

when viewed as an element of $H_{\text{dR}}^0(X_{N'}) = \prod_t \mathbb{C}$ is given by

$$\begin{aligned} t \mapsto \sum_{a \in (\mathbb{Z}/N\mathbb{Z})^\times} \psi(a) e^{-2\pi i a t / N} &= \psi(-t)^{-1} \sum_{a \in (\mathbb{Z}/N\mathbb{Z})^\times} \psi(a) e^{2\pi i a / N} \\ &= \psi(-t)^{-1} G_\psi = \psi(-t)^{-1} i^\eta \epsilon(M_{\psi^{-1}}, \tau). \end{aligned}$$

If now $\phi^\rho \in S_k(U_{N'})$ corresponds to f^ρ then $\beta(\phi^\rho)_t(z) = f^\rho(z)$ is again independent of t and the right hand side of (46) is given by

$$(47) \quad t \mapsto \frac{i^k \epsilon(M_f \otimes M_{\psi^{-1}}, \tau)}{N} \psi(t)^{-1} f^\rho.$$

The perfect pairing $M_f \otimes_K M_f \rightarrow M_\psi(1 - k)$ and the identity of Hecke eigenvalues [63, (4.6.17)] induce an isomorphism $M_f^* \cong M_f \otimes_K M_{\psi^{-1}}(k - 1) \cong M_{f^\rho}(k - 1)$ so that the functional equation for $\Lambda(M_f \otimes M_{\psi^{-1}}, \tau, s)$ can be written

$$(48) \quad \Lambda(M_f \otimes M_{\psi^{-1}}, \tau, s) = \epsilon(M_f \otimes M_{\psi^{-1}}, \tau) N^{-s} \Lambda(M_{f^\rho} \otimes M_\psi, \tau, k - s).$$

Recall that the definition $(g|_k \gamma)(z) = \det(\gamma)^{k/2} j(\gamma, z)^{-k} g(\gamma(z))$ for $\gamma \in \text{GL}_2(\mathbb{R})^+$ defines a right action of $\text{GL}_2(\mathbb{R})^+$ on functions $g : \mathfrak{H} \rightarrow \mathbb{C}$. Put $W_N = \begin{pmatrix} 0 & 1 \\ N & 0 \end{pmatrix}$. By [63, Theorem 4.3.6] we have

$$\Lambda(M_f \otimes M_{\psi^{-1}}, \tau, s) = \Lambda(f, s) = i^k N^{-s+k/2} \Lambda(f|_k W_N, k - s)$$

which together with (48) yields

$$f^\rho = \epsilon(M_f \otimes M_{\psi^{-1}}, \tau)^{-1} i^k N^{k/2} f|_k W_N.$$

Hence (47) becomes

$$(49) \quad t \mapsto (-1)^k N^{k/2-1} \psi(t)^{-1} f|_k W_N.$$

Turning to the left hand side of (46) we have

$$(W\phi)(x) := \phi(xw) \quad \text{and} \quad \phi(wh) = \phi(W_{\mathbb{Q}} W_N^{-1} h) = \phi(W_N^{-1} h)$$

where $h \in \text{GL}_2(\mathbb{R})^+$ and $W_{\mathbb{Q}} \in \text{GL}_2(\mathbb{Q})$ is the matrix with image w (resp. W_N) in $\text{GL}_2(\mathbb{A}_f)$ (resp. $\text{GL}_2(\mathbb{R})$). For $\gamma \in \text{GL}_2(\mathbb{R})^+$ we have

$$\begin{aligned} \det(h)^{-1} j(h, i)^k \phi(\gamma h) &= \det(\gamma) \det(\gamma h)^{-1} j(\gamma, h(i))^{-k} j(\gamma h, i)^k \phi(\gamma h) \\ &= \det(\gamma) j(\gamma, h(i))^{-k} f(\gamma h(i)) \\ &= \det(\gamma)^{1-k/2} (f|_k \gamma)(h(i)). \end{aligned}$$

Combining these equations we find that $W\phi$ corresponds to

$$\begin{aligned} t \mapsto \det(h)^{-1} j(h, i)^k (W\phi)(g_t h) &= \det(h)^{-1} j(h, i)^k \phi(w w^{-1} g_t w h) \\ &= \det(h)^{-1} j(h, i)^k \sigma^{-1}(w^{-1} g_t w) \phi(wh) \\ &= \det(h)^{-1} j(h, i)^k \sigma^{-1}(w^{-1} g_t w) \phi(W_N^{-1} h) \\ &= \sigma^{-1}(w^{-1} g_t w) \det(W_N^{-1})^{1-k/2} (f|_k W_N^{-1})(h(i)). \end{aligned}$$

Since $f|_k W_N^2 = (-1)^k f$ this last expression equals

$$(50) \quad \sigma^{-1}(w^{-1} g_t w) (-1)^k N^{k/2-1} (f|_k W_N)(h(i)).$$

For $g_t \equiv \begin{pmatrix} 1 & 0 \\ 0 & t^{-1} \end{pmatrix} \pmod N$, we have $w^{-1} g_t w \equiv \begin{pmatrix} t^{-1} & * \\ 0 & 1 \end{pmatrix} \pmod N$ and $\sigma^{-1}(w^{-1} g_t w) = \psi(t^{-1}) = \psi(t)^{-1}$. So (49) and (50) agree which finishes the proof of the lemma. \square

The definition (11) of the pairing on σ -constructions shows that $(x, \alpha \cup y)_{N'} = (x, y) \otimes_K \alpha$ where $\alpha \in M_\psi$ and (x, y) is the K -linear extension of the $\mathbb{Q}(1-k)$ -valued pairing on $\mathcal{M}(N')L$ in (9). Combining this with Lemma 2.11 the last term in (44) equals

$$(51) \quad [U : U_{N'}]^{-1} (f, (I^\infty)^{-1} (1 \otimes F_\infty) I^\infty f^\rho) \otimes_K \alpha_{\text{dR}}$$

in $\mathbb{C} \otimes K(1-r)_{\text{dR}} \otimes_K M_{\psi, \text{dR}}$ where

$$\begin{aligned} \alpha_{\text{dR}} &= \psi(-1) \frac{i^{k-\eta} \epsilon(M_f \otimes M_{\psi^{-1}})}{N \epsilon(M_{\psi^{-1}})} (I^\infty)^{-1} (1 \otimes F_\infty) I^\infty b_{\text{dR}} \\ &= \psi(-1) \frac{i^{k-\eta} \epsilon(M_f \otimes M_{\psi^{-1}})}{N \epsilon(M_{\psi^{-1}})} \psi(-1)^{-1} b_{\text{dR}} \end{aligned}$$

(with I^∞ also denoting the comparison isomorphism for M_ψ). For any premotivic structure we have $(F_\infty \otimes F_\infty) I^\infty = I^\infty (F_\infty \otimes 1)$ and we have $(F_\infty \otimes 1)(f^\rho) = f^\rho$ since $f^\rho \in K \otimes M(N')_{\text{dR}} \subset \mathbb{C} \otimes K \otimes M(N')_{\text{dR}}$. Hence

$$(I^\infty)^{-1} (1 \otimes F_\infty) I^\infty f^\rho = (I^\infty)^{-1} (F_\infty \otimes 1) I^\infty f^\rho.$$

Under the natural isomorphism $\mathbb{C} \otimes K \otimes M(N')_B \cong (\mathbb{C} \otimes M(N')_B)^{\mathbf{I}^K}$ the action of $F_\infty \otimes 1 \otimes 1$ on the left hand side gets transformed into the action sending (x_τ) to $\tau \mapsto (F_{\infty, \tau} \otimes 1)(x_{\overline{\tau}})$ where $F_{\infty, \tau}$ is complex conjugation acting on \mathbb{C} in the factor indexed by τ . Hence the τ -component of (51) equals

$$[U : U_{N'}]^{-1}(\tau(f), (I^\infty)^{-1}(F_{\infty, \tau} \otimes 1)I^\infty \overline{\tau}(f^\rho)) \otimes_{\mathbb{C}} \tau(\alpha_{\text{dR}}) \\ = [U : U_{N'}]^{-1}(k-2)!(4\pi)^{k-1} \phi(N')(\tau(f), \tau(f))_{\Gamma(N')} \tau(\alpha_{\text{dR}}) \otimes \iota^{k-1}$$

where ϕ is Euler’s function and we have used (10). Therefore (51) equals

$$(52) \quad [U : U_{N'}]^{-1}(k-2)!(4\pi)^{k-1} \phi(N')(f, f)_{\Gamma(N')} \cdot \alpha_{\text{dR}} \otimes \iota^{k-1} \\ = \frac{[\overline{\Gamma}_1(N) : \overline{\Gamma}(N')]}{[U : U_{N'}]} \phi(N')(k-2)!(4\pi)^{k-1} (f, f)_{\Gamma_1(N)} \cdot \alpha_{\text{dR}} \otimes \iota^{k-1}$$

in $\mathbb{C} \otimes M_\psi(1-k)_{\text{dR}}$ where $[\overline{\Gamma}_1(N) : \overline{\Gamma}(N')]$ is the degree of the covering $\Gamma(N') \backslash \mathfrak{H} \rightarrow \Gamma_1(N) \backslash \mathfrak{H}$. Since the maps $\det : U \rightarrow (\mathbb{Z}/N'\mathbb{Z})^\times$ and $\text{SL}_2(\mathbb{Z}) \rightarrow \text{SL}_2(\mathbb{Z}/N'\mathbb{Z})$ are surjective one finds

$$(53) \quad [U : U_{N'}] = \phi(N') [\text{SL}_2(\mathbb{Z}) \cap U : \text{SL}_2(\mathbb{Z}) \cap U_{N'}] \\ = \phi(N') [\Gamma_0(N) : \Gamma(N')] \\ = \phi(N') \phi(N) [\Gamma_1(N) : \Gamma(N')] \\ = \phi(N') \phi(N) \delta(N) [\overline{\Gamma}_1(N) : \overline{\Gamma}(N')]$$

where $\delta(N) = 1$ if $N > 2$ and $\delta(N) = 2$ if $N \leq 2$ (note that $-1 \in \Gamma_1(N)$ iff $N \leq 2$ whereas $-1 \notin \Gamma(N')$). Combining this with Lemma 2.12 below we find that (52) equals

$$\frac{(k-2)!(4\pi)^{k-1}}{\phi(N)\delta(N)} \cdot \frac{(k-1)!\delta(N)N\phi(N)L^{\text{nv}}(A_f, 1)}{4^k \pi^{k+1}} \cdot \frac{i^{k-\eta} \epsilon(M_f \otimes M_{\psi^{-1}})}{N\epsilon(M_{\psi^{-1}})} \cdot b_{\text{dR}} \otimes \iota^{k-1} \\ = \frac{i^{k-\eta}(k-2)!(k-1)!L^{\text{nv}}(A_f, 1)\epsilon(M_f \otimes M_{\psi^{-1}})}{4\pi^2 \epsilon(M_{\psi^{-1}})} \cdot b_{\text{dR}} \otimes \iota^{k-1}.$$

This finishes the proof of Theorem 2.10. \square

LEMMA 2.12. – *If f is a newform of conductor N , weight k and with coefficients in the number field K , we have*

$$(\tau(f), \tau(f))_{\Gamma_1(N)} = \frac{(k-1)!\delta(N)N\phi(N)L^{\text{nv}}(A_f, \tau, 1)}{4^k \pi^{k+1}}$$

for any embedding $\tau : K \rightarrow \mathbb{C}$ and $\delta(N)$ as in (53).

Proof. – We fix τ and write f for $\tau(f)$ to ease notation. By Theorem 5.1 of [51] (essentially a reformulation of a theorem of Rankin and Shimura), we have

$$L(k, f, \overline{\psi}) = \frac{4^k \pi^{k+1} (f, f)_{\Gamma_1(N)}}{(k-1)!\delta(N)N N_\psi \phi(N/N_\psi)}$$

where $L(s, f, \overline{\psi}) = \prod_p L_p(s, f, \overline{\psi})$,

$$L_p(s, f, \overline{\psi})^{-1} = (1 - \overline{\psi}(p)\alpha_p^2 p^{-s})(1 - \overline{\psi}(p)\alpha_p \beta_p p^{-s})(1 - \overline{\psi}(p)\beta_p^2 p^{-s})$$

and α_p, β_p are defined as in Section 1.7.2 for $p \nmid N$ and $\alpha_p + \beta_p = a_p, \alpha_p \beta_p = 0$ for $p|N$. Denote by M_p the exact power of p dividing an integer M . To show the lemma it suffices to show that

$$(54) \quad L_p(k, f, \bar{\psi}) \frac{\phi(N_p/N_{\psi,p})}{N_p/N_{\psi,p}} = L_p^{\text{nv}}(A_f, \tau, 1) \frac{\phi(N_p)}{N_p}$$

for all primes p . If $p \nmid N$, this is immediate from Section 1.7.2. If $N_p = p$ and $N_{\psi,p} = 1$, we have $a_p^2 = \psi(p)p^{k-2}$ by [63, Theorem 4.6.17(2)] and $\pi_p(f)$ is special so that (54) holds true by (23). The only other case in which $a_p \neq 0$ is when $N_p = N_{\psi,p}$ [63, Theorem 4.6.17]. In this case $\bar{\psi}(p) = 0$ and hence $L_p(k, f, \psi) = 1$ whereas $\pi_p(f)$ is principal series so that $L_p^{\text{nv}}(A_f, \tau, 1) = (1 - p^{-1})^{-1} = N_p/\phi(N_p)$ by (23). Finally, if $N_p > 1$ and $a_p = 0$, then $L_p(k, f, \bar{\psi}) = L_p^{\text{nv}}(A_f, \tau, 1) = 1, N_p/N_{\psi,p} > 1$ and both sides in (54) equal $(1 - p^{-1})$. \square

Remark. – In the following, we shall not need the full precision of Theorem 2.10 but only the fact that $i^{k-\eta}((k-2)!)^2 \epsilon(M_f \otimes M_{\psi^{-1}})/2\epsilon(M_{\psi^{-1}})\epsilon(A_f)$ is a unit in $\mathcal{O} = \mathcal{O}_K[(Nk!)^{-1}]$. This in turn is a consequence of Lemma 2.13 below.

LEMMA 2.13. – *Let M be an object of \mathbf{PM}_K which is L -admissible everywhere and let $\tau : K \rightarrow \mathbb{C}$ be an embedding. Then $\epsilon(M, \tau) = \epsilon(M, \tau, 0)$ is a unit in $\overline{\mathbb{Z}}[c(M)^{-1}]$ where $\overline{\mathbb{Z}}$ is the ring of algebraic integers.*

Proof. – By definition $\epsilon(M, \tau) = \prod_p \epsilon(D_{\text{pst}}(M_\lambda|G_p) \otimes_{K_\lambda, \tau'} \mathbb{C}, \psi_p, dx_p)$ is a product over all places p of \mathbb{Q} where the additive characters ψ_p and the Haar measures dx_p are chosen as in [17, 5.3] and $\tau' : K_\lambda \rightarrow \mathbb{C}$ is any extension of τ . The assumption that M is L -admissible at p implies that the isomorphism class of $D_{\text{pst}}(M_\lambda|G_p) \otimes_{K_\lambda, \tau'} \mathbb{C}$ is independent of τ' . The definition of ϵ in [16, (8.12)] and [16, Theorem 6.5 (a),(b)] show that

$$\epsilon(D_{\text{pst}}(M_\lambda|G_p) \otimes_{K_\lambda, \tau'} \mathbb{C}, \psi_p, dx_p) = \tau' \epsilon(D_{\text{pst}}(M_\lambda|G_p), \psi_p, dx_p) \in \tau'(K_\lambda(\mu_{p^\infty})).$$

Replacing τ' by $\gamma\tau', \gamma \in \text{Aut}(\mathbb{C}/K(\mu_{p^\infty}))$, and using the L -admissibility again, we deduce from this formula that $\epsilon(D_{\text{pst}}(M_\lambda|G_p) \otimes_{K_\lambda, \tau'} \mathbb{C}, \psi_p, dx_p) \in K(\mu_{p^\infty})$. The remark after [16, (8.12.4)] shows that ϵ can be directly expressed in terms of the λ -adic representation M_λ for $\lambda \nmid p$. Namely

$$\epsilon(D_{\text{pst}}(M_\lambda|G_p), \psi_p, dx_p) = \epsilon_0((M_\lambda|W_p)^{ss}, \psi_p, dx_p) \det(-\text{Frob}|M_\lambda^{I_p})^{-1}$$

where ϵ_0 is introduced in [16, §5] and $(M_\lambda|W_p)^{ss}$ is the semisimplification of M_λ as a representation of W_p . Now for any $\lambda \nmid p$ the W_p -representation M_λ is the restriction of a continuous G_p -representation, hence carries a W_p -stable \mathcal{O}_λ -lattice. This implies, on the one hand, that $\det(-\text{Frob}|M_\lambda^{I_p}) \in \mathcal{O}_\lambda^\times$ and on the other hand, via [16, Theorem 6.5(c)], that $\epsilon_0((M_\lambda|W_p)^{ss}, \psi_p, dx_p) \in \mathcal{O}_\lambda[\mu_{p^\infty}]^\times$. Noting that with our choice of ψ_p, dx_p the epsilon factor equals 1 (resp. a power of i) for $p \nmid c(M)$ (resp. $p = \infty$) the lemma follows. \square

2.4. Bloch–Kato conjecture

We now recall the formulation of the λ -part of the Bloch–Kato conjecture. We assume that M is a premotivic structure in \mathbf{PM}_K such that M is critical, $L(M, 0) \neq 0$ and Conjecture 2.8 holds. We assume that λ is a prime of K such that

$$(55) \quad H^0(\mathbb{Q}, M_\lambda) \cong H_{\mathbf{f}}^1(\mathbb{Q}, M_\lambda) \cong H_{\mathbf{f}}^1(\mathbb{Q}, M_\lambda^D) \cong H^0(\mathbb{Q}, M_\lambda^D) \cong 0.$$

This is conjectured to hold for all λ under our hypotheses on M and it implies Conjecture 2.6. If $M = A_f$ or $A_f(1)$ and $\lambda \notin S_f$ then (55) holds by Theorem 2.7.

Fontaine and Perrin-Riou [41, II.4] define an \mathcal{O}_λ -lattice $\delta_{f,\lambda}(M)$ in $K_\lambda \otimes_K \Delta_f(M)$. They assume $K = \mathbb{Q}$, denote their lattice $\Delta_S(T)$ (where S is a finite set of primes and T is a Galois-stable lattice in M_λ) and then prove it is independent of the choice of S and T . One checks that the definition and independence argument carry over to arbitrary K by taking determinants relative to \mathcal{O}_λ and K_λ instead of \mathbb{Z}_ℓ and \mathbb{Q}_ℓ . The arguments of [41, II.5] carry over as well, giving another description of $\delta_{f,\lambda}(M)$ for which we need more notation. Choose a Galois stable lattice $\mathcal{M}_\lambda \subset M_\lambda$ and a free rank one \mathcal{O}_λ -module $\omega \subset K_\lambda \otimes_K \det_K t_M$. We let $\theta(\mathcal{M}_\lambda) = \det_{\mathcal{O}_\lambda} \mathcal{M}_\lambda^+$, regarded as a lattice in $K_\lambda \otimes_K \det_K M_B^+$ via the comparison isomorphism I_λ^B . We let $\mathcal{M}_\lambda^D = \text{Hom}_{\mathcal{O}_\lambda}(\mathcal{M}_\lambda, \mathcal{O}_\lambda(1)) \subset M_\lambda^D$. The Tate–Shafarevich group of \mathcal{M}

$$\text{III}(\mathcal{M}_\lambda) := \frac{H_f^1(\mathbb{Q}, M_\lambda/\mathcal{M}_\lambda)}{H_f^1(\mathbb{Q}, M_\lambda) \otimes (K_\lambda/\mathcal{O}_\lambda)}$$

is always finite and can be identified with $H_f^1(\mathbb{Q}, M_\lambda/\mathcal{M}_\lambda)$ under our hypothesis $H_f^1(\mathbb{Q}, M_\lambda) = 0$. The same holds for \mathcal{M}_λ^D . Furthermore, by the main result of [33] (also [41, II.5.4.2]), $\text{III}(\mathcal{M}_\lambda)$ and $\text{III}(\mathcal{M}_\lambda^D)$ have the same length. In fact, there is an \mathcal{O}_λ -linear isomorphism

$$(56) \quad \text{III}(\mathcal{M}_\lambda^D) \cong \text{Hom}_{\mathbb{Z}_\ell}(\text{III}(\mathcal{M}_\lambda), \mathbb{Q}_\ell/\mathbb{Z}_\ell).$$

Finally, the Tamagawa ideal of \mathcal{M}_λ relative to ω is defined as

$$\text{Tam}_\omega^0(\mathcal{M}_\lambda) = \text{Tam}_{\ell,\omega}^0(\mathcal{M}_\lambda) \cdot \text{Tam}_\infty^0(\mathcal{M}_\lambda) \cdot \prod_{p \neq \ell} \text{Tam}_p^0(\mathcal{M}_\lambda),$$

where the factors are defined as in I.4.1 (and II.5.3.3) of [41]. Recall that $\text{Tam}_p^0(\mathcal{M}_\lambda) = 1$ if \mathcal{M}_λ is unramified at $p \neq \ell$ and that

$$\text{Tam}_\infty^0(\mathcal{M}_\lambda) = \text{Fitt}_{\mathcal{O}_\lambda} H^1(\mathbb{R}, \mathcal{M}_\lambda) = \mathcal{O}_\lambda$$

if ℓ is odd. The argument of [41, I.4.2.2] shows that if $p \neq \ell$, then

$$\text{Tam}_p^0(\mathcal{M}_\lambda) = \text{Fitt}_{\mathcal{O}_\lambda} H^1(I_p, \mathcal{M}_\lambda)_{\text{tor}}^{G_{\mathbb{Q}_p}}$$

from which it is not hard to deduce that

$$(57) \quad \text{Tam}_p^0(\mathcal{M}_\lambda) = \text{Tam}_p^0(\mathcal{M}_\lambda^D).$$

Viewing $\text{Hom}_{\mathcal{O}_\lambda}(\theta(\mathcal{M}_\lambda), \omega)$ as a lattice in $K_\lambda \otimes_K \Delta_f(M)$, we have by [41, Theorem II.5.3.6]

$$(58) \quad \delta_{f,\lambda}(M) = \frac{\text{Fitt}_{\mathcal{O}_\lambda} H^0(\mathbb{Q}, M_\lambda/\mathcal{M}_\lambda) \cdot \text{Fitt}_{\mathcal{O}_\lambda} H^0(\mathbb{Q}, M_\lambda^D/\mathcal{M}_\lambda^D)}{\text{Fitt}_{\mathcal{O}_\lambda} \text{III}(\mathcal{M}_\lambda^D) \cdot \text{Tam}_\omega^0(\mathcal{M}_\lambda)} \text{Hom}_{\mathcal{O}_\lambda}(\theta(\mathcal{M}_\lambda), \omega).$$

The λ -part of the Bloch–Kato conjecture can then be formulated as follows:

CONJECTURE 2.14. – *Let M in \mathbf{PM}_K be critical, $b(M)$ as in Conjecture 2.8, λ a place of K such that (55) holds and $\delta_{f,\lambda}(M)$ as in (58). Then*

$$\delta_{f,\lambda}(M) = (1 \otimes b(M))\mathcal{O}_\lambda.$$

THEOREM 2.15. – *Let f be a newform and S_f the set of places defined in (31). Then Conjecture 2.14 holds for both $M = A_f$ and $M = A_f(1)$ and any $\lambda \notin S_f$.*

Proof. – Suppose M , $b(M)$ and λ are as in Conjecture 2.14 and S is a set of places of \mathbb{Q} containing ℓ , ∞ and those where M_λ is ramified. Assume $\Sigma := S \setminus \{\ell, \infty\}$ is nonempty and $L_p(M, 0)^{-1} \neq 0$ for all $p \in \Sigma$. Put $b^\Sigma(M) = \prod_{p \in \Sigma} L_p(M, 0)b(M)$. By [41, Proof of I.4.2.2] we have for $p \in \Sigma$,

$$\text{Fitt}_{\mathcal{O}_\lambda} H_{\mathbf{f}}^1(\mathbb{Q}_p, \mathcal{M}_\lambda) = L_p(M, 0)^{-1} \text{Tam}_p^0(\mathcal{M}_\lambda)$$

since $L_p(M, 0)^{-1} \neq 0$. The exact sequence of Lemma 2.1 applied to $W = M_\lambda^D / \mathcal{M}_\lambda^D$ then implies that Conjecture 2.14 is equivalent to

$$(59) \quad \frac{\text{Fitt}_{\mathcal{O}_\lambda} H^0(\mathbb{Q}, M_\lambda^D / \mathcal{M}_\lambda^D)}{\text{Fitt}_{\mathcal{O}_\lambda} H_\Sigma^1(\mathbb{Q}, M_\lambda^D / \mathcal{M}_\lambda^D) \text{Tam}_{\ell, \omega}^0(\mathcal{M}_\lambda)} \text{Hom}_{\mathcal{O}_\lambda}(\theta(\mathcal{M}_\lambda), \omega) = (1 \otimes b^\Sigma(M)) \mathcal{O}_\lambda$$

where $H_\Sigma^1(\mathbb{Q}, M_\lambda^D / \mathcal{M}_\lambda^D)$ was defined in Section 2.1. We shall first prove Theorem 2.15 for $M = B_f = A_f(1)$ in which case the condition $L_p(M, 0)^{-1} \neq 0$ for the reformulation (59) of Conjecture 2.14 is satisfied.

Recall that $\mathcal{A}_{f, \lambda} = \text{ad}_{\mathcal{O}_\lambda}^0 \mathcal{M}_{f, \lambda}$ and put $\mathcal{B}_{f, \lambda} = \mathcal{A}_{f, \lambda}(1)$. Using the identification (39) of $t_{A_f} = \det_K t_{A_f}$ we let

$$\begin{aligned} \omega_A &= \mathcal{O}_\lambda \otimes_{\mathcal{O}} \text{Hom}_{\mathcal{O}}(\text{Fil}^{k-1} \mathcal{M}_{f, \text{dR}}, \mathcal{M}_{f, \text{dR}} / \text{Fil}^{k-1} \mathcal{M}_{f, \text{dR}}) \\ &\cong \text{Hom}_{\mathcal{O}_\lambda}(\text{Fil}^{k-1} \mathcal{M}_{f, \text{dR}} \otimes_{\mathcal{O}} \mathcal{O}_\lambda, \mathcal{M}_{f, \text{dR}} \otimes_{\mathcal{O}} \mathcal{O}_\lambda / \text{Fil}^{k-1} \mathcal{M}_{f, \text{dR}} \otimes_{\mathcal{O}} \mathcal{O}_\lambda) \\ &\cong \text{Hom}_{\mathcal{O}_\lambda}(\text{Fil}^{k-1} \mathcal{M}_{f, \lambda\text{-crys}}, \mathcal{M}_{f, \lambda\text{-crys}} / \text{Fil}^{k-1} \mathcal{M}_{f, \lambda\text{-crys}}). \end{aligned}$$

Similarly, identifying $\det_K t_{B_f}$ with $\det_K t_{A_f} \otimes \mathbb{Q}(2)_{\text{dR}}$ we define ω_B as $\omega_A \otimes \iota^{-2}$.

Fix a prime $\lambda \notin S_f$ and let Σ be the set of primes dividing N if $N > 1$ or put $\Sigma = \{p\}$ for some prime $\lambda \nmid p$ if $N = 1$. The isomorphism $\gamma : M_f \rightarrow M_f^\Sigma$ of Proposition 1.4 satisfies

$$\gamma^t = \gamma^{-1} \phi \prod_{p \in \Sigma} L_p^{\text{nv}}(B_f, 0)^{-1}$$

by Proposition 1.4 where $\phi = \prod_{\delta_p=1} (-a_p) \prod_{\delta_p=2} \psi(p)p^{k-1} \in \mathcal{O}_\lambda^\times$ (it is well known that $a_p^2 = \psi(p)p^{k-1}$ or $a_p^2 = \psi(p)p^{k-2}$ if $\delta_p = 1$ [63, 4.6.17]). Moreover, γ induces an isomorphism

$$B_f = \text{Hom}_K(M_f, M_f(1)) \rightarrow B_f^\Sigma := \text{Hom}_K(M_f^\Sigma, M_f^\Sigma(1))$$

and an isomorphism $\gamma : \Delta_{\mathbf{f}}(B_f) \rightarrow \Delta_{\mathbf{f}}(B_f^\Sigma)$ so that

$$\gamma(b)(x \otimes (2\pi i)^2) \otimes \iota^2 = \gamma_{\text{dR}} b(\gamma_{\text{dR}}^{-1}(x) \otimes (2\pi i)^2) \otimes \iota^2$$

for $b \in \Delta_{\mathbf{f}}(B_f)$ and $x \in M_{f, \text{dR}}^\Sigma$. For such b and x we have

$$\begin{aligned} \langle x, \gamma(b)(x \otimes (2\pi i)^2) \otimes \iota^2 \rangle^\Sigma &= \langle \gamma_{\text{dR}}^{-1}(x), \gamma_{\text{dR}}^t \gamma(b)(x \otimes (2\pi i)^2) \otimes \iota^2 \rangle \\ &= \langle \gamma_{\text{dR}}^{-1}(x), \gamma_{\text{dR}}^{-1} \gamma(b)(x \otimes (2\pi i)^2) \otimes \iota^2 \rangle \phi \prod_{p \in \Sigma} L_p^{\text{nv}}(B_f, 0)^{-1} \\ (60) \quad &= \langle \gamma_{\text{dR}}^{-1}(x), b(\gamma_{\text{dR}}^{-1}(x) \otimes (2\pi i)^2) \otimes \iota^2 \rangle \phi \prod_{p \in \Sigma} L_p^{\text{nv}}(B_f, 0)^{-1}. \end{aligned}$$

Recall that $\text{Fil}^{k-1} \mathcal{M}_{f,\text{dR}}^\Sigma = \mathcal{O} \cdot f^\Sigma = \mathcal{O} \cdot \gamma(f)$ by Propositions 1.4 where $\mathcal{O} = \bigcap_{\lambda \in S_f} K \cap \mathcal{O}_\lambda$. Note that if b' is an \mathcal{O}_λ -basis for

$$\begin{aligned} & \text{Hom}_{\mathcal{O}_\lambda}(\theta(\mathcal{B}_{f,\lambda}^\Sigma), \omega_B) \\ & \cong \text{Hom}_{\mathcal{O}_\lambda}(\text{Fil}^{k-1} \mathcal{M}_{f,\text{dR}}^\Sigma \otimes_{\mathcal{O}} \theta(\mathcal{B}_{f,\lambda}), (\mathcal{M}_{f,\text{dR}}^\Sigma / \text{Fil}^{k-1} \mathcal{M}_{f,\text{dR}}^\Sigma) \otimes_{\mathcal{O}} \mathcal{O}_\lambda \otimes \iota^{-2}), \end{aligned}$$

then

$$\mathcal{O}_\lambda \cdot b'(f^\Sigma \otimes (2\pi i)^2) \otimes \iota^2 = (\mathcal{M}_{f,\text{dR}}^\Sigma / \text{Fil}^{k-1} \mathcal{M}_{f,\text{dR}}^\Sigma) \otimes_{\mathcal{O}} \mathcal{O}_\lambda$$

and hence

$$\mathcal{O}_\lambda \cdot \langle f^\Sigma, b'(f^\Sigma \otimes (2\pi i)^2) \otimes \iota^2 \rangle^\Sigma = \mathcal{O}_\lambda \eta_f^\Sigma \mathcal{M}_\psi(1-k)_{\text{dR}}$$

where η_f^Σ was defined before Proposition 1.4. On the other hand by (60), Theorem 2.10 and the remark after the proof of Theorem 2.10, we have for $\lambda \notin S_f$,

$$\begin{aligned} & \mathcal{O}_\lambda \cdot \langle f^\Sigma, \gamma b^\Sigma(B_f)(f^\Sigma \otimes (2\pi i)^2) \otimes \iota^2 \rangle^\Sigma \\ & = \mathcal{O}_\lambda \cdot \langle f, b^\Sigma(B_f)(f \otimes (2\pi i)^2) \otimes \iota^2 \rangle \prod_{p \in \Sigma} L_p^{\text{nv}}(B_f, 0)^{-1} \\ & = \mathcal{O}_\lambda \cdot \langle f, b(B_f)(f \otimes (2\pi i)^2) \otimes \iota^2 \rangle \prod_{p \in \Sigma_e(f)} L_p(B_f, 0) \\ & = \mathcal{O}_\lambda(b_{\text{dR}} \otimes \iota^{k-1}) = \mathcal{O}_\lambda \mathcal{M}_\psi(1-k)_{\text{dR}}. \end{aligned}$$

Eq. (59) for $M = B_f$ therefore reduces to

$$\frac{\text{Fitt}_{\mathcal{O}_\lambda} H^0(\mathbb{Q}, B_{f,\lambda}^D / \mathcal{B}_{f,\lambda}^D)}{\text{Fitt}_{\mathcal{O}_\lambda} H_\Sigma^1(\mathbb{Q}, B_{f,\lambda}^D / \mathcal{B}_{f,\lambda}^D) \text{Tam}_{\ell,\omega_B}^0(\mathcal{B}_{f,\lambda})} \eta_f^\Sigma = \mathcal{O}_\lambda.$$

Using Proposition 2.16 below, the fact that $\mathcal{A}_f = \mathcal{B}_f^D$ and the vanishing of the group $H^0(\mathbb{Q}, \mathcal{A}_{f,\lambda} / \mathcal{A}_{f,\lambda})$ for $\lambda \in S_f$, this identity reduces to

$$\text{Fitt}_{\mathcal{O}_\lambda} H_\Sigma^1(\mathbb{Q}, \mathcal{A}_{f,\lambda} / \mathcal{A}_{f,\lambda}) = \mathcal{O}_\lambda \eta_f^\Sigma$$

which is Theorem 3.7.

By (56), (57) and Proposition 2.16, the factor in front of $\text{Hom}(\theta(\mathcal{M}_\lambda), \omega)$ in (58) is the same for $M = A_f$ and $M = B_f$. The isomorphism tw defined in (42) maps $\text{Hom}_{\mathcal{O}_\lambda}(\theta(\mathcal{A}_{f,\lambda}), \omega_A)$ to $\text{Hom}_{\mathcal{O}_\lambda}(\theta(\mathcal{B}_{f,\lambda}), \omega_B)$, hence $\delta_{f,\lambda}(A_f)$ to $\delta_{f,\lambda}(B_f)$. Theorem 2.15 for $M = A_f$ therefore follows from Theorem 2.15 for $M = B_f$, together with Theorem 2.10 and the fact that $(1-k)\epsilon(A_f)$ is a unit in \mathcal{O}_λ . \square

PROPOSITION 2.16. – *We have $\text{Tam}_{\ell,\omega_A}^0(\mathcal{A}_{f,\lambda}) = \text{Tam}_{\ell,\omega_B}^0(\mathcal{B}_{f,\lambda}) = \mathcal{O}_\lambda$ for $\lambda \notin S_f$.*

Proof. – With the notation in Section 1.1.1, we further denote by MF the additive category of filtered ϕ -modules as defined in [36, 1.2.1], by K_λ -MF the category of K_λ -modules in MF and by K_λ -MF^a the full subcategory of K_λ -MF with filtration restrictions as in Section 1.1.1. Scalar extension $- \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ induces an exact functor \mathcal{O}_λ -MF \rightarrow K_λ -MF where the notion of exactness in MF is defined in [36, 1.2.3].

Now assume that $\mathcal{D}_1, \mathcal{D}_2$ are torsion free objects of \mathcal{O}_λ -MF^a for some a and put $D_i = \mathcal{D}_i \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. Set $\mathcal{D} = \text{Hom}_{\mathcal{O}_\lambda}(\mathcal{D}_1, \mathcal{D}_2)$ and $D = \text{Hom}_{K_\lambda}(D_1, D_2)$ which are objects of MF and

MF respectively. We also have $D \cong \mathcal{D} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. An elementary computation shows that the first two rows in the following commutative diagram are exact

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & \mathrm{Hom}_{\mathcal{O}_\lambda\text{-MF}}(\mathcal{D}_1, \mathcal{D}_2) & \longrightarrow & \mathrm{Fil}^0 D & \xrightarrow{1-\phi^0} & D & \xrightarrow{\pi} & \mathrm{Ext}_{\mathcal{O}_\lambda\text{-MF}}^1(\mathcal{D}_1, \mathcal{D}_2) & \longrightarrow & 0 \\
 & & \downarrow \epsilon^0 & & \downarrow & & \downarrow & & \downarrow \epsilon^1 & & \\
 0 & \longrightarrow & \mathrm{Hom}_{K_\lambda\text{-MF}}(\mathcal{D}_1, \mathcal{D}_2) & \longrightarrow & \mathrm{Fil}^0 D & \xrightarrow{1-\phi} & D & \xrightarrow{\pi} & \mathrm{Ext}_{K_\lambda\text{-MF}}^1(\mathcal{D}_1, \mathcal{D}_2) & \longrightarrow & 0 \\
 & & \downarrow \theta^0 & & \downarrow \iota & & \downarrow \iota & & \downarrow \theta^1 & & \\
 0 & \longrightarrow & H^0(\mathbb{Q}_\ell, V) & \longrightarrow & \mathrm{Fil}^0 D(V) & \xrightarrow{1-\phi} & D(V) & \xrightarrow{e} & H_f^1(\mathbb{Q}_\ell, V) & \longrightarrow & 0 \\
 & & \parallel & & \downarrow & & \downarrow (\mathrm{id}, 0) & & \parallel & & \\
 0 & \longrightarrow & H^0(\mathbb{Q}_\ell, V) & \longrightarrow & D(V) & \longrightarrow & D(V) \oplus t_V & \longrightarrow & H_f^1(\mathbb{Q}_\ell, V) & \longrightarrow & 0
 \end{array}$$

(61)

where π is defined as follows. For $\eta \in D$, define an extension \mathcal{E}_η of \mathcal{D}_1 by \mathcal{D}_2 in $\mathcal{O}_\lambda\text{-MF}$ with underlying \mathcal{O}_λ -module $\mathcal{D}_2 \oplus \mathcal{D}_1$, filtration

$$\mathrm{Fil}^i \mathcal{E}_\eta := \mathrm{Fil}^i \mathcal{D}_2 \oplus \mathrm{Fil}^i \mathcal{D}_1$$

and Frobenius maps $\phi^i : \mathrm{Fil}^i \mathcal{E}_\eta \rightarrow \mathcal{E}_\eta$

$$(62) \quad \phi^i(x, y) = (\phi^i(x) + \eta\phi^i(y), \phi^i(y)).$$

The same definitions for $\eta \in D$ lead to an extension in $K_\lambda\text{-MF}$. Then $\pi(\eta)$ is the class of the Yoneda extension \mathcal{E}_η in Ext^1 (we shall identify Ext^1 with the group of Yoneda extensions throughout).

To explain the remaining part of diagram (61), we first recall the notion of admissibility from [36, 3.6.4]. A filtered ϕ -module D' in MF is called admissible if the natural map $B_{\mathrm{crys}} \otimes_{\mathbb{Q}_\ell} D' \cong B_{\mathrm{crys}} \otimes_{\mathbb{Q}_\ell} V(D')$ is an isomorphism where $V(D')$ is the G_ℓ -representation

$$V(D') = \mathrm{Fil}^0(D' \otimes_{B_{\mathrm{crys}}} \phi^{\otimes \phi=1}).$$

The functor $D' \rightarrow V(D')$ is fully faithful and exact on the category of admissible filtered ϕ -modules, and induces an equivalence of this category with the category $\mathrm{Rep}_{\mathrm{cris}}(G_\ell)$ of crystalline $K_\lambda[G_\ell]$ -representations (see [36, 3.6.5]). If $D' = \mathcal{D}' \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ for some object \mathcal{D}' of \mathcal{MF}^0 , then D' is admissible by [39, Theorem 8.4], and for such D' we have a natural isomorphism $V(D') \cong \mathbb{V}(\mathcal{D}') \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ by (1). If $D' = \mathcal{D}' \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ for some object \mathcal{D}' of \mathcal{MF}^a then we can extend the definition of \mathbb{V} by $\mathbb{V}(\mathcal{D}') = \mathbb{V}(\mathcal{D}'[-a])(a)$ (Tate twist) and we deduce again that D' is admissible. In particular, D_1 and D_2 are admissible, and then D is admissible by [36, Proposition 3.4.3]. Putting $V := V(D)$ and $V_i := V(D_i)$ we have an isomorphism of G_ℓ -representations $V = \mathrm{Hom}_{K_\lambda}(V_1, V_2)$ by [36, 3.6].

Coming back to diagram (61), the map ι is just the natural map induced by

$$D \xrightarrow{1 \otimes \bar{\iota}} B_{\mathrm{crys}} \otimes_{\mathbb{Q}_\ell} D \cong B_{\mathrm{crys}} \otimes_{\mathbb{Q}_\ell} V \leftarrow H^0(\mathbb{Q}_\ell, B_{\mathrm{crys}} \otimes_{\mathbb{Q}_\ell} V) =: D(V)$$

and e is the boundary map in Galois cohomology induced from the short exact sequence of $G_{\mathbb{Q}_\ell}$ -modules

$$(63) \quad 0 \rightarrow V(D) \rightarrow \text{Fil}^0(B_{\text{crys}} \otimes_{\mathbb{Q}_\ell} D) \xrightarrow{1-\phi \otimes \phi} B_{\text{crys}} \otimes_{\mathbb{Q}_\ell} D \rightarrow 0$$

as in the proof of [4, Lemma 4.5(b)]. It is clear that $K_\lambda\text{-MF}^a$ is closed under extensions inside $K_\lambda\text{-MF}$, hence we obtain a chain of isomorphisms

$$\begin{aligned} \theta^i : \text{Ext}_{K_\lambda\text{-MF}}^i(D_1, D_2) &\leftarrow \text{Ext}_{K_\lambda\text{-MF}^a}^i(D_1, D_2) \\ &\xrightarrow{v^i} \text{Ext}_{\text{Rep}_{\text{cris}}(G_\ell)}^i(V_1, V_2) \xrightarrow{\Delta^i} \text{Ext}_{\text{Rep}_{\text{cris}}(G_\ell)}^i(K_\lambda, \text{Hom}_{K_\lambda}(V_1, V_2)) \rightarrow H_f^i(\mathbb{Q}_\ell, V) \end{aligned}$$

for $i = 0, 1$. Here Δ^1 sends a Yoneda extension

$$0 \rightarrow V_2 \rightarrow V_3 \rightarrow V_1 \rightarrow 0$$

to the pull back to $K_\lambda \cdot 1_{V_1} \subseteq \text{Hom}_{K_\lambda}(V_1, V_1)$ of the induced extension

$$0 \rightarrow \text{Hom}_{K_\lambda}(V_1, V_2) \rightarrow \text{Hom}_{K_\lambda}(V_1, V_3) \rightarrow \text{Hom}_{K_\lambda}(V_1, V_1) \rightarrow 0.$$

The maps v^i (defined by applying V to a Yoneda extension) are isomorphisms because V is fully faithful and exact.

The three lower rows in (61) with the indicated maps form a commutative diagram, and all these rows are exact (see [4, Lemma 4.5(b)] for the two lower rows). We shall verify the identity $\theta^1 \pi = e\iota$, all the others being straightforward. Consider the commutative diagram

$$(64) \quad \begin{array}{ccccccc} 0 & \longrightarrow & V_2 & \longrightarrow & \text{Fil}^0(B_{\text{crys}} \otimes_{\mathbb{Q}_\ell} D_2) & \xrightarrow{1-\phi \otimes \phi} & B_{\text{crys}} \otimes_{\mathbb{Q}_\ell} D_2 & \longrightarrow & 0 \\ & & \parallel & & \uparrow & & \uparrow 1 \otimes \psi & & \\ 0 & \longrightarrow & V_2 & \longrightarrow & \text{Fil}^0(B_{\text{crys}} \otimes_{\mathbb{Q}_\ell} (D_2 \oplus D_1))^{\phi=1} & \longrightarrow & \text{Fil}^0(B_{\text{crys}} \otimes_{\mathbb{Q}_\ell} D_1)^{\phi=1} & \longrightarrow & 0 \end{array}$$

where all unnamed arrows are natural projection or inclusion maps, the top row is (63) with D replaced by D_2 , and the action of ϕ on $D_2 \oplus D_1$ is given by (62). For $\psi \in D$, the extension $e\iota(\psi)$ is the pullback of (63) under $K_\lambda(1 \otimes \psi) \subset B_{\text{crys}} \otimes_{\mathbb{Q}_\ell} D$. To compute $(\Delta^1)^{-1}e\iota(\psi)$ apply the exact functor $\text{Hom}_{K_\lambda}(V_1, -)$ to diagram (64). Via the isomorphisms

$$\begin{aligned} \text{Hom}_{K_\lambda}(V_1, B_{\text{crys}} \otimes_{\mathbb{Q}_\ell} D_2) &\cong \text{Hom}_{B_{\text{crys}} \otimes_{\mathbb{Q}_\ell} K_\lambda}(B_{\text{crys}} \otimes_{\mathbb{Q}_\ell} V_1, B_{\text{crys}} \otimes_{\mathbb{Q}_\ell} D_2) \\ &\cong \text{Hom}_{B_{\text{crys}} \otimes_{\mathbb{Q}_\ell} K_\lambda}(B_{\text{crys}} \otimes_{\mathbb{Q}_\ell} D_1, B_{\text{crys}} \otimes_{\mathbb{Q}_\ell} D_2) \\ &\cong B_{\text{crys}} \otimes_{\mathbb{Q}_\ell} \text{Hom}_{K_\lambda}(D_1, D_2), \\ \text{Hom}_{K_\lambda}(V_1, \text{Fil}^0(B_{\text{crys}} \otimes_{\mathbb{Q}_\ell} D_2)) &\cong \text{Fil}^0(B_{\text{crys}} \otimes_{\mathbb{Q}_\ell} \text{Hom}_{K_\lambda}(D_1, D_2)), \end{aligned}$$

the first row becomes isomorphic to (63) and the image of

$$1_{V_1} \in \text{Hom}_{K_\lambda}(V_1, V_1) = \text{Hom}_{K_\lambda}(V_1, \text{Fil}^0(B_{\text{crys}} \otimes_{\mathbb{Q}_\ell} D_1)^{\phi=1})$$

in $B_{\text{crys}} \otimes_{\mathbb{Q}_\ell} D$ is $1 \otimes \psi$. Hence $(\Delta^1)^{-1}e\iota(\psi)$ is represented by the lower row in (64). But from the definition of π it is immediate that the lower row in (64) is the image of $\pi(\psi)$ under the functor V . This gives the identity $\theta^1 \pi = e\iota$.

Put $T_i = \mathbb{V}(\mathcal{D}_i)$ for $i = 1, 2$ and $T = \text{Hom}_{\mathcal{O}_\lambda}(T_1, T_2)$ with its natural G_ℓ -action. Then, since T_1 is torsion free, $H^1(\mathbb{Q}_\ell, T)$ naturally identifies with the set of equivalence classes of extensions of $\mathcal{O}_\lambda[G_\ell]$ -modules

$$(65) \quad 0 \rightarrow T_2 \rightarrow T_3 \rightarrow T_1 \rightarrow 0.$$

Since the functor \mathbb{V} is exact on $\mathcal{O}_\lambda\text{-MF}^a$, and since $\mathcal{O}_\lambda\text{-MF}^a$ is closed under extensions in $\mathcal{O}_\lambda\text{-MF}$, we obtain maps

$$\Theta^i : \text{Ext}_{\mathcal{O}_\lambda\text{-MF}}^i(\mathcal{D}_1, \mathcal{D}_2) \xrightarrow{\sim} \text{Ext}_{\mathcal{O}_\lambda\text{-MF}^a}^i(\mathcal{D}_1, \mathcal{D}_2) \rightarrow \text{Ext}_{\mathcal{O}_\lambda[G_\ell]}^i(T_1, T_2) \cong H^i(\mathbb{Q}_\ell, T)$$

analogous to the maps θ^i . The faithfulness of \mathbb{V} implies that Θ^0 is injective and fullness of \mathbb{V} implies that Θ^0 is surjective and that Θ^1 is injective. The image of Θ^1 lies in the subgroup

$$H_f^1(\mathbb{Q}_\ell, T) := \{[T_3] \in H^1(\mathbb{Q}_\ell, T) \mid [V_3] := [T_3 \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell] \in H_f^1(\mathbb{Q}_\ell, V)\}$$

since $\mathbb{V}(\mathcal{D}_3) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \cong V(\mathcal{D}_3 \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell)$ is a crystalline representation. Conversely, if (65) lies in $H_f^1(\mathbb{Q}_\ell, T)$, the G_ℓ -module T_3 is a submodule of a crystalline representation V_3 so that $D(V_3)$ lies in $K_\lambda\text{-MF}^a$ and hence T_3 lies in the essential image of the Fontaine–Laffaille functor \mathbb{V} , $T_3 = \mathbb{V}(\mathcal{D}_3)$, say. Since \mathbb{V} is full the extension (65) is the image of a sequence $0 \rightarrow \mathcal{D}_2 \rightarrow \mathcal{D}_3 \rightarrow \mathcal{D}_1 \rightarrow 0$ in $\mathcal{O}_\lambda\text{-MF}^a$ and since \mathbb{V} is fully faithful and exact, this sequence is exact, hence represents an element of $\text{Ext}_{\mathcal{O}_\lambda\text{-MF}}^1(\mathcal{D}_1, \mathcal{D}_2)$. We conclude that

$$(66) \quad \Theta^1 : \text{Ext}_{\mathcal{O}_\lambda\text{-MF}}^1(\mathcal{D}_1, \mathcal{D}_2) \cong H_f^1(\mathbb{Q}_\ell, T)$$

is an isomorphism. It is clear that $\theta^i \epsilon^i = \tilde{\epsilon}^i \Theta^i$ where $\tilde{\epsilon}^i : H^i(\mathbb{Q}_\ell, T) \rightarrow H^i(\mathbb{Q}_\ell, V)$ are the natural maps. The last row in (61) induces an isomorphism

$$\begin{aligned} \det_{K_\lambda} H^0(\mathbb{Q}_\ell, V) \otimes_{K_\lambda} \det_{K_\lambda}^{-1} H_f^1(\mathbb{Q}_\ell, V) &\cong \det_{K_\lambda} D \otimes_{K_\lambda} \det_{K_\lambda}^{-1} D \otimes_{K_\lambda} \det_{K_\lambda}^{-1} t_V \\ &\cong \det_{K_\lambda}^{-1} t_V \end{aligned}$$

and the Tamagawa ideal is defined in [41, I.4.1.1] so that

$$(67) \quad \det_{\mathcal{O}_\lambda} H^0(\mathbb{Q}_\ell, T) \otimes_{\mathcal{O}_\lambda} \det_{\mathcal{O}_\lambda}^{-1} H_f^1(\mathbb{Q}_\ell, T) \cong \text{Tam}_{\ell, \omega}^0(T) \omega^{-1}.$$

Using the fact that Θ^0 is an isomorphism together with (66) and (61) one computes that the left hand side in (67) equals $\det_{\mathcal{O}_\lambda}^{-1} \mathcal{D} / \text{Fil}^0 \mathcal{D}$ so that $\text{Tam}_{\ell, \omega}^0(T) = \mathcal{O}_\lambda$ if ω is a basis of $\det_{\mathcal{O}_\lambda} \mathcal{D} / \text{Fil}^0 \mathcal{D}$.

These arguments apply to $\mathcal{D}_1 = \mathcal{D}_2 = \mathcal{M}_{f, \lambda\text{-crys}}$ which is an object of $\mathcal{O}_\lambda\text{-MF}^0$ if $\lambda \notin S_f$, more specifically if $\ell \nmid N$ and $\ell > k$. We have $T_1 = T_2 = \mathcal{M}_{f, \lambda}$ and $T = \mathcal{A}_{f, \lambda} \oplus \mathcal{O}_\lambda$. Our choice of ω_A then ensures that $\text{Tam}_{\ell, \omega_A}^0(\mathcal{A}_{f, \lambda}) = \mathcal{O}_\lambda$. For $B_f = \mathcal{A}_f(1)$ we can use the same argument as long as both $\mathcal{D}_1 = \mathcal{M}_{f, \lambda\text{-crys}}$ and $\mathcal{D}_2 = \mathcal{M}_{f, \lambda\text{-crys}}[1]$ are objects of $\mathcal{O}_\lambda\text{-MF}^1$. This is the case if $\ell > k + 1$ or if $\ell = k + 1$ and $\mathcal{M}_{f, \lambda\text{-crys}}$ has no nonzero quotient A in $\mathcal{O}_\lambda\text{-MF}$ with $\text{Fil}^{k-1} A = A$.

LEMMA 2.17. – *If $a_\ell \equiv 0 \pmod{\lambda}$, then $\mathcal{M}_{f, \lambda\text{-crys}}$ has no nonzero quotient A in $\mathcal{O}_\lambda\text{-MF}$ with $\text{Fil}^{k-1} A = A$.*

Proof. – By [75] we know that the characteristic polynomial of ϕ on

$$\mathcal{M} := \mathcal{M}_{f, \lambda\text{-crys}} \text{ is } X^2 - \psi^{-1}(\ell) a_\ell X + \psi^{-1}(\ell) \ell^{k-1},$$

hence $\bar{\phi}$ has characteristic polynomial X^2 on $\bar{\mathcal{M}} := \mathcal{M} \otimes_{\mathcal{O}} (\mathcal{O}_\lambda/\lambda)$. Since

$$(68) \quad \bar{\mathcal{M}} = \bar{\phi}(\bar{\mathcal{M}}) + \bar{\phi}^{k-1}(\text{Fil}^{k-1} \bar{\mathcal{M}})$$

and $\dim_{\mathcal{O}_\lambda/\lambda} \text{Fil}^{k-1} \bar{\mathcal{M}} = 1$, the map $\bar{\phi}$ is nonzero, hence conjugate to $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Since $\bar{\phi} = \ell^{k-1} \bar{\phi}^{k-1} = 0$ on $\text{Fil}^{k-1} \bar{\mathcal{M}}$ and because of (68), we have $\text{Fil}^{k-1} \bar{\mathcal{M}} = \ker(\bar{\phi}) = \bar{\phi}(\bar{\mathcal{M}})$. It is now easy to see that $\bar{\mathcal{M}}$ is a simple object in $\mathcal{O}_\lambda\text{-MF}$: Any proper subobject $N \subset \bar{\mathcal{M}}$ is $\bar{\phi}$ -stable, hence contained in $\ker(\bar{\phi}) = \text{Fil}^{k-1} \bar{\mathcal{M}}$ and we have $\text{Fil}^{k-1} N = N$. But again by (68) we find $\bar{\phi}^{k-1}(\text{Fil}^{k-1} \bar{\mathcal{M}}) \not\subset \ker(\bar{\phi}) = \bar{\phi}(\bar{\mathcal{M}})$ so that $N = \text{Fil}^{k-1} N = 0$. If A is a nonzero quotient of \mathcal{M} then \bar{A} is a nonzero quotient of $\bar{\mathcal{M}}$ hence equal to $\bar{\mathcal{M}}$ and we find $\text{Fil}^{k-1} \bar{A} \neq \bar{A}$ and $\text{Fil}^{k-1} A \neq A$. \square

It remains to prove Proposition 2.16 for $\mathcal{B}_{f,\lambda}$ in the ordinary case $a_\ell \neq 0 \pmod{\lambda}$ (and $\ell = k + 1$). We use the fact that $B_f \cong A_f^*(1)$ and appeal to the following conjecture, a slight generalization (from \mathbb{Z}_ℓ to \mathcal{O}_λ) of conjecture $C_{EP}(V)$ of [67] (we also use a similar generalization of [67, Proposition C.2.6]).

Let V be a crystalline representation of G_ℓ over K_λ and $T \subseteq V$ a G_ℓ -stable \mathcal{O}_λ -lattice. Let ω (resp. ω^*) be a lattice of

$$\det_{K_\lambda} D(V) / \text{Fil}^0 D(V) \quad (\text{resp. } \det_{K_\lambda} D(V^*(1)) / \text{Fil}^0 D(V^*(1)))$$

so that we obtain a lattice $\omega \otimes \omega^{*, -1}$ of $\det_{K_\lambda} D(V)$ via the exact sequence

$$(69) \quad 0 \rightarrow (D(V^*(1)) / \text{Fil}^0 D(V^*(1)))^* \rightarrow D(V) \rightarrow D(V) / \text{Fil}^0 D(V) \rightarrow 0.$$

Let $\eta(T, \omega, \omega^*) \in B_{\text{crys}} \otimes_{\mathbb{Q}_\ell} K_\lambda$ be such that $\det_{\mathcal{O}_\lambda} T = \eta(T, \omega, \omega^*) \omega \otimes \omega^{*, -1}$ under the comparison isomorphism $B_{\text{crys}} \otimes_{\mathbb{Q}_\ell} \det_{K_\lambda} V \cong B_{\text{crys}} \otimes_{\mathbb{Q}_\ell} \det_{K_\lambda} D(V)$. One shows that $\eta(T, \omega, \omega^*) \in \mathbb{Q}_\ell^{ur} \otimes_{\mathbb{Q}_\ell} K_\lambda$ [67, Lemme C.2.8] and that in fact $\eta(T, \omega, \omega^*) \in 1 \otimes K_\lambda$ up to an element in $(\mathbb{Z}_\ell^{ur} \otimes_{\mathbb{Z}_\ell} \mathcal{O}_\lambda)^\times$.

CONJECTURE 2.18. – For $j \in \mathbb{Z}$, put $h_j(V) = \dim_{K_\lambda} \text{Fil}^j D(V) / \text{Fil}^{j+1} D(V)$, and put $\Gamma^*(j) = (j - 1)!$ if $j > 0$ and $\Gamma^*(j) = (-1)^j ((-j)!)^{-1}$ if $j \leq 0$. Then

$$\mathcal{O}_\lambda \frac{\text{Tam}_{\ell, \omega}^0(T)}{\text{Tam}_{\ell, \omega^*}^0(T^*(1))} = \mathcal{O}_\lambda \prod_j \Gamma^*(-j)^{-h_j(V)} \eta(T, \omega, \omega^*).$$

Remark. – One can show that upon taking the norm from K_λ to \mathbb{Q}_ℓ all quantities in this formula transform into the corresponding quantities obtained by viewing V as a representation over \mathbb{Q}_ℓ rather than K_λ . Since the norm map $K_\lambda^\times / \mathcal{O}_\lambda^\times \rightarrow \mathbb{Q}_\ell^\times / \mathbb{Z}_\ell^\times$ is injective it suffices to prove the conjecture for $K_\lambda = \mathbb{Q}_\ell$.

We make Conjecture 2.18 more explicit for $V = A_{f,\lambda}$. In this case we have $h_j(V) = 1$ for $i = -1, 0, 1$ and $h_j(V) = 0$ otherwise so that $\prod_j \Gamma^*(-j)^{-h_j(V)} = -1$. For $\lambda \notin S_f$, equation (20) shows that the isomorphism $B_{\text{crys}} \otimes_{\mathbb{Q}_\ell} \det_{K_\lambda} V \cong B_{\text{crys}} \otimes_{\mathbb{Q}_\ell} \det_{K_\lambda} D(V)$ is induced by the functor \mathbb{V} for the unit object in \mathcal{PM}_K^S , hence sends $\det_{\mathcal{O}_\lambda} \mathcal{A}_{f,\lambda}$ to $\det_{\mathcal{O}_\lambda} \mathcal{A}_{f,\text{dR}} \otimes_{\mathcal{O}} \mathcal{O}_\lambda$. The computation of t_{B_f} in (40) works with $M_{f,\text{dR}}$ replaced by $\mathcal{M}_{f,\text{dR}} \otimes_{\mathcal{O}} \mathcal{O}_\lambda$ and the pairing (21) on A_f gives a perfect pairing $(\mathcal{A}_{f,\text{dR}} \otimes_{\mathcal{O}} \mathcal{O}_\lambda) \otimes (\mathcal{A}_{f,\text{dR}} \otimes_{\mathcal{O}} \mathcal{O}_\lambda) \rightarrow \mathcal{O}_\lambda$. Hence we find that $\omega_A \otimes \omega_B^{-1}$ is a basis of $\det_{\mathcal{O}_\lambda} \mathcal{A}_{f,\text{dR}} \otimes_{\mathcal{O}} \mathcal{O}_\lambda$ via the exact sequence (69). We conclude that $\eta(\mathcal{A}_{f,\lambda}, \omega_A, \omega_B) = 1$ and that Conjecture 2.18 reduces to the assertion

$$\mathcal{O}_\lambda \text{Tam}_{\ell, \omega_A}^0(\mathcal{A}_{f,\lambda}) = \mathcal{O}_\lambda \text{Tam}_{\ell, \omega_B}^0(\mathcal{A}_{f,\lambda}(1)).$$

Moreover, we know from the first part of the proof that the left hand side equals \mathcal{O}_λ if $\lambda \notin S_f$. Now Conjecture 2.18 is shown in [66] for $K_\lambda = \mathbb{Q}_\ell$ and V an ordinary representation of $G_{\mathbb{Q}_\ell}$ (combine Proposition 4.2.5, Theorem 3.5.4 of loc. cit.) under the assumption of another conjecture $\text{Rec}(V)$ which has meanwhile been proved in [12]. Ordinarity of $A_{f,\lambda}$ is implied by ordinarity of $M_{f,\lambda}$ which in turn is implied by $a_\ell \not\equiv 0 \pmod{\lambda}$. This finishes the proof of Proposition 2.16. \square

3. The Taylor–Wiles construction

Our method for computing the Selmer group of $A_{f,\lambda}$ is based on that of Wiles [88] and his work with Taylor [86]. We first give an axiomatic formulation of the method of [88] and [86], made possible by the simplifications due to Faltings ([86], appendix), Lenstra [60], Fujiwara [42] and one of the authors [24]. This formulation makes no reference to deformation rings and group rings that appear in other axiomatizations of the method. We then verify these axioms in the context of modular forms of higher weight.

3.1. An axiomatic formulation

In this section, we fix a prime λ of a number field K and let $\kappa = \mathcal{O}_K/\lambda$. We let ℓ denote the rational prime in λ and F the quadratic subfield of $\mathbb{Q}(\mu_\ell)$. We also fix a continuous, odd, irreducible representation

$$\rho_0 : G_{\mathbb{Q}} \rightarrow \text{Aut}_\kappa(V_0)$$

where V_0 is two-dimensional over κ . We define the Serre weight k of ρ_0 as in [78], but using geometric normalizations. (Thus k is the integer associated in [78] to the representation on $\text{Hom}_\kappa(V_0, \kappa)$.) We impose the following three conditions on the representations ρ_0 we consider:

- ρ_0 has minimal conductor among its twists.
- The restriction of ρ_0 to G_F is absolutely irreducible.
- The Serre weight k of ρ_0 satisfies $2 \leq k \leq \ell - 1$.

The last condition is equivalent to $\rho_0|_{I_\ell}$ being equivalent over $\bar{\kappa}$ to a representation of the form

- $\psi_\ell^{1-k} \oplus \psi_\ell^{\ell(1-k)}$ where ψ_ℓ is a fundamental character of level two,
- or $\begin{pmatrix} 1 & * \\ 0 & \chi_\ell^{1-k} \end{pmatrix}$, peu ramifié if $k = 2$.

We let $\psi : G_{\mathbb{Q}} \rightarrow \mathcal{O}_\lambda^\times$ denote the Teichmüller lift of $\chi_\ell^{1-k}(\det \rho_0^{-1})$; thus ψ is unramified at ℓ and has order prime to ℓ , and $\psi^{-1}\chi_\ell^{1-k}$ is a lift of $\det \rho_0$. We let δ denote $\psi^{-1}\chi_\ell^{1-k}$.

We consider continuous geometric ℓ -adic representations

$$\rho : G_{\mathbb{Q}} \rightarrow \text{Aut}_{K_\rho}(V_\rho)$$

where V_ρ is two-dimensional over a finite extension K_ρ of K_λ contained in \bar{K}_λ , ρ has determinant δ and reduction isomorphic to ρ_0 over $\bar{\kappa}$. We let \mathcal{O}_ρ denote the ring of integers of K_ρ . We say such a representation ρ is an *allowable lift* of ρ_0 if its restriction to G_ℓ is short and crystalline. For a prime $p \neq \ell$, we say ρ is *minimally ramified* at p if the following hold:

- If $\#\rho_0(I_p) \neq \ell$, then $\rho(I_p) \cong \rho_0(I_p)$.
- If $\#\rho_0(I_p) = \ell$, then $\dim_{K_\rho} V_\rho^{I_p} = 1$.

Suppose we are given a set \mathcal{N} of allowable lifts. We assume the \bar{K}_λ -isomorphism classes of the elements of \mathcal{N} are distinct. For each ρ , we let Σ_ρ denote the set of primes at which ρ is not minimally ramified. For each set of primes

$$\Sigma \subseteq \Sigma_0 := \{p \mid p \neq \ell\}$$

we let \mathcal{N}^Σ denote the set of ρ in \mathcal{N} such that $\Sigma_\rho \subset \Sigma$ and write V^Σ for the direct sum over \mathcal{N}^Σ of $\bar{V}_\rho = \bar{K}_\lambda \otimes_{K_\rho} V_\rho$. We assume that \mathcal{N}^Σ is finite if Σ is finite.

A *trellis* for \mathcal{N} is an $\mathcal{O}_\lambda G_{\mathbb{Q}}$ -submodule L of V^{Σ_0} such that for each finite set $\Sigma \subset \Sigma_0$, the \mathcal{O}_λ -module $L^\Sigma := L \cap V^\Sigma$ is finitely generated and the map $\bar{K}_\lambda \otimes_{\mathcal{O}_\lambda} L^\Sigma \rightarrow V^\Sigma$ is an isomorphism. One checks that if ρ in \mathcal{N} is such that $K_\rho = K_\lambda$, then $L_\rho := L \cap \bar{V}_\rho$ satisfies $\bar{K}_\lambda \otimes_{\mathcal{O}_\lambda} L_\rho \xrightarrow{\sim} V_\rho$. (To see this, for each $\sigma \neq \rho \in \mathcal{N}^\Sigma$, choose g_σ such that $\text{tr } \rho(g_\sigma) \neq \text{tr } \sigma(g_\sigma)$. Then the map $V^\Sigma \rightarrow \prod_{\sigma \neq \rho} V^\Sigma$ defined by $(g_\sigma^2 - \text{tr } \rho(g_\sigma)g_\sigma - \det \rho(g_\sigma))_\sigma$ has kernel \bar{V}_ρ . It follows that its restriction to a map $L^\Sigma \rightarrow \prod_{\sigma \neq \rho} L^\Sigma$ has kernel L_ρ , and therefore that $\bar{K}_\lambda \otimes_{\mathcal{O}_\lambda} L_\rho \xrightarrow{\sim} V_\rho$.) For such ρ , we let

$$A_\rho = (\text{ad}_{K_\lambda}^0 V_\rho) / (\text{ad}_{\mathcal{O}_\lambda}^0 L_\rho).$$

One checks that if ρ is minimally ramified at p , then $A_\rho^{I_p}$ is divisible.

A *system of perfect pairings* for L is an $\mathcal{O}_\lambda[G_{\mathbb{Q}}]$ -isomorphism

$$\varphi^\Sigma : L^\Sigma \rightarrow \text{Hom}_{\mathcal{O}_\lambda}(L^\Sigma, \mathcal{O}_\lambda(\psi^{-1}\chi_\ell^{1-k}))$$

for each finite $\Sigma \subset \Sigma_0$. Since the \bar{V}_ρ are irreducible, non-isomorphic and have determinant δ , we see that for each ρ in \mathcal{N}^Σ , φ^Σ induces a $\bar{K}_\lambda G_{\mathbb{Q}}$ -isomorphism

$$\wedge_{\bar{K}_\lambda}^2 \bar{V}_\rho \rightarrow \bar{K}_\lambda(\delta)$$

which we denote by φ_ρ^Σ . Moreover if $K_\rho = K_\lambda$, then φ_ρ^Σ arises from an injection

$$\wedge_{\mathcal{O}_\lambda}^2 L_\rho \rightarrow \mathcal{O}_\lambda(\delta).$$

We say that a prime q is *horizontal* if the following hold

- $q \equiv 1 \pmod{\ell}$;
- ρ_0 is unramified at q ;
- $\rho_0(\text{Frob}_q)$ has distinct eigenvalues.

If Q is a finite set of horizontal primes, we let Δ_Q denote the maximal quotient of $\prod_{q \in Q} (\mathbb{Z}/q\mathbb{Z})^\times$ of ℓ -power order. For each $q \in Q$, we choose an eigenvalue $\alpha_q \in \bar{\kappa}$ of $\rho_0(\text{Frob}_q)$ and let $\mu_{q,0}$ denote the unramified character $G_q \rightarrow \bar{\kappa}^\times$ sending Frob_q to α_q . Suppose that ξ is a character $\Delta_Q \rightarrow \bar{K}_\lambda^\times$. We say that $\rho \in \mathcal{N}^Q$ is a ξ -lift of ρ_0 if for each $q \in Q$, we have

$$\bar{V}_\rho \cong \bar{K}_\lambda(\mu_{q,\rho}) \oplus \bar{K}_\lambda(\delta/\mu_{q,\rho})$$

as $\bar{K}_\lambda G_q$ -modules for some lift $\mu_{q,\rho} : G_q \rightarrow \bar{K}_\lambda^\times$ of $\mu_{q,0}$ with $\mu_q | I_q$ corresponding via local class field theory to $\xi|_{\Delta_{\{q\}}}$.

THEOREM 3.1. – *Let \mathcal{N} be a set of allowable lifts of ρ_0 (with distinct \bar{K}_λ -isomorphism classes and finite \mathcal{N}^Σ for each finite $\Sigma \subset \Sigma_0$), L a trellis for \mathcal{N} and φ a system of perfect pairings for L . Suppose that*

- $\mathcal{N}^\emptyset \neq \emptyset$;
- if $\Sigma \subset \Sigma^0$ is a finite set of primes and $\rho \in \mathcal{N}^\emptyset$, then

$$\varphi_\rho^\Sigma = \varphi_\rho^\emptyset \beta_\rho^\Sigma \prod_{p \in \Sigma} L_p(\text{ad}_{K_\rho}^0 V_\rho, 1)^{-1}$$

for some β_ρ^Σ in \mathcal{O}_ρ ;

• if Q is a finite set of horizontal primes, then

- (i) $\beta_\rho^Q \in \mathcal{O}_\lambda$ is independent of $\rho \in \mathcal{N}^\emptyset$,
- (ii) $\#\mathcal{N}^Q \leq \#\mathcal{N}^\emptyset \cdot \#\Delta_Q$, and
- (iii) there is a ξ -lift of ρ_0 in \mathcal{N}^Q for each $\xi: \Delta_Q \rightarrow \bar{K}_\lambda^\times$.

Then every allowable lift of ρ_0 is isomorphic over \bar{K}_λ to some ρ in \mathcal{N} . Furthermore if $K_\rho = K_\lambda$, then the lengths of

$$H_\Sigma^1(\mathbb{Q}, A_\rho) \quad \text{and} \quad \mathcal{O}_\lambda(\delta)/\varphi_\rho^\Sigma(\wedge_{\mathcal{O}_\lambda}^2 L_\rho)$$

coincide for any finite subset Σ of Σ_0 containing Σ_ρ .

Proof. – One checks that to prove the theorem, we can replace K_λ by any finite extension and so assume that κ contains the eigenvalues of the elements of the image of ρ_0 . Note also that the hypotheses ensure the existence of an element ρ_{\min} of \mathcal{N}^\emptyset . We may assume also that $K_\lambda = K_{\rho_{\min}}$, and we write simply V_{\min} , L_{\min} and A_{\min} for $V_{\rho_{\min}}$, $L_{\rho_{\min}}$ and $A_{\rho_{\min}}$.

We first recall the results we need from the deformation theory of Galois representations. See [19,61] and Appendix A of [13] for more details. We let \mathcal{C} denote the category of complete local Noetherian \mathcal{O}_λ -algebras. Recall that if A is an object of \mathcal{C} with maximal ideal \mathfrak{m} , then an A -deformation of V_0 is an isomorphism class of free A -modules M endowed with continuous $AG_{\mathbb{Q}}$ -action $\rho_M: G_{\mathbb{Q}} \rightarrow \text{Aut}_A M$ such that $M/\mathfrak{m}M$ is $(A/\mathfrak{m})G_{\mathbb{Q}}$ -isomorphic to $(A/\mathfrak{m}) \otimes_\kappa V_0$. For a prime $p \neq \ell$, we say that an A -deformation of V_0 is *minimally ramified* at p if the following hold:

- If $\#\rho_0(I_p) \neq \ell$, then $\rho_M(I_p) \cong \rho_0(I_p)$.
- If $\#\rho_0(I_p) = \ell$, then M/M^{I_p} is free of rank one over A .

Suppose that Σ is a finite subset of Σ_0 . We say that M is of *type* Σ if the following hold:

- the $AG_{\mathbb{Q}}$ -module M is minimally ramified outside Σ ;
- the $AG_{\mathbb{Q}}$ -module $\wedge_A^2 M$ is isomorphic to $A \otimes_{\mathcal{O}_\lambda} \mathcal{O}_\lambda(\delta)$;
- there exists an object A_0 of \mathcal{C} with maximal ideal \mathfrak{m}_0 and finite residue field so that $M \cong A \otimes_{A_0} M_0$ and for every $n > 0$, the $\mathbb{Z}_\ell G_\ell$ -module $M_0/\mathfrak{m}_0^n M_0$ is an object of the category $\mathcal{MF}_{\text{tor}}^0$.

Consider the functor on \mathcal{C} which associates to A the set of A -deformations of ρ_0 of type Σ . By the results of Mazur and Ramakrishna, this functor is representable by an object of \mathcal{C} .² We denote this object R^Σ and let M^Σ denote the universal deformation. We recall also that R^Σ is topologically generated over \mathcal{O}_λ by the elements t_g^Σ for g in $G_{\mathbb{Q}}$, where t_g^Σ denotes the trace of the endomorphism g of the free R^Σ -module M^Σ . In particular, R^Σ has residue field κ .

If $\Sigma_1 \subset \Sigma_2$, then M^{Σ_1} is an R^{Σ_1} -deformation of V_0 of type Σ_2 and hence gives rise to a natural surjection $R^{\Sigma_2} \rightarrow R^{\Sigma_1}$.

Suppose now that ρ is in \mathcal{N} and $\Sigma_\rho \subset \Sigma$. Then \mathcal{O}_ρ is an object of \mathcal{C} and there is an \mathcal{O}_ρ -deformation M of ρ_0 of type Σ so that V_ρ is $K_\rho G_{\mathbb{Q}}$ -isomorphic to $K_\rho \otimes_{\mathcal{O}_\rho} M$. We thus obtain a continuous \mathcal{O}_λ -algebra homomorphism

$$\theta_\rho^\Sigma: R^\Sigma \rightarrow K_\rho$$

so that $K_\rho \otimes_{R^\Sigma} M^\Sigma$ is isomorphic to V_ρ . The maps θ_ρ^Σ for varying $\Sigma \supset \Sigma_\rho$ are compatible with the natural surjections $R^{\Sigma_2} \rightarrow R^{\Sigma_1}$ defined above. Note also that if $K_\rho = K_\lambda$, then $A = \mathcal{O}_\lambda$ and θ_ρ^Σ defines a surjection $R^\Sigma \rightarrow \mathcal{O}_\lambda$. In that case we have a natural isomorphism

$$(70) \quad \text{Hom}_{\mathcal{O}_\lambda}(\mathfrak{p}_\rho^\Sigma/(\mathfrak{p}_\rho^\Sigma)^2, K_\lambda/\mathcal{O}_\lambda) \cong H_\Sigma^1(\mathbb{Q}, A_\rho)$$

² Following [19] and of [13], we note that it is not necessary to assume A has residue field κ or to use strict equivalence classes of deformations.

of \mathcal{O}_λ -modules where \mathfrak{p}_ρ^Σ is the kernel of θ_ρ^Σ . (We omit the proof, which is now standard; see for example Proposition 1.2 of [88] or Section 2 of [14], and use Proposition 2.2 above to identify the local condition on ℓ .) In particular this is the case for $\rho = \rho_{\min}$ and any finite $\Sigma \subset \Sigma_0$.

We regard V^Σ as a module for R^Σ via

$$(71) \quad R^\Sigma \rightarrow \prod_{\rho \in \mathcal{N}^\Sigma} K_\rho$$

defined by the maps θ_ρ^Σ . Note that if g is in $G_\mathbb{Q}$, then t_g^Σ acts on V^Σ via the endomorphism

$$\text{tr}(\rho(g)) = g + \delta(g)g^{-1}$$

which is given by an element of $\mathcal{O}_\lambda G_\mathbb{Q}$. It follows that L^Σ is stable under the action of R^Σ and that ϕ^Σ is R^Σ -linear. If $\Sigma_1 \subset \Sigma_2$, then regarding L^{Σ_1} as an R^{Σ_2} -module via the natural surjection to R^Σ , we see that the inclusion $L^{\Sigma_1} \rightarrow L^{\Sigma_2}$ is R^{Σ_2} -linear, as is its adjoint with respect to φ^{Σ_1} and φ^{Σ_2} .

We define the finite flat \mathcal{O}_λ -algebra T^Σ to be the image of R^Σ in $\text{End}_{\mathcal{O}_\lambda} L^\Sigma$. The maps θ_ρ^Σ induce an isomorphism of finite \bar{K}_λ -algebras

$$\bar{K}_\lambda \otimes_{\mathcal{O}_\lambda} T^\Sigma \rightarrow \prod_{\rho \in \mathcal{N}^\Sigma} \bar{K}_\lambda$$

such that $t_g^\Sigma \mapsto (\text{tr } \rho(g))_{\rho \in \mathcal{N}^\Sigma}$ for g in $G_\mathbb{Q}$. (The injectivity follows from that of

$$\bar{K}_\lambda \otimes_{\mathcal{O}_\lambda} T^\Sigma \rightarrow \bar{K}_\lambda \otimes_{\mathcal{O}_\lambda} \text{End}_{\mathcal{O}_\lambda} L^\Sigma,$$

and the surjectivity from the distinctness of the θ_ρ^Σ .) In particular T^Σ is reduced and

$$(72) \quad \text{rank}_{\mathcal{O}_\lambda} L^\Sigma = 2 \cdot \#\mathcal{N}^\Sigma = 2 \cdot \text{rank}_{\mathcal{O}_\lambda} T^\Sigma.$$

Suppose that ρ is an element of \mathcal{N}^Σ such that $K_\rho = K_\lambda$. Write \mathcal{P}_ρ^Σ for the image of \mathfrak{p}_ρ^Σ in T^Σ and I_ρ^Σ for the annihilator of \mathcal{P}_ρ^Σ in T^Σ . Note that \mathcal{P}_ρ^Σ (resp., I_ρ^Σ) is the set of elements in T^Σ whose image in $\prod \bar{K}_\lambda$ has trivial component at ρ (resp., at each $\rho' \neq \rho$).

Now consider the \mathcal{O}_λ -module

$$\Omega_\rho^\Sigma = L^\Sigma / (L^\Sigma[\mathcal{P}_\rho^\Sigma] + L^\Sigma[I_\rho^\Sigma]).$$

We define η_ρ^Σ as the annihilator of the finite torsion \mathcal{O}_λ -module

$$\mathcal{O}_\lambda(\delta) / \varphi_\rho^\Sigma (\wedge_{\mathcal{O}_\lambda}^2 L_\rho).$$

We shall write $\mathfrak{p}_{\min}^\Sigma$, Ω_{\min}^Σ and η_{\min}^Σ for $\mathfrak{p}_{\rho_{\min}}^\Sigma$, $\Omega_{\rho_{\min}}^\Sigma$ and $\eta_{\rho_{\min}}^\Sigma$.

LEMMA 3.2. – *The \mathcal{O}_λ -module Ω_ρ^Σ is isomorphic to $(\mathcal{O}_\lambda / \eta_\rho^\Sigma)^2$.*

Proof. – Note that the kernel of the projection $L^\Sigma \rightarrow V_\rho$ coincides with that of the surjective composite

$$L^\Sigma \xrightarrow{\sim} \text{Hom}_{\mathcal{O}_\lambda}(L^\Sigma, \mathcal{O}_\lambda(\delta)) \rightarrow \text{Hom}_{\mathcal{O}_\lambda}(L_\rho, \mathcal{O}_\lambda(\delta)),$$

where the first map is φ^Σ and the second is the natural surjection. Denoting this kernel by L_ρ^\perp , we have $L_\rho \subset L^\Sigma[\mathcal{P}_\rho^\Sigma]$ and $L_\rho^\perp \subset L^\Sigma[I_\rho^\Sigma]$. Furthermore both inclusions are equalities since they become so after tensoring with \bar{K}_λ and L^Σ/L_ρ and L^Σ/L_ρ^\perp are torsion-free. Therefore the \mathcal{O}_λ -module Ω_ρ^Σ is isomorphic to the cokernel of the map

$$L_\rho \rightarrow L^\Sigma/L_\rho^\perp \xrightarrow{\sim} \text{Hom}_{\mathcal{O}_\lambda}(L_\rho, \mathcal{O}_\lambda(\delta))$$

induced by φ^Σ , which in turn is isomorphic to

$$\text{Hom}_{\mathcal{O}_\lambda}(L_\rho, \mathcal{O}_\lambda(\delta)) \otimes_{\mathcal{O}_\lambda} \mathcal{O}_\lambda/\eta_\rho^\Sigma$$

(in fact, canonically so as an $\mathcal{O}_\lambda G_{\mathbb{Q}}$ -module). \square

Suppose now that Q is a finite set of horizontal primes. For each $q \in Q$, we have chosen an eigenvalue α_q of $\rho_0(\text{Frob}_q)$. As in Lemma 2.44 of [14],

$$M^Q \cong R^Q(\mu_q^Q) \oplus R^Q(\delta/\mu_q^Q)$$

as an $R^Q G_q$ -module for some lift $\mu_q^Q : G_q \rightarrow (R^Q)^\times$ of $\mu_{q,0}$. (Recall that the character $\mu_{q,0}$ was defined before the statement of the theorem and is κ^\times -valued since we enlarged K_λ .) The restriction of μ_q^Q to the inertia group I_q factors through

$$I_q \rightarrow \mathbb{Z}_q^\times \rightarrow \Delta_{\{q\}}$$

where the first map is gotten from local class field theory and the second is the natural projection. We thus obtain a homomorphism $\Delta_{\{q\}} \rightarrow (R^Q)^\times$ for each $q \in Q$. We can thus regard R^Q as an $\mathcal{O}_\lambda[\Delta_Q]$ -algebra, and so regard L^Q as an $\mathcal{O}_\lambda[\Delta_Q]$ -module. Note that every $\rho \in \mathcal{N}^Q$ is a ξ -lift for a unique $\xi = \xi_\rho : \Delta_Q \rightarrow \bar{K}_\lambda^\times$, and then Δ_Q acts on V_ρ via ξ_ρ .

Now let \mathcal{P}^Q denote the augmentation ideal of $\mathcal{O}_\lambda[\Delta_Q]$, i.e., the kernel of the map $\mathcal{O}_\lambda[\Delta_Q] \rightarrow \mathcal{O}_\lambda$ defined by $g \mapsto 1$ for g in Δ_Q . Let I^Q denote the annihilator of \mathcal{P}^Q in $\mathcal{O}_\lambda[\Delta_Q]$, i.e., the principal ideal generated by $t_{\Delta_Q} = \sum_{g \in \Delta_Q} g$. Now consider the \mathcal{O}_λ -module

$$\Omega^Q = L^Q / (L^Q[\mathcal{P}^Q] + L^Q[I^Q]).$$

LEMMA 3.3. – L^Q is free over $\mathcal{O}_\lambda[\Delta_Q]$, and $L^Q/\mathcal{P}^Q L^Q$ is isomorphic over R^Q to L^θ .

Proof. – Note that $\bar{K}_\lambda[\Delta_Q] \cong \prod_\xi \bar{K}_\lambda$ via $g \mapsto (\xi(g))_\xi$, the product being over all characters $\xi : \Delta_Q \rightarrow \bar{K}_\lambda^\times$. Hypothesis (c) of the theorem ensures that this algebra acts faithfully on V^Q , and hence that $\mathcal{O}_\lambda[\Delta_Q]$ acts faithfully on L^Q . Furthermore, if ρ is in \mathcal{N}^Q , then ρ is in \mathcal{N}^θ if and only if Δ_Q acts trivially on V_ρ , so we have $L^Q[\mathcal{P}^Q] = L^\theta$. It follows also that $\#\Delta_Q L^\theta \subset t_{\Delta_Q} L^Q \subset L^\theta$. Thus $L^Q[I^Q] = (L^\theta)^\perp$, so Ω^Q is isomorphic to the cokernel of the endomorphism ν_Q of L^θ obtained by composing the inclusion $L^\theta \rightarrow L^Q$ with its adjoint with respect to φ^Q and φ^θ . Our hypotheses on the pairings (including (a)) ensure that

$$\nu_Q = \#\Delta_Q \beta^Q \prod_{q \in Q} q^{-3} (qt_{\text{Frob}_q}^2 \delta^{-1}(\text{Frob}_q) - (q+1)^2) \in \#\Delta_Q R^Q.$$

Therefore Ω^Q has length at least that of $L^\theta/\#\Delta_Q L^\theta$. Since $\mathcal{P}^Q/(\mathcal{P}^Q)^2 \cong \mathcal{O}_\lambda \otimes \Delta_Q$, we get

$$\text{length}_{\mathcal{O}_\lambda} \Omega^Q \geq d \text{length}_{\mathcal{O}_\lambda} \mathcal{P}^Q/(\mathcal{P}^Q)^2$$

where d is the \mathcal{O}_λ -rank of L^θ . On the other hand, hypothesis (b) gives

$$\text{rank}_{\mathcal{O}_\lambda} L^Q = 2 \cdot \#\mathcal{N}^Q \leq 2 \cdot \#\Delta_Q \cdot \#\mathcal{N}^\theta = d \text{rank}_{\mathcal{O}_\lambda} \mathcal{O}_\lambda[\Delta_Q].$$

Theorem 2.4 of [24] therefore implies that L^Q is free over $\mathcal{O}[\Delta_Q]$ of rank d . It follows that $L^Q/\mathcal{P}^Q L^Q$ is free of rank d over \mathcal{O}_λ . Since the adjoint of $L^\theta \rightarrow L^Q$ is surjective with kernel containing $\mathcal{P}^Q L^Q$, it follows that $L^Q/\mathcal{P}^Q L^Q \cong L^\theta$. \square

The following is proved exactly as in Chapter 3 of [88] (see Theorem 2.49 of [14]), except that we use Corollary 2.3 above instead of Proposition 1.9 of [88] (or Proposition 2.27 of [14]).

LEMMA 3.4. – *There exists an integer $r \geq 0$ and sets of horizontal primes Q_n for each $n \geq 1$ such that the following hold:*

- $\#Q_n = r$;
- $q \equiv 1 \pmod{\ell^n}$ for each $q \in Q_n$;
- R^{Q_n} is generated by r elements as an \mathcal{O} -algebra.

We are now ready to prove that R^θ is a complete intersection over which L^θ is free of rank two. We let r and $Q = Q_n$ for $n \geq 1$ be as in Lemma 3.4. Setting $A = \kappa[[S_1, \dots, S_r]]$, $B = \kappa[[X_1, \dots, X_r]]$, $R = \kappa \otimes_{\mathcal{O}} R^\theta$ and $H = \kappa \otimes_{\mathcal{O}} L^\theta$, we shall define B -modules H_n and maps $\phi_n : A \rightarrow B$, $\psi_n : B \rightarrow R$ and $\pi_n : H_n \rightarrow H$ satisfying the hypotheses of Theorem 1.3 of [24]. We first choose surjective κ -algebra homomorphisms $A \rightarrow \kappa[\Delta_{Q_n}]$ and $B \rightarrow R_n$ where $R_n = \kappa \otimes_{\mathcal{O}} R^{Q_n}$. Note that the kernel of $A \rightarrow \kappa[\Delta_{Q_n}]$ is contained in $\mathfrak{m}_A^{\ell^n} \subset \mathfrak{m}_A^n$. Define ψ_n as the composite $B \rightarrow R_n \rightarrow R$ and define $\phi_n : A \rightarrow B$ so the diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ \kappa[\Delta_{Q_n}] & \longrightarrow & R_n \end{array}$$

commutes. We consider $L_n = \kappa \otimes_{\mathcal{O}} L^{Q_n}$ as a B -module via $B \rightarrow R_n$, and define H_n as $L_n/\mathfrak{m}_A^n L_n$ and π_n as the map induced by $L^{Q_n} \rightarrow L^\theta$. Then H_n is free over A/\mathfrak{m}_A^n , and π_n induces $H_n/\mathfrak{m}_A H_n \xrightarrow{\sim} H$. We can therefore apply Theorem 1.3 of [24] to conclude that R is a complete intersection over which H is a free module. Since T^θ is finite and flat over \mathcal{O}_λ , it follows that $R^\theta \rightarrow T^\theta$ is an isomorphism since it is so after tensoring with κ . Moreover these rings are complete intersections over which L^θ is a free module of rank 2.

We now apply the implication (c) \Rightarrow (b) of Theorem 2.4 of [24] to the R^θ -module L^θ and prime ideal $\mathfrak{p}_{\min}^\theta$. We thus obtain the formula

$$2 \cdot \text{length}_{\mathcal{O}_\lambda} H_\theta^1(\mathbb{Q}, A_{\min}) = 2 \cdot \text{length}_{\mathcal{O}_\lambda} \mathfrak{p}_{\min}^\theta / (\mathfrak{p}_{\min}^\theta)^2 = \text{length}_{\mathcal{O}_\lambda} \Omega_{\min}^\theta = 2 \cdot v_\lambda(\eta_{\min}^\theta)$$

where the first equality follows from (70) and the last from Lemma 3.2.

Suppose now that Σ is a finite subset of Σ_0 . Applying (70) and Lemma 3.2 again, together with the inequality

$$\text{length}_{\mathcal{O}_\lambda} H_\Sigma^1(\mathbb{Q}, A_{\min}) \leq \text{length}_{\mathcal{O}_\lambda} H_\theta^1(\mathbb{Q}, A_{\min}) - \sum_{p \in \Sigma} v_\lambda(L_p(\text{ad}_K^0 V_{\min}, 1))$$

obtained from the exact sequence of Lemma 2.1, we find that

$$2 \cdot \text{length}_{\mathcal{O}_\lambda} \mathfrak{p}_{\min}^\Sigma / (\mathfrak{p}_{\min}^\Sigma)^2 = 2 \cdot \text{length}_{\mathcal{O}_\lambda} H_\Sigma^1(\mathbb{Q}, A_{\min}) \leq 2 \cdot v(\eta_{\min}^\Sigma) = \text{length}_{\mathcal{O}_\lambda} \Omega_{\min}^\Sigma.$$

We can then apply the implication (a) \Rightarrow (c) of Theorem 2.4 of [24] to conclude R^Σ is a complete intersection over which L^Σ is a free module of rank 2.

The second assertion of Theorem 3.1 follows from another application of Theorem 2.4 of [24], (70) and Lemma 3.2. To deduce the first assertion of Theorem 3.1, note that the map

$$\bar{K}_\lambda \otimes_{\mathcal{O}_\lambda} R^\Sigma \rightarrow \prod_{\rho \in \mathcal{N}^\Sigma} \bar{K}_\lambda$$

induced by (71) is an isomorphism. Every allowable lift of ρ_0 arises, up to \bar{K}_λ -isomorphism, from a \bar{K}_λ -linear map $\bar{K}_\lambda \otimes_{\mathcal{O}_\lambda} R^\Sigma$ for some Σ . It therefore arises from θ_ρ^Σ for some ρ in \mathcal{N} . \square

3.2. Consequences

Let us now return to the setting of Theorem 2.15, namely that f is a newform of weight $k \geq 2$, character ψ and conductor N with coefficients in a number field K , and λ is a prime of K not in the set S_f defined in (31). We let ℓ denote the rational prime in λ ; let $\kappa = \mathcal{O}_K/\lambda$ and $\bar{\mathcal{M}}_{f,\lambda} = \kappa \otimes_{\mathcal{O}_{K,\lambda}} \mathcal{M}_{f,\lambda}$ where $\mathcal{M}_{f,\lambda}$ is defined in Section 1.6.2. We then consider the representation

$$\rho_0 : G_{\mathbb{Q}} \rightarrow \text{Aut}_{\kappa} \bar{\mathcal{M}}_{f,\lambda}.$$

Enlarging K and replacing f by a twist if necessary, we can assume that κ contains the eigenvalues of all elements of $\rho_0(G_{\mathbb{Q}})$ and ρ_0 has minimal conductor among its twists. We shall now construct a set of lifts of ρ_0 from modular forms satisfying the hypotheses of Theorem 3.1.

Suppose that g is a newform of the same weight k and character ψ , but any conductor N_g not divisible by ℓ . We suppose that g has coefficients in a subfield K_g of \bar{K}_λ generated over K by the coefficients of g . The inclusion of K_g in \bar{K}_λ determines a prime λ_g of K_g over ℓ and identifies K_{g,λ_g} with a finite extension of K_λ in \bar{K}_λ . The representation

$$\rho_g : G_{\mathbb{Q}} \rightarrow \text{Aut}_{K_{g,\lambda_g}} M_{g,\lambda_g}$$

is an allowable lift of ρ_0 if and only if

$$(73) \quad a_p(g) \equiv a_p(f) \pmod{\lambda_g}$$

for all but finitely many p . We let \mathcal{N} denote the set of ρ_g such that (73) holds. Note that the $\rho_g \in \mathcal{N}$ are inequivalent for distinct g .

From the work of Ribet and others, one knows that \mathcal{N}_\emptyset is non-empty. (See the discussions following Theorem 1 of [27] and Corollary 1.2 of [21].) Choose $f_{\min} \in \mathcal{N}^\emptyset$ and enlarge K if necessary so that Lemma 1.5 holds for $f = f_{\min}$ and all primes $p \in \Sigma_1$. For each finite subset Σ of $\Sigma_0 = S_f(\mathbb{Q}) \setminus \{\ell\}$, we consider the $\mathcal{O}_\lambda[G_{\mathbb{Q}}]$ -module $\mathcal{M}(\sigma^\Sigma)_{!,\lambda}$ defined in Section 1.8.1, and endowed with an action of $\tilde{\mathbb{T}}^{\Psi_\Sigma}$ in Section 1.8.2. Let \mathfrak{m}^Σ denote the maximal ideal of $\tilde{\mathbb{T}}^{\Psi_\Sigma}$ defined there, i.e., the kernel of the map $\tilde{\mathbb{T}}^{\Psi_\Sigma} \rightarrow \kappa$ defined by $t_p \mapsto a_p(f_{\min}^{\Sigma \cup \{r\}}) \pmod{\lambda}$ for $p \notin \Psi_\Sigma$. We let $L^\Sigma = \mathcal{M}(\sigma^\Sigma)_{!,\lambda,\mathfrak{m}^\Sigma}$.

If Q is a finite set of horizontal primes for ρ_0 , then we let $D_Q = \prod_{q \in Q} q$ and define U_1^Q as the kernel of the homomorphism

$$U_0(N_1^\emptyset D_Q) \rightarrow \prod_{q \in Q} (\mathbb{Z}/q\mathbb{Z})^\times \rightarrow \Delta_Q$$

where the first map sends $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to $(a_q)_q$ and the second is the natural projection. We let σ_1^Q denote the restriction of σ^θ to U_1^Q . Then the $\mathcal{O}_\lambda[G_\mathbb{Q}]$ -module $\mathcal{M}(\sigma_1^Q)_{!,\lambda}$ is endowed with an action of $\tilde{\mathbb{T}}^{\Psi_\theta}$ and we let \mathfrak{m}_1^Q denote the kernel of the map $\tilde{\mathbb{T}}^{\Psi_\theta} \rightarrow \kappa$ defined by $t_p \mapsto a_p(f_{\min}^{\{r\}}) \bmod \lambda$ for $p \notin \Psi_\theta \cup Q$ and $t_q \mapsto \alpha_q$ for $q \in Q$, where α_q is a chosen eigenvalue of $\rho_0(\text{Frob}_q)$. We let $L_1^Q = \mathcal{M}(\sigma_1^Q)_{!,\lambda,\mathfrak{m}_1^Q}$. Recall that for each $\rho \in \mathcal{N}^Q$, there is a unique character $\xi_\rho : \Delta_Q \rightarrow \bar{K}_\lambda^\times$ such that ρ is a ξ_ρ -lift of ρ_0 . We also use ξ_ρ to denote the corresponding character of $G_\mathbb{Q}$ factoring through $\text{Gal}(\mathbb{Q}(\mu_{D_Q})/\mathbb{Q}) \cong (\mathbb{Z}/D_Q\mathbb{Z})^\times \rightarrow \Delta_Q$.

LEMMA 3.5. – *There is a $\bar{K}_\lambda[G_\mathbb{Q}]$ -linear isomorphism*

$$\iota^\Sigma : \bar{K}_\lambda \otimes_{\mathcal{O}_\lambda} L^\Sigma \xrightarrow{\sim} V^\Sigma$$

for each finite subset Σ of Σ_0 , and

$$\iota_1^Q : \bar{K}_\lambda \otimes_{\mathcal{O}_\lambda} L_1^Q \xrightarrow{\sim} \bigoplus_{\rho \in \mathcal{N}_Q} \bar{V}_\rho(\xi_\rho^{-1})$$

for each finite set Q of horizontal primes for ρ_0 .

Proof. – Let \mathbb{T}^Σ denote the image of $\mathcal{O}_\lambda \otimes_{\mathcal{O}_K} \tilde{\mathbb{T}}^{\Psi_\Sigma}$ in $\text{End}_{\mathcal{O}_\lambda} \mathcal{M}(\sigma^\Sigma)_{!,\lambda}$. Identifying $K_\lambda \otimes_{\mathcal{O}_\lambda} \mathbb{T}_{\mathfrak{m}^\Sigma}^\Sigma$ with the product of $\mathbb{T}_{\mathfrak{p}}^\Sigma$ over minimal primes \mathfrak{p} contained in $\mathfrak{m}^\Sigma \mathbb{T}^\Sigma$, we see that $K_\lambda \otimes_{\mathcal{O}_\lambda} L^\Sigma$ is isomorphic to the direct sum of $\mathcal{M}(\sigma^\Sigma)_{!,\lambda,\mathfrak{p}}$ for such \mathfrak{p} .

Suppose that g is a newform with coefficients in K_g and $\rho_g \in \mathcal{N}^\Sigma$ (where $K \subset K_g \subset \bar{K}_\lambda$ as in the definition of \mathcal{N}). The Σ -level structures σ^Σ defined in Section 1.8.1 are then the same for f^{\min} and g (in fact, if $p \notin \Sigma$ then $c_p(f_{\min}) = c_p(g)$ and $\delta_p(f_{\min}) = \delta_p(g)$, if $p \in \Sigma$ then $c_p(f_{\min}) + \delta_p(f_{\min}) = c_p(g) + \delta_p(g)$, and if $p \in \Sigma_1 \setminus \Sigma$ then the representation \mathcal{V}'_p defined in Lemma 1.5 for f^{\min} works for g as well). We thus have

$$M_{g,1,\lambda_g}^\Sigma \subset K_{g,\lambda_g} \otimes_{K_\lambda} M(\sigma^\Sigma)_{!,\lambda}$$

giving rise a homomorphism $\mathbb{T}^\Sigma \rightarrow K_{g,\lambda_g}$ defined by $T_p \mapsto a_p(g^{\Sigma \cup \{r\}})$ for $p \notin \Psi_\Sigma$. Letting \mathfrak{p}_g^Σ denote its kernel, we have $\mathfrak{p}_g^\Sigma \subset \mathfrak{m}^\Sigma \mathbb{T}^\Sigma$. Moreover $\mathfrak{p}_g^\Sigma = \mathfrak{p}_{g'}^\Sigma$ if and only if g and g' are conjugate under G_{K_λ} .

Next we check that every such minimal $\mathfrak{p} \subset \mathfrak{m}^\Sigma \mathbb{T}^\Sigma$ arises this way. Indeed every minimal prime \mathfrak{p} of \mathbb{T}^Σ is the kernel of a homomorphism $\tilde{\mathbb{T}}^{\Psi_\Sigma} \rightarrow K_h$ arising from an eigenform h of level $N^{\Sigma \cup \{r\}}$. Moreover the newform g associated to h satisfies $(\mathcal{V}'_p \otimes_{\mathcal{O}_{K,\tau}} \pi_p(g))^{GL_2(\mathbb{Z}_p)} \neq 0$ for all $p \in \Sigma_1 \setminus \Sigma$, $\tau : K \rightarrow \mathbb{C}$ (where $N^{\Sigma \cup \{r\}}$ and \mathcal{V}'_p are defined using f_{\min}). If $\mathfrak{p} \subset \mathfrak{m}^\Sigma \mathbb{T}^\Sigma$, then (73) holds, so ρ_g is an allowable lift of ρ_0 . In particular, we have that ρ_g is unramified at r , so $c_r(g) = 0$. Combining the inequality $c_p(g) \leq c_p(f_{\min})$ for $p \notin \Sigma$ with the condition on $\pi_p(g)$ for $p \in \Sigma_1 \setminus \Sigma$, we conclude that ρ_g is minimally ramified outside Σ . Finally, the condition that $T_p \in \mathfrak{m}^\Sigma$ for $p \in \Sigma \cup \{r\}$ implies that $a_p(h) \in \lambda_h$ for such p , from which we deduce that $a_p(h) = 0$ and therefore that $h = g^{\Sigma \cup \{r\}}$.

We have now shown that the set of minimal $\mathfrak{p} \subset \mathfrak{m}^\Sigma \mathbb{T}^\Sigma$ is precisely the set of \mathfrak{p}_g^Σ where g runs over G_{K_λ} -orbits of newforms g such that $\rho_g \in \mathcal{N}^\Sigma$. For such \mathfrak{p} , we have that $\mathbb{T}_{\mathfrak{p}}^\Sigma$ is reduced, so that $K_\lambda \otimes_{\mathcal{O}_\lambda} \mathcal{M}(\sigma^\Sigma)_{!,\lambda,\mathfrak{p}} \cong \mathcal{M}(\sigma^\Sigma)_{!,\lambda,\mathfrak{p}}$. Extending scalars to \bar{K}_λ gives

$$\bar{K}_\lambda \otimes_{K_\lambda} \mathcal{M}(\sigma^\Sigma)_{!,\lambda,\mathfrak{p}} \cong \bigoplus_{\{g | \mathfrak{p}_g^\Sigma = \mathfrak{p}\}} \bar{K}_\lambda \otimes_{K_{g,\lambda_g}} M_{g,1,\lambda_g}^\Sigma,$$

and summing over \mathfrak{p} gives the desired isomorphism.

The construction of ι_1^Q is similar, so we omit the details and note the only significant difference in the proof. Starting with a newform g such that ρ_g is a ξ -lift of ρ_0 in \mathcal{N}^Q , we consider the newform associated to $g \otimes \xi$. This in turn gives an eigenform g_1^Q of level $N^\Sigma D_Q r^2$ with $a_r(g_1^Q) = 0$ and $a_q(g_1^Q)$ reducing to α_q for all $q \in Q$. Working with a Hecke algebra \mathbb{T}_1^Q defined analogously to \mathbb{T}^Σ above, one checks that the minimal primes of \mathbb{T}_1^Q contained in $\mathfrak{m}_1^Q \mathbb{T}_1^Q$ correspond to the G_{K_λ} -orbits of the eigenforms g_1^Q . \square

Recall that in Section 1.8.3 we defined an $\mathcal{O}_\lambda[G_\mathbb{Q}]$ -linear homomorphism $\hat{\gamma}_{\Sigma'}^\Sigma : L^\Sigma \rightarrow L^{\Sigma'}$ for $\Sigma \subset \Sigma'$ and proved in Lemma 1.11 that it is injective with torsion-free cokernel as an \mathcal{O}_λ -module. Recall also that $\hat{\gamma}_{\Sigma''}^\Sigma = \hat{\gamma}_{\Sigma''}^{\Sigma'} \circ \hat{\gamma}_{\Sigma'}^\Sigma$ if $\Sigma \subset \Sigma' \subset \Sigma''$, so we can consider $L := \varinjlim L^\Sigma$ over all finite $\Sigma \subset \Sigma_0$ with respect to the inclusions $\hat{\gamma}_{\Sigma'}^\Sigma$ for $\Sigma \subset \Sigma'$. Note that the isomorphisms ι^Σ in Lemma 3.5 can be chosen so that the $\hat{\gamma}_{\Sigma'}^\Sigma$ are compatible with the inclusions $V^\Sigma \subset V^{\Sigma'}$. Taking their direct limit, we get an isomorphism

$$\iota : \bar{K}_\lambda \otimes_{\mathcal{O}_\lambda} L_0 \cong \bigoplus_{\rho \in \mathcal{N}} V_\rho$$

so that $\iota(L)$ is a trellis with $\iota(L)^\Sigma = \iota^\Sigma(L^\Sigma)$ and with a system of perfect pairings provided by Corollary 1.6.

We now verify the remaining hypotheses of Theorem 3.1. The second bullet and part (a) of the third follow from Proposition 1.12(b). To establish parts (b) and (c), we appeal to Lemma 1.10 with $\Psi = \Sigma_1 \cup Q$. Combined with Lemma 3.5, this implies that $\bigoplus_{\rho \in \mathcal{N}^Q} V_\rho(\xi_\rho^{-1})^-$ is a free $\bar{K}_\lambda[\Delta_Q]$ -module with Δ_Q acting on each $V_\rho(\xi_\rho^{-1})$ via ξ_ρ^2 . We conclude that the number of ξ -lifts of ρ_0 in \mathcal{N}^Q is independent of ξ , giving (b) and (c).

THEOREM 3.6. – *Suppose $\rho : G_\mathbb{Q} \rightarrow \text{Aut}_{K_\lambda} V$ is a continuous geometric representation whose restriction to G_ℓ is ramified, crystalline and short. If ρ_0 is modular and its restriction to G_F is absolutely irreducible, where F is the quadratic subfield of $\mathbb{Q}(\mu_\ell)$, then ρ is modular.*

Proof. – Note that we may enlarge K_λ in order to prove the theorem. We can then apply Theorem 3.1 to the set \mathcal{N} just constructed for the twist $\rho_0 \otimes_\kappa \psi'$ of minimal conductor, where ψ' is unramified at ℓ . Writing $\tilde{\cdot}$ for Teichmüller liftings, we conclude that $\rho \otimes_{K_\lambda} \tilde{\psi}' \psi$ is modular, where ψ is a character of ℓ -power order such that $\chi_\ell^{1-k} \psi^2 \det \rho$ has order not divisible by ℓ . \square

THEOREM 3.7. – *Let f be a newform of weight $k \geq 2$ and level N with coefficients in the number field K . Suppose that λ is a prime of K not in the set S_f defined in (31), and let \mathcal{O}_λ be the ring of integers in K_λ . Suppose that Σ is a finite set of primes not containing ℓ such that $M_{f,\lambda}$ is minimally ramified outside Σ . Then the \mathcal{O}_λ -module*

$$H_\Sigma^1(\mathbb{Q}, A_{f,\lambda}/\mathcal{A}_{f,\lambda})$$

has length $v_\lambda(\eta_f^\Sigma)$ where η_f^Σ was defined before Proposition 1.4.

Proof. – Enlarging K and applying Theorem 3.1 to the set \mathcal{N} just constructed for the twist $\rho_0 \otimes_\kappa \psi'$ of minimal conductor, we conclude that the theorem holds for a twist of f , hence for f itself. \square

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