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HECKE CURVES AND HITCHIN DISCRIMINANT

BY JUN-MUK HWANG [1] AND S. RAMANAN

ABSTRACT. – Let C be a smooth projective curve of genus $g \geqslant 4$ over the complex numbers and $\mathcal{SU}_C^s(r,d)$ be the moduli space of stable vector bundles of rank r with a fixed determinant of degree d. In the projectivized cotangent space at a general point E of $\mathcal{SU}_C^s(r,d)$, there exists a distinguished hypersurface $\mathcal{S}_E$ consisting of cotangent vectors with singular spectral curves. In the projectivized tangent space at E, there exists a distinguished subvariety $\mathcal{C}_E$ consisting of vectors tangent to Hecke curves in $\mathcal{SU}_C^s(r,d)$ through E. Our main result establishes that the hypersurface $\mathcal{S}_E$ and the variety $\mathcal{C}_E$ are dual to each other. As an application of this duality relation, we prove that any surjective morphism $\mathcal{SU}_C^s(r,d) \to \mathcal{SU}_{C'}^s(r,d)$, where C' is another curve of genus g, is biregular. This confirms, for $\mathcal{SU}_C^s(r,d)$, the general expectation that a Fano variety of Picard number 1, excepting the projective space, has no non-trivial self-morphism and that morphisms between Fano varieties of Picard number 1 are rare. The duality relation also gives simple proofs of the non-abelian Torelli theorem and the result of Kouvidakis–Pantev on the automorphisms of $\mathcal{SU}_C^s(r,d)$.

RÉSUMÉ. – Soient C une courbe projective lisse de genre $g \geqslant 4$ sur les nombres complexes, $\mathcal{SU}_C^s(r,d)$ la variété de modules des fibrés stables de rang r de déterminant fixé de degré d. Dans l'espace cotangent projectivisé en un point général E de $\mathcal{SU}_C^s(r,d)$, il existe une hypersurface distinguée $\mathcal{S}_E$ qui correspond aux vecteurs cotangents avec courbes spectrales singulières. Dans l'espace tangent projectivisé en E, il existe une sous-variété distinguée $\mathcal{C}_E$ comprenant des vecteurs tangents aux courbes de Hecke dans $\mathcal{SU}_C^s(r,d)$ passant par E. Notre résultat principal établit que l'hypersurface $\mathcal{S}_E$ et la variété $\mathcal{C}_E$ sont duales l'une à l'autre. Comme application de cette relation de dualité, nous démontrons qu'un morphisme surjectif $\mathcal{SU}_C^s(r,d) \to \mathcal{SU}_{C'}^s(r,d)$ est birégulière, où C' est une courbe quelconque de genre g. Ceci confirme, pour $\mathcal{SU}_C^s(r,d)$, l'espérance générale qu'une variété de Fano de nombre de Picard égal à 1, à l'exception des espaces projectifs, n'a pas de morphisme non trivial sur elle-même et que les morphismes entre les variétés de Fano de nombre de Picard égal à 1 sont rares. La relation de dualité fournit en même temps des preuves simples du théorème de Torelli non abélien et du résultat de Kouvidakis–Pantev concernant les automorphismes de $\mathcal{SU}_C^s(r,d)$.

1. Introduction

Any smooth projective variety with Picard group isomorphic to $\mathbf{Z}$ is usually classified into one of three classes, namely *general type, Calabi–Yau or Fano* according as the canonical line bundle is positive, trivial or negative. Fano varieties are somewhat special among varieties, and algebraic homogeneous spaces fall in that class. If we leave out projective spaces, morphisms between two such varieties of the same dimension seem to be rare [5]. In particular, there is a

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conjecture, originating from a related question of Lazarsfeld, that there are no nonconstant self maps of these varieties except automorphisms.

Let C be a smooth projective curve of genus g over the complex numbers and $\mathcal{SU}_C^s(r,d)$ be the moduli space of stable vector bundles of rank r with a fixed determinant of degree d. When r and d are coprime, these are smooth Fano varieties with Picard group $\mathbf{Z}$. Thus these provide examples against which the above kind of conjectures can be tested.

Our main aim in this paper is to prove the following theorem.

THEOREM 5.6. – *Let C and C' be two smooth projective curves of genus $g \geqslant 4$. Let $f : \mathcal{SU}_C^s(r,d) \to \mathcal{SU}_{C'}^s(r,d)$ be a surjective morphism. Then f is biregular.*

Note that we do not assume in this theorem that r and d are coprime, but take only the smooth locus of the varieties in question. The theorem is perhaps also valid for $g = 3$ but our method does not cover that case. The method synthesizes three different strands.

Firstly, the moduli spaces of vector bundles have been studied by Hitchin [3] from the viewpoint of symplectic geometry of its cotangent bundle. On the other hand this study has been used as a tool to derive results on the moduli spaces themselves in [2]. These ideas can be codified in the terms ‘spectral curves’, ‘Higgs moduli’, ‘nonabelian theta functions’, etc.

Secondly a certain amount of rigidity in the moduli spaces were established by [12] and [13] by a study of the geometry of the moduli spaces. Here the main ingredient is the notion of ‘Hecke cycles’. For our purposes it is more fruitful to consider what we call ‘Hecke curves’ [9].

Finally, the moduli space may be investigated by tools commonly used in the study of higher-dimensional Fano varieties. This leads to the study of rational curves on it [7,9] and the Hecke curves provide the means for doing it. The result quoted above is obtained by studying interesting relationship between these aspects.

Let us now briefly describe our approach.

Associated to the Hitchin map on the cotangent bundle of $\mathcal{SU}_C^s(r,d)$, there exists a canonically defined hypersurface $\mathcal{S} \subset \mathbf{P}T^*(\mathcal{SU}_C^s(r,d))$ corresponding to twisted endomorphisms of stable vector bundles whose spectral curves are singular. For a general point $E \in \mathcal{SU}_C^s(r,d)$, the corresponding hypersurface $\mathcal{S}_E$ in the projectivized cotangent space $\mathbf{P}T_E^*(\mathcal{SU}_C^s(r,d))$ will be called *the Hitchin discriminant* at E.

On the other hand, there are naturally defined rational curves on $\mathcal{SU}_C^s(r,d)$, which (as referred to above) we call Hecke curves. For a general $E \in \mathcal{SU}_C^s(r,d)$, let $\mathcal{C}_E$ be the subvariety of $\mathbf{P}T_E(\mathcal{SU}_C^s(r,d))$ consisting of tangent vectors to Hecke curves through E. This subvariety $\mathcal{C}_E$ will be called the *variety of Hecke tangents* at E.

The key point in our proof is the following result which we hope is sufficiently interesting in itself.

THEOREM 4.4. – *Let $g \geqslant 4$ and let E be a general point of $\mathcal{SU}_C^s(r,d)$. Then the Hitchin discriminant $\mathcal{S}_E$ is the dual variety of the variety of Hecke tangents $\mathcal{C}_E$.*

This has other interesting consequences. It gives simple proofs, for $g \geqslant 4$, of non-abelian Torelli theorem (Theorem 5.1) and the description due to Kouvidakis and Pantev, of the automorphisms of $\mathcal{SU}_C^s(r,d)$ (Theorem 5.4). Our proof of the non-abelian Torelli theorem is reminiscent of Andreotti’s proof of the abelian Torelli theorem [1]. Recall that in Andreotti’s proof the curve is recovered as the dual variety of a certain discriminantal hypersurface associated to the Gauss map of the Riemann theta divisor. In our proof of non-abelian Torelli theorem, the curve is recovered from the dual variety of a certain discriminantal hypersurface associated to the Hitchin map.

2. Variety of minimal rational tangents

In this preliminary section, we recall some results concerning minimal rational curves (cf. [8]). Let M be a smooth quasi-projective variety of dimension n. We will assume that there exists a component $\mathcal{K}$ of the Hilbert scheme of complete curves on M such that

(†) the subscheme $\mathcal{K}_y \subset \mathcal{K}$ consisting of members of $\mathcal{K}$ passing through a general point $y \in M$ is a non-empty irreducible smooth projective variety of which every member is an irreducible smooth rational curve lying in M.

A member of $\mathcal{K}$ is called a *minimal rational curve* on M. For a point $y \in M$, let $T_y(M)$ be the tangent space to M at y. Define the *tangent morphism*

$$\tau_y : \mathcal{K}_y \to \mathbf{P}T_y(M)$$

by sending $\ell \in \mathcal{K}_y$, a smooth rational curve $\ell \subset M$, to

$$\tau_y(\ell) := \mathbf{P}T_y(\ell).$$

For a general member ℓ of $\mathcal{K}_y$,

$$T(M)|_\ell \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^{n-1-p}$$

where p is the dimension of $\mathcal{K}_y$ and $\mathcal{O}(2)$ corresponds to $T(\ell)$ [8, Theorem 1.2]. This implies that τ_y is generically finite over its image. The image of τ_y is denoted by $\mathcal{C}_y$ and called the *variety of minimal rational tangents* at the general point y associated to the family $\mathcal{K}$. The following proposition is a consequence of basic deformation theory.

PROPOSITION 2.1 [8, Theorem 1.4]. – *Let ℓ be a general member of $\mathcal{K}_y$ with*

$$T(M)|_\ell \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^{n-1-p}.$$

Then τ_y is an immersion at $\ell \in \mathcal{K}_y$ and the tangent space to $\mathcal{C}_y$ at $\tau_y(\ell)$ corresponds to the subspace of $T_y(M)$ defined by the $\mathcal{O}(2) \oplus \mathcal{O}(1)^p$-part of $T(M)|_\ell$.

Recall that when X is an irreducible subvariety of a projective space $\mathbf{P}_N$, its dual variety X^* is the irreducible subvariety of the dual projective space $\mathbf{P}_N^*$ which is the closure of the set of hyperplanes containing the projective tangent space of a smooth point of X. Note that for ℓ as above,

$$H^0\big(\ell, T^*(M)|_\ell\big) \cong H^0\big(\ell, \mathcal{O}(-2) \oplus \mathcal{O}(-1)^p \oplus \mathcal{O}^{n-1-p}\big) = H^0(\ell, \mathcal{O}^{n-1-p})$$

are exactly cotangent vectors annihilating $\mathcal{O}(2) \oplus \mathcal{O}(1)^p$-part of $T(M)|_\ell$. Also note that sections of $T^*(M)$ over ℓ give smooth rational curves in $T^*(M)$. As a consequence, we get the following.

COROLLARY 2.2. – *Let $\hat{\mathcal{S}} \subset T^*(M)$ be the closure of the union of the smooth rational curves in $T^*(M)$ given by $H^0(\ell, T^*(M))$ as ℓ varies over $\mathcal{K}$. Let $\mathcal{S} \subset \mathbf{P}T^*(M)$ be the corresponding projective subvariety. For a point $y \in M$ let $\mathcal{S}_y$ be the intersection $\mathcal{S} \cap \mathbf{P}T^*_y(M)$. Then for general y, $\mathcal{S}_y$ is the dual variety of $\mathcal{C}_y$.*

We recall the following result from [6].

THEOREM 2.3 [6, Theorem 1]. – *In the situation above, the tangent morphism $\tau_y : \mathcal{K}_y \to \mathcal{C}_y$ is birational for a general point $y \in M$.*

This was proved in [6] when M is a projective variety, but the proof there works even when M is quasi-projective, as long as the assumption (†) holds.

We will also need the following which is essentially [4, Proposition 2].

PROPOSITION 2.4. – *Let M and $\mathcal{K}$ be as above. Suppose there exists an open subset A' of an abelian variety A and a generically finite morphism $f: A' \to M$. Let $y \in M$ be a general point and $\ell \subset M$ be a general member of $\mathcal{K}_y$. Assume that there exists a complete curve $\ell' \subset A'$ such that $f(\ell') = \ell$. Then the variety of minimal rational tangents $\mathcal{C}_y$ is a linear subvariety in $\mathbf{P}T_y(M)$.*

The proof uses the following lemma about curves on abelian varieties, which is exactly [4, Lemma 3].

LEMMA 2.5. – *Let $C_t \subset A$ be a p-dimensional irreducible family of curves on an n-dimensional abelian variety A passing through a common point $a \in A$. If the constructible set in A consisting of the union of C_t's is of dimension $(p+1)$ and the subspace of $H^0(C_t, T^*(A))$ consisting of elements annihilating tangent vectors to C_t is of dimension $\geqslant n-1-p$ for a general member C_t, then the closure of the union of these curves is a translate of a $(p+1)$-dimensional abelian subvariety.*

Proof of Proposition 2.4. – Let $a \in \ell'$ be a point with $f(a) = y$. Note that elements of $H^0(\ell, T^*(M))$ annihilate the tangent vectors to ℓ and $h^0(\ell, T^*(M)) = n - 1 - p$ where p is the number of $\mathcal{O}(1)$-factors in $T(M)|_\ell$, or equivalently, the dimension of $\mathcal{K}_y$. The pull-back of elements of $H^0(\ell, T^*(M))$ to $H^0(\ell', T^*(A))$ gives a subspace of dimension $\geqslant n - 1 - p$, annihilating tangent vectors to ℓ', because ℓ passes through the general point $y \in M$. By Lemma 2.5, the closure of the union of all such choices of ℓ' is a translate of a $(p+1)$-dimensional abelian subvariety. In particular, the closure of their tangent vectors at a must be a linear subvariety of $\mathbf{P}T_a(A)$. This implies that $\mathcal{C}_y$ is a linear subvariety of $\mathbf{P}T_y(M)$. □

Remark 2.6. – Since some of our applications, namely, Theorems 5.1 and 5.3 below, will be simpler proofs of some results which have been proved by other means, it is worth pointing out that the preliminary results reviewed in this section are not so difficult to prove. The proofs of Proposition 2.1 and Corollary 2.2 are quite straightforward and use only basic deformation theory due to Kodaira. Proposition 2.4, whose proof is also easy, will not be needed for Theorems 5.1 and 5.3. The proof of Theorem 2.3 is more involved, but Theorem 2.3 will be needed in this paper only when the genus of C is 4.

3. Variety of Hecke tangents

Let C be a smooth projective curve of genus $g \geqslant 4$. Let $\mathcal{SU}^s_C(r,d)$ be the moduli space of stable bundles of rank r with a fixed determinant of degree d over C. For $M = \mathcal{SU}^s_C(r,d)$, there exists a family of rational curves satisfying the condition (†), called Hecke curves. Let us briefly recall the definition (see [13] and [9] for details).

Let $E \in \mathcal{SU}^s_C(r,d)$ be a stable bundle over C. Denote by E^* the dual bundle and $\mathbf{P}E$ the projectivization consisting of lines through the origin on each fiber. For $x \in C$ and $\zeta \in \mathbf{P}E^*_x$, consider a new vector bundle E^ζ defined by

$$0 \longrightarrow E^\zeta \longrightarrow E \longrightarrow (E_x/\zeta^\perp) \otimes \mathcal{O}_x \longrightarrow 0$$

where $\zeta^\perp$ denotes the hyperplane in E_x annihilated by ζ. Let $\iota: E^\zeta_x \to E_x$ be the homomorphism between the fibers at x induced by the sheaf map $E^\zeta \to E$. The kernel of ι, $\mathrm{Ker}(\iota)$, is a

1-dimensional subspace of the fiber E^ζ_x and its annihilator $(\mathrm{Ker}(\iota))^\perp$ is a hyperplane in $(E^\zeta)^*_x$. Let $\mathbf{l}$ be a line in $\mathbf{P}E^\zeta_x$ containing the point $[\mathrm{Ker}(\iota)]$. For each point $l \in \mathbf{l}$ corresponding to a 1-dimensional subspace $l \subset E^\zeta_x$, consider the vector bundle $\widetilde{E}^l$ defined by

$$0 \longrightarrow \widetilde{E}^l \longrightarrow (E^\zeta)^* \longrightarrow \left[(E^\zeta)^*_x/l^\perp\right] \otimes \mathcal{O}_x \longrightarrow 0$$

where $l^\perp \subset (E^\zeta)^*_x$ is the hyperplane annihilating l. This vector bundle $\widetilde{E}^l$ is stable for each $[l] \in \mathbf{l}$ if E is a general point of $\mathcal{SU}^s_C(r,d)$ and $g \geqslant 4$ [9, Proposition 2]. It is easy to check that for $l = \mathrm{Ker}(\iota)$,

$$\widetilde{E}^{\mathrm{Ker}(\iota)} \cong E^*.$$

It follows that $\{(\widetilde{E}^l)^*; l \in \mathbf{l}\}$ defines a rational curve passing through E in $\mathcal{SU}^s_C(r,d)$. A rational curve on $\mathcal{SU}^s_C(r,d)$ constructed this way is called a *Hecke curve*. Using [13, 5.9], one can show that a Hecke curve is smooth. In view of [13, 5.16], it is easy to check that a Hecke curve has degree $2r$ with respect to $K^{-1}_{\mathcal{SU}^s_C(r,d)}$.

On $\mathbf{P}E^*$, consider the relative cotangent bundle Ω_E of the fibration $\varpi : \mathbf{P}E^* \to C$. The projective bundle $\mathbf{P}\Omega_E$ over $\mathbf{P}E^*$ is a smooth projective variety of dimension $2r-2$. The set of all lines in $\mathbf{P}E^\zeta_x$ containing the point $[\mathrm{Ker}(\iota)]$ is naturally isomorphic to $\mathbf{P}(E^\zeta_x/\mathrm{Ker}(\iota)) \cong \mathbf{P}\Omega_{E,\zeta}$. In other words, each point of $\mathbf{P}\Omega_E$ defines a Hecke curve through E for a general point $E \in \mathcal{SU}^s_C(r,d)$. The argument of [13, 5.13] shows that Hecke curves associated to two distinct points of $\mathbf{P}\Omega_E$ are distinct rational curves on $\mathcal{SU}^s_C(r,d)$. Thus $\mathbf{P}\Omega_E$ is naturally isomorphic to the variety of all Hecke curves through E. A simple dimension-counting shows that Hecke curves are dense in an irreducible component of the Hilbert scheme of curves on $\mathcal{SU}^s_C(r,d)$ [9, Proposition 3]. It follows that the component $\mathcal{K}$ of the Hilbert scheme of $\mathcal{SU}^s_C(r,d)$ corresponding to Hecke curves satisfies the condition $(\dagger)$, i.e., Hecke curves are minimal rational curves of $\mathcal{SU}^s_C(r,d)$.

Let us describe the tangent morphism associated to Hecke curves through a general point $E \in \mathcal{SU}^s_C(r,d)$. Let $\varphi : \mathbf{P}\Omega_E \to \mathbf{P}E^*$ be the projectivization of Ω_E and ξ_E be the $\mathcal{O}(1)$-bundle of the projectivization so that $\varphi_*\xi_E = \Omega^*_E$ is the relative tangent bundle of ϖ. Recall that $\varpi_*\Omega^*_E$ is the bundle ad_E of traceless endomorphisms of E. Let $\pi : \mathbf{P}\Omega_E \to C$ be the composition $\pi = \varpi \circ \varphi$. Note that

$$\begin{aligned} H^0(\mathbf{P}\Omega_E, \xi_E \otimes \pi^*\omega_C) &= H^0(\mathbf{P}E^*, \varphi_*\xi_E \otimes \varpi^*\omega_C) \\ &= H^0(\mathbf{P}E^*, \Omega^*_E \otimes \varpi^*\omega_C) \\ &= H^0(C, \varpi_*\Omega^*_E \otimes \omega_C) \\ &= H^0(C, \mathrm{ad}_E \otimes \omega_C) \end{aligned}$$

is the dual of the tangent space of $\mathcal{SU}^s_C(r,d)$ at E. Thus the line bundle $\xi_E \otimes \pi^*\omega_C$ defines a rational map

$$\tau_E : \mathbf{P}\Omega_E \to \mathbf{P}T_E\big(\mathcal{SU}^s_C(r,d)\big).$$

For a general E, this rational map is exactly the tangent morphism assigning to each Hecke curve through E its tangent vector at E [9, Theorem 3]. We denote the image of τ_E by $\mathcal{C}_E$ and call it the *variety of Hecke tangents*.

THEOREM 3.1. – *Let $g \geqslant 5$. Then for a general stable bundle $E \in \mathcal{SU}^s_C(r,d)$, the line bundle $\xi_E \otimes \pi^*\omega_C$ is very ample, i.e., $\tau_E : \mathbf{P}\Omega_E \to \mathcal{C}_E$ is a biregular morphism.*

Proof. – Write L for $\xi_E \otimes \pi^*\omega_C$. For any $x \in C$, the line bundle L restricted to the fiber $\pi^{-1}(x)$ is very ample. Thus L is very ample on $\mathbf{P}\Omega_E$ if for any $x, y \in C$, the case $x = y$ included, the restriction map

$$H^0(\mathbf{P}\Omega_E, L) \longrightarrow H^0\big(\pi^{-1}(x+y), L|_{\pi^{-1}(x+y)}\big)$$

is surjective. From the exact sequence

$$0 \longrightarrow L \otimes \pi^*\mathcal{O}(-x-y) \longrightarrow L \longrightarrow L|_{\pi^{-1}(x+y)} \longrightarrow 0,$$

the surjectivity is guaranteed if

$$H^1\big(\mathbf{P}\Omega_E, L \otimes \pi^*\mathcal{O}(-x-y)\big) = H^1\big(C, \mathrm{ad}_E \otimes K_X(-x-y)\big)$$

or its dual $H^0(C, \mathrm{ad}_E \otimes \mathcal{O}(x+y))$ vanishes. Thus Theorem 3.1 follows from Proposition 3.2 below. □

PROPOSITION 3.2. – *Let ℓ be a positive integer satisfying $g \geqslant \frac{3}{2}\ell + 2$. Then for a general stable bundle F of arbitrary rank and degree $H^0(C, \mathrm{ad}_F(D)) = 0$ for any effective divisor D of degree ℓ.*

We need a few lemmas.

LEMMA 3.3. – *For a general stable bundle E on C of rank r and degree d, $H^0(C, E) = 0$ if $d \leqslant r(g-1)$.*

Proof. – Let us count the dimension of the space of stable bundles which have non-zero sections. If E has a non-zero section, there exists a line subbundle $L \subset E$ with $d' := \deg(L) \geqslant 0$. Thus E can be realized as an extension of the type

$$0 \longrightarrow L \longrightarrow E \longrightarrow G \longrightarrow 0$$

where L is a line bundle of degree $d' \geqslant 0$ with $H^0(C, L) \neq 0$ and G is a vector bundle of rank $r-1$ and degree $d'' = d - d'$. Since non-stable bundles can be deformed to stable bundles [12, Proposition 2.6], we may assume that G is stable in dimension-counting. Recall that the moduli space $\mathcal{U}_C(r, d)$ of semi-stable bundles of rank r and degree d on C has dimension $r^2(g-1)+1$. Thus the dimension of deformation of G is equal to

$$\dim \mathcal{U}_C(r-1, d'') = (r-1)^2(g-1) + 1.$$

The dimension of possible choices of the line bundle L is $\leqslant d'$. For a fixed G and a fixed L, the dimension of extensions of G by L is $h^1(C, G^* \otimes L)$. We claim that $H^0(C, G^* \otimes L) = 0$. In fact, assuming that $G = E/L$ for some stable bundle E, if there exists a homomorphism $\eta : G \to L$, the composition

$$E \longrightarrow E/L \xrightarrow{\eta} L \longrightarrow E$$

must be identically zero because any endomorphism of E must be a homothety. Hence $\eta \equiv 0$. It follows that

$$\begin{aligned} h^1(C, G^* \otimes L) &= -\chi(G^* \otimes L) \\ &= d'' - (r-1)d' + (r-1)(g-1) \\ &= d - rd' + (r-1)(g-1). \end{aligned}$$

Thus the space of stable bundles which have non-zero sections has dimension at most

$$d' + (r-1)^2(g-1) + 1 + h^1(C, G^* \otimes L) - 1 = (r^2 - r)(g-1) + d + (1-r)d'.$$

Since $\dim \mathcal{U}_C(r,d) = r^2(g-1) + 1$ and

$$\begin{aligned}\left[r^2(g-1)+1\right] - \left[(r^2-r)(g-1) + d + (1-r)d'\right] &= r(g-1) - d + (r-1)d' + 1 \\ &\geqslant r(g-1) - d + 1,\end{aligned}$$

a general stable bundle cannot have a non-zero section if $r(g-1) - d \geqslant 0$. □

LEMMA 3.4. – *Let E be a general stable bundle of rank $r-1$ and degree d. Assume ℓ is a positive integer satisfying $d > -(r-1)(g-1-\ell) + \ell$. Then there exists an element $\epsilon \in H^1(C, E^*)$ such that for any effective divisor D of degree ℓ on C, $\psi^D(\epsilon) \neq 0$ where $\psi^D : H^1(C, E^*) \to H^1(C, E^*(D))$ is the homomorphism arising from the short exact sequence*

$$0 \longrightarrow E^* \longrightarrow E^*(D) \longrightarrow E^*(D)|_D \longrightarrow 0.$$

Proof. – From the exact sequence

$$H^0\big(C, E^*(D)|_D\big) \longrightarrow H^1(C, E^*) \longrightarrow H^1\big(C, E^*(D)\big),$$

it suffices to show that

$$h^1(C, E^*) - h^0\big(C, E^*(D)|_D\big) - \ell > 0,$$

where ℓ is interpreted as the dimension of possible choices of D. Note $h^0(C, E^*) = 0$ by Lemma 3.3 because $\deg(E^*) = -d \leqslant (r-1)(g-1)$. Thus

$$h^1(C, E^*) = -\chi(E^*) = d + (r-1)(g-1).$$

Also $h^0(C, E^*(D)|_D) = \ell(r-1)$. So

$$h^1(C, E^*) - h^0\big(C, E^*(D)|_D\big) - \ell = d + (r-1)(g-1) - \ell(r-1) - \ell$$

which is positive from $d > -(r-1)(g-1-\ell) + \ell$. □

LEMMA 3.5. – *Let E be a general stable bundle on C of rank $r-1$ and degree d. Let ℓ be a positive integer satisfying $|d| \leqslant (r-1)(g-1-\ell)$. Assume that $H^0(C, \mathrm{ad}_E(Z)) = 0$ for any effective divisor Z of degree ℓ. Suppose there exists an extension F of E by $\mathcal{O}$,*

$$0 \longrightarrow \mathcal{O} \longrightarrow F \longrightarrow E \longrightarrow 0$$

such that $H^0(C, \mathrm{ad}_F(D)) \neq 0$ for some effective divisor D of degree ℓ. Then the extension class $[F] \in H^1(C, E^*)$ satisfies $\psi^{D'}([F]) = 0$ for some effective divisor D' of length ℓ.

Proof. – Let $\phi : F \to F(D)$ be a non-zero element of $H^0(C, \mathrm{ad}_F(D))$. The composition $\beta \circ \phi \circ \alpha$ in

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O} & \xrightarrow{\alpha} & F & \longrightarrow & E & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \phi} & & & & \\ 0 & \longrightarrow & \mathcal{O}(D) & \longrightarrow & F(D) & \xrightarrow{\beta} & E(D) & \longrightarrow & 0 \end{array}$$

defines a section of $E(D)$. Since $\deg(E(D)) = d + \ell(r-1) \leqslant (r-1)(g-1)$, $E(D)$ cannot have a non-zero section by Lemma 3.3 and consequently $\beta \circ \phi \circ \alpha = 0$. Thus there exists γ satisfying

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O} & \xrightarrow{\alpha} & F & \longrightarrow & E & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \phi} & & & & \\ 0 & \longrightarrow & \mathcal{O}(D) & \longrightarrow & F(D) & \xrightarrow{\beta} & E(D) & \longrightarrow & 0. \end{array}$$

Let $s \in H^0(C, \mathcal{O}(D))$ be the section defined by γ. Consider

$$\phi' := \phi - I_F \cdot s : F \to F(D)$$

where I_F denotes the identity map of F. Then ϕ' annihilates the subbundle α: $\mathcal{O} \subset F$, inducing a non-zero homomorphism $\zeta : E \to F(D)$ satisfying

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O} & \xrightarrow{\alpha} & F & \longrightarrow & E & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \beta\circ\zeta} & & \\ 0 & \longrightarrow & \mathcal{O}(D) & \xrightarrow{\alpha'} & F(D) & \xrightarrow{\beta} & E(D) & \longrightarrow & 0. \end{array}$$

If $\beta \circ \zeta = 0$, then there exists $\xi : E \to \mathcal{O}(D)$ such that $\alpha' \circ \xi = \zeta$. Then we get a non-zero element ξ^* in $H^0(C, E^*(D))$ which is not possible by Lemma 3.3 because

$$\deg\bigl(E^*(D)\bigr) = -d + \ell(r-1) \leqslant (r-1)(g-1).$$

Thus $\beta \circ \zeta \neq 0$. By the assumption $H^0(C, \mathrm{ad}_E(D)) = 0$, we conclude that

$$\beta \circ \zeta = I_E \cdot s'$$

for some non-zero $s' \in H^0(C, \mathcal{O}(D))$. Let D' be the effective divisor defined by s'. We claim that $\psi^{D'}([F]) = 0$ in $H^1(C, E^*(D'))$, which proves the lemma.

To prove the claim, let us recall the definition of the extension class $[F]$ and $\psi^{D'}([F])$. Let $\delta : H^0(C, E^* \otimes F) \to H^1(C, E^*)$ be the boundary map associated to the short exact sequence

$$0 \longrightarrow E^* \longrightarrow E^* \otimes F \longrightarrow E^* \otimes E \longrightarrow 0.$$

Then $[F] := \delta(I_E)$ for the identity map $I_E \in H^0(C, E^* \otimes E)$. The multiplications by s'

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E^* & \longrightarrow & E^* \otimes F & \longrightarrow & E^* \otimes E & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \cdot s'} & & \downarrow{\scriptstyle \cdot s'} & & \downarrow{\scriptstyle \cdot s'} & & \\ 0 & \longrightarrow & E^*(D') & \longrightarrow & E^* \otimes F(D') & \longrightarrow & E^* \otimes E(D') & \longrightarrow & 0 \end{array}$$

induce a commutative diagram

$$\begin{array}{ccccc} H^0(C, E^* \otimes F) & \longrightarrow & H^0(C, E^* \otimes E) & \xrightarrow{\delta} & H^1(C, E^*) \\ \downarrow{\scriptstyle \cdot s'} & & \downarrow{\scriptstyle \cdot s'} & & \downarrow{\scriptstyle \psi^{D'}} \\ H^0(C, E^* \otimes F(D')) & \xrightarrow{\tilde{\beta}} & H^0(C, E^* \otimes E(D')) & \xrightarrow{\delta^{s'}} & H^1(C, E^*(D')). \end{array}$$

It follows that

$$\psi^{D'}\big([F]\big) = \psi^{D'}\big(\delta(I_E)\big) = \delta^{s'}(I_E \cdot s').$$

But we know that

$$I_E \cdot s' = \beta \circ \zeta = \tilde{\beta}(\zeta)$$

for some $\zeta \in H^0(C, E^* \otimes F(D))$. Thus $\psi^{D'}([F]) = 0$. □

LEMMA 3.6. – *Let $r \geqslant 2$, $\ell \geqslant 1$ and d be integers satisfying*

$$-(r-1)(g-1-\ell) + \ell < d \leqslant (r-1)(g-1-\ell).$$

Suppose for a general stable bundle E of rank $r-1$ and degree d, $H^0(C, \mathrm{ad}_E(D)) = 0$ for any effective divisor D of degree ℓ. Then for a general stable bundle F of rank r and degree d, $H^0(C, \mathrm{ad}_F(D)) = 0$ for any effective divisor D of degree ℓ.

Proof. – By Lemma 3.4, we can choose $[F_0] \in H^1(C, E^*)$ such that $\psi^D([F_0]) \neq 0$ for any effective divisor D of degree ℓ. Then by Lemma 3.5, $H^0(C, \mathrm{ad}_{F_0}(D)) = 0$ for any effective divisor D of degree ℓ. By [12, Proposition 2.6], F_0 can be approximated by stable bundles, i.e., there exists a flat family of bundles $\{F_t, t \in T\}$ parametrized by an affine curve T with a base point $0 \in T$ such that F_t is stable for $t \neq 0$. On $C^{(\ell)} \times T$ where $C^{(\ell)}$ is the set of effective divisors of degree ℓ, consider the loci of points $(D, t) \in C^{(\ell)} \times T$ such that $H^0(C, \mathrm{ad}_{F_t}(D)) \neq 0$. This loci is a closed subvariety of $C^{(\ell)} \times T$ and is disjoint from $C^{(\ell)} \times \{0\}$ since $H^0(C, \mathrm{ad}_{F_0}(D)) = 0$ for all $D \in C^{(\ell)}$. It follows that there exists $t \neq 0$ such that $H^0(C, \mathrm{ad}_{F_t}(D)) = 0$ for all $D \in C^{(\ell)}$. In particular, for a general stable bundle F, $H^0(C, \mathrm{ad}_F(D)) = 0$ for all $D \in C^{(\ell)}$. □

Proof of Proposition 3.2. – The proof is by induction on the rank r of F. If $r = 1$ this is obvious. Assume that the result holds for a general stable bundle E of rank $r-1$ and degree $d = \deg(F)$. By Lemma 3.6, the result follows if

$$-(r-1)(g-1-\ell) + \ell < d \leqslant (r-1)(g-1-\ell).$$

Note that there are $2(r-1)(g-1-\ell) - \ell$ consecutive integers d satisfying this and

$$2(r-1)(g-1-\ell) - \ell \geqslant r \quad \text{for } r \geqslant 2 \text{ and } g \geqslant \frac{3}{2}\ell + 2.$$

If $H^0(C, \mathrm{ad}_F(D)) = 0$ for some vector bundle F then $H^0(C, \mathrm{ad}_{F'}(D)) = 0$ for any vector bundle F' of the form $F' = F \otimes L$ for a line bundle L. Thus we may assume that the degree d of F belongs to any set of r consecutive integers. This finishes the proof of Proposition 3.2. □

THEOREM 3.7. – *Let $g = 4$. Then for a general stable bundle $E \in \mathcal{SU}_C^s(r, d)$,*

$$\tau_E : \mathbf{P}\Omega_E \to \mathcal{C}_E$$

is a birational morphism and is unramified in a neighborhood of a general fiber of $\pi : \mathbf{P}\Omega_E \to C$.

Proof. – The birationality of τ_E over its image is from Theorem 2.3. That τ_E is unramified in a neighborhood of a general fiber of π follows from Proposition 3.8 below, in the same way that Theorem 3.1 followed from Proposition 3.2. □

PROPOSITION 3.8. – *Let $g \geqslant 4$ and F be a general stable bundle of arbitrary rank and degree. Then there exists a point $x \in C$ such that $H^0(C, \mathrm{ad}_F(2x)) = 0$.*

For the proof of Proposition 3.8, we need the following three lemmas, Lemmas 3.9, 3.10 and 3.11, which are just slight modifications of Lemmas 3.4, 3.5 and 3.6, respectively.

LEMMA 3.9. – *Let E be a general stable bundle of rank $r-1$ and degree d satisfying $d > -(r-1)(g-3)$. Then there exists an element $\epsilon \in H^1(C, E^*)$ such that for a given $x \in C$, $\psi^{2x}(\epsilon) \neq 0$ where $\psi^{2x}: H^1(C, E^*) \to H^1(C, E^*(2x))$ is as defined in Lemma* 3.4 *with $D = 2x$.*

Proof. – As in the proof of Lemma 3.4, it suffices to show

$$h^1(C, E^*) - h^0\big(C, E^*(2x)|_{2x}\big) > 0.$$

But this is obvious from

$$h^1(C, E^*) = d + (r-1)(g-1), \qquad h^0\big(C, E^*(2x)|_{2x}\big) = 2(r-1),$$

as in the proof of Lemma 3.4. □

LEMMA 3.10. – *Let x be a point satisfying $h^0(C, \mathcal{O}(2x)) = 1$, which is certainly true for a general $x \in C$. Let E be a vector bundle of rank $r-1$ and degree d satisfying $|d| \leqslant (r-1)(g-3)$. Assume that $H^0(C, \mathrm{ad}_E(2x)) = 0$. Suppose F is an extension of E by $\mathcal{O}$ with $H^0(C, \mathrm{ad}_F(2x)) \neq 0$. Then the extension class $[F] \in H^1(C, E^*)$ satisfies $\psi^{2x}([F]) = 0$.*

Proof. – The proof of Lemma 3.5 works almost verbatim. It suffices to replace the divisors D and D' by $2x$ and the sections s and s' by the unique section (up to scalar) of $\mathcal{O}(2x)$. □

LEMMA 3.11. – *Let $r \geqslant 2$ and d be integers satisfying*

$$-(r-1)(g-3) < d \leqslant (r-1)(g-3).$$

Suppose for a general stable bundle E of rank $r-1$ and degree d, $H^0(C, \mathrm{ad}_E(2x)) = 0$ for some $x \in C$. Then for a general stable bundle F of rank r and $\det(F) = \det(E)$,

$$H^0\big(C, \mathrm{ad}_F(2x)\big) = 0.$$

Proof. – A simple modification of the proof of Lemma 3.6 works. It suffices to take $\{F_t\}$ with $\det(F_t) = \det(E)$, replace $C^{(\ell)}$ by C and use Lemmas 3.9 and 3.10 in place of Lemmas 3.4 and 3.5, respectively. □

Proof of Proposition 3.8. – The proof is by induction on the rank r of F as in the proof of Proposition 3.2. If $r = 1$, it is obvious. Assume that the result holds for a general stable bundle E of rank $r-1$ and $\det(E) = \det(F)$. By Lemma 3.11, the result follows if

$$-(r-1)(g-3) < d \leqslant (r-1)(g-3).$$

Note that there are $2(r-1)(g-3)$ consecutive integers d satisfying these inequalities and

$$2(r-1)(g-3) \geqslant r \quad \text{if } r \geqslant 2 \text{ and } g \geqslant 4.$$

If a vector bundle F satisfies $H^0(C, \mathrm{ad}_F(2x)) = 0$ for some $x \in C$, then $H^0(C, \mathrm{ad}_{F'}(2x)) = 0$ for any vector bundle F' of the form $F' = F \otimes L$ for a line bundle L. Thus we may assume that the degree d of F belongs to any set of r consecutive integers. This finishes the proof of Proposition 3.8. □

4. Hitchin discriminant and its dual variety

Let us briefly recall the definition of the Hitchin map and spectral curves. See [2,3] and [10] for details. As before, C is a smooth projective curve of genus $\geqslant 4$. Let

$$W := H^0(C, \omega_C^{\otimes 2}) \oplus \cdots \oplus H^0(C, \omega_C^{\otimes r})$$

be the space of characteristic polynomials and $h : T^*(\mathcal{SU}_C^s(r,d)) \to W$ be the *Hitchin map* defined by

$$h(\theta) := \big(s_2(\theta), \ldots, s_r(\theta)\big)$$

where for $\theta \in T_E^*(\mathcal{SU}_C^s(r,d)) = H^0(C, \mathrm{ad}_E \otimes \omega_C)$,

$$s_i(\theta) := (-1)^i \mathrm{tr}(\wedge^i \theta).$$

Let K_C be the total space of the canonical line bundle ω_C and $\alpha : K_C \to C$ be the natural projection. For an element $s = (s_2, \ldots, s_r) \in W$, the *spectral curve* C_s associated to s is the curve in the total space K_C defined by the equation

$$x^r + s_2 x^{r-2} + \cdots + s_{r-1} x + s_r$$

where x is the tautological section of $\alpha^* \omega_C$. Let $\mathcal{D} \subset W$ be the set of characteristic polynomials with singular spectral curves. The following two facts are standard.

PROPOSITION 4.1 [10, Corollary 1.5 and Remark 1.7]. – *$\mathcal{D}$ is an irreducible hypersurface in W and for a general point $s \in \mathcal{D}$, C_s is an integral curve with a unique ordinary double point over a general point of C.*

PROPOSITION 4.2 [2, 3.6 and 3.7]. – *If $s \in W$ has an integral spectral curve, then $h^{-1}(s)$ is irreducible and for a general $\alpha \in h^{-1}(s)$ regarded as an element of $H^0(C, \mathrm{ad}_E \otimes \omega_C)$ for some $E \in \mathcal{SU}_C^s(r,d)$, each eigenvalue of $\alpha_x : E_x \to E_x \otimes \omega_C$ has one-dimensional eigenspace for each $x \in C$. If furthermore the spectral curve is smooth, i.e., $s \in W \setminus \mathcal{D}$, then $h^{-1}(s)$ is an open subset of an abelian variety and is dominant over $\mathcal{SU}_C^s(r,d)$.*

PROPOSITION 4.3. – *The hypersurface $h^{-1}(\mathcal{D})$ in $T^*(\mathcal{SU}_C^s(r,d))$ is irreducible.*

Proof. – Since $h^{-1}(s)$ for a general $s \in \mathcal{D}$ is irreducible by Propositions 4.1 and 4.2, there exists a unique irreducible component S_1 of $h^{-1}(\mathcal{D})$ which is dominant over $\mathcal{D}$. Suppose there exists another component S_2 which is not dominant over $\mathcal{D}$. We will get a contradiction.

For $E \in \mathcal{SU}_C^s(r,d)$, let us denote the restriction of h to the cotangent space $T_E^*(\mathcal{SU}_C^s(r,d))$ by

$$h_E : T_E^*(\mathcal{SU}_C^s(r,d)) \longrightarrow W.$$

There is a natural $\mathbf{C}^\times$-action on $T_E^*(\mathcal{SU}_C^s(r,d))$ by the scalar multiplication and a natural $\mathbf{C}^\times$-action on W by the weighted scalar multiplication. Clearly, h_E is equivariant with respect to these actions of $\mathbf{C}^\times$. Suppose that $T_E^*(\mathcal{SU}_C^s(r,d)) \cap h^{-1}(0) = 0$. Then h_E descends to a morphism

$$\check{h}_E : \mathbf{P}T_E^*\big(\mathcal{SU}_C^s(r,d)\big) \longrightarrow \mathbf{P}_{\mathrm{weight}} W$$

where $\mathbf{P}_{\mathrm{weight}} W$ is the weighted projective space obtained as the quotient of $W \setminus 0$ by the weighted $\mathbf{C}^\times$-action. This $\check{h}_E$ must be a finite morphism. It follows that h_E is a finite morphism.

If S_2 intersects $T^*_E(\mathcal{SU}^s_C(r,d))$ for some E with $T^*_E(\mathcal{SU}^s_C(r,d)) \cap h^{-1}(0) = 0$, the intersection $S'_2 := S_2 \cap T^*_E(\mathcal{SU}^s_C(r,d))$ must be a hypersurface in $T^*_E(\mathcal{SU}^s_C(r,d))$. But then S'_2 is dominant over the irreducible hypersurface $\mathcal{D}$ because h is finite on $T^*_E(\mathcal{SU}^s_C(r,d))$, a contradiction. Thus the image $\mathrm{pr}(S_2)$ under the natural projection

$$\mathrm{pr}: T^*\big(\mathcal{SU}^s_C(r,d)\big) \to \mathcal{SU}^s_C(r,d)$$

is contained in the subvariety

$$\mathcal{N} := \big\{E \in \mathcal{SU}^s_C(r,d)\colon \dim\big(T^*_E\big(\mathcal{SU}^s_C(r,d)\big) \cap h^{-1}(0)\big) \geqslant 1\big\}.$$

Recall that $\mathcal{N} \neq \mathcal{SU}^s_C(r,d)$ by [11]. Thus $\mathrm{pr}(S_2)$ is a hypersurface in $\mathcal{SU}^s_C(r,d)$ and $S_2 = \mathrm{pr}^{-1}(\mathrm{pr}(S_2))$. Since $\dim(h^{-1}(0)) = \dim(\mathcal{SU}^s_C(r,d))$ from [11],

$$\dim\big(T^*_E\big(\mathcal{SU}^s_C(r,d)\big) \cap h^{-1}(0)\big) = 1$$

for a general $E \in \mathrm{pr}(S_2)$. Thus $h_E : T^*_E(\mathcal{SU}^s_C(r,d)) \to W$ must have general fiber dimension $\leqslant 1$. This implies that $h_E(T^*_E(\mathcal{SU}^s_C(r,d)))$ is a hypersurface in W. Since $T^*_E(\mathcal{SU}^s_C(r,d)) \subset S_2$, this is a contradiction to the fact that S_2 is not dominant over $\mathcal{D}$. □

Let $\mathcal{S}$ be the hypersurface in $\mathbf{P}T^*(\mathcal{SU}^s_C(r,d))$ corresponding to $h^{-1}(\mathcal{D})$ in $T^*(\mathcal{SU}^s_C(r,d))$. For a general point $E \in \mathcal{SU}^s_C(r,d)$, the hypersurface $\mathcal{S}_E := \mathcal{S} \cap \mathbf{P}T^*_E(\mathcal{SU}^s_C(r,d))$ will be called the *Hitchin discriminant* at E.

Recall that when $X \subset \mathbf{P}_N$ is a smooth subvariety, its dual variety is the subvariety of the dual projective space $\mathbf{P}^*_N$ corresponding to singular hyperplane sections of X. Suppose the normalization $\hat{X}$ of X is smooth and $\tau : \hat{X} \to X \subset \mathbf{P}_N$ is the normalization morphism. Then X^* is the closure of the set of hyperplanes containing the projective tangent space of a point of X where τ is an immersion. This observation will be used implicitly in the proof of the following theorem, for the case of $g = 4$.

THEOREM 4.4. – *Assume $g \geqslant 4$. Let $E \in \mathcal{SU}^s_C(r,d)$ be a general point. Then the Hitchin discriminant $\mathcal{S}_E \subset \mathbf{P}T^*_E(\mathcal{SU}^s_C(r,d))$ is the dual variety of the variety of Hecke tangents $\mathcal{C}_E \subset \mathbf{P}T_E(\mathcal{SU}^s_C(r,d))$. In other words, $\mathcal{S}$ defined above agrees with $\mathcal{S}$ in Corollary* 2.2.

Proof. – Let $\theta \in h^{-1}(\mathcal{D})$ be a general point. Then $\theta : E \to E \otimes \omega_C$ is an endomorphism of a general stable bundle E such that its spectral curve $C_{h(\theta)}$ has a unique ordinary double point singularity which lies over a general point of C. It suffices to show that $\theta \in \mathbf{P}T^*_E(\mathcal{SU}^s_C(r,d))$ belongs to the dual variety of $\mathcal{C}_E$. By Proposition 4.2, for each $x \in C$, each eigenvalue of θ_x has a 1-dimensional eigenspace. Thus we have a curve

$$C_\theta \subset \mathbf{P}E^*$$

biregular to the spectral curve $C_{h(\theta)}$ corresponding to the 1-dimensional eigenspaces.

Let Ω^*_E be the relative tangent bundle of the projective bundle $\varpi : \mathbf{P}E^* \to C$. Recall that when an endomorphism of a vector space V is regarded as a vector field on $\mathbf{P}V$, the zero set of the vector field corresponds to the set of eigenvectors of the endomorphism. When θ is regarded as a vertical vector field on $\mathbf{P}E^*$ twisted by $\varpi^*\omega_C$ via the isomorphism

$$H^0(C, \mathrm{ad}_E \otimes \omega_C) \cong H^0(\mathbf{P}E^*, \Omega^*_E \otimes \varpi^*\omega_C),$$

θ vanishes exactly on C_θ. Thus, when we regard it as a section of $\xi_E \otimes \pi^*\omega_C$ on $\mathbf{P}\Omega_E$, it defines an element of the linear system $|\xi_E \otimes \pi^*\omega_C|$ with a singular point lying over the singular point of C_θ by Lemma 4.5 below. This implies that θ belongs to the dual variety of $\mathcal{C}_E$ because τ_E is an immersion over a general point of C by Theorems 3.1 and 3.7. □

LEMMA 4.5. – *Let*

$$V = a_1(z_1,\dots,z_n)\frac{\partial}{\partial z_1} + \cdots + a_{n-1}(z_1,\dots,z_n)\frac{\partial}{\partial z_{n-1}}$$

be a holomorphic vector field on the polydisc Δ^n. Assume that the zero set of the vector field

$$a_1(z) = \cdots = a_{n-1}(z) = 0$$

is a curve with a singularity at 0. *Let Ω be the relative cotangent bundle of the projection $p:\Delta^n \to \Delta,\ p(z_1,\dots,z_n) = z_n$. Then the hypersurface in $\mathbf{P}(\Omega) \cong \mathbf{P}^{n-2} \times \Delta^n$ defined by*

$$a_1(z_1,\dots,z_n)y_1 + \cdots + a_{n-1}(z_1,\dots,z_n)y_{n-1} = 0$$

for the homogeneous coordinates $[y_1 : \cdots : y_{n-1}] \in \mathbf{P}^{n-2}$ has a singular point over $z_1 = \cdots = z_n = 0$.

Proof. – Since

$$a_1(z) = \cdots = a_{n-1}(z) = 0$$

is a curve with a singular point at $0 \in \Delta^n$, the matrix $(\frac{\partial a_i}{\partial z_j})|_{z=0}$ has rank $\leqslant n-2$, by Jacobian criterion of smoothness. Thus there exist complex numbers $c_1,\dots,c_{n-1}$, with $c_i \neq 0$ for some i, satisfying

$$\sum_{i=1}^{n-1}\left(\frac{\partial a_i}{\partial z_j}\bigg|_{z=0}\right)c_i = 0 \quad \text{for each } 1 \leqslant j \leqslant n.$$

It is straightforward to check that the point

$$z_1 = \cdots = z_n = 0, \qquad [y_1 : \cdots : y_{n-1}] = [c_1 : \cdots : c_{n-1}]$$

is a singular point of the hypersurface

$$a_1(z_1,\dots,z_n)y_1 + \cdots + a_{n-1}(z_1,\dots,z_n)y_{n-1} = 0. \qquad \square$$

COROLLARY 4.6. – *The irreducible hypersurface $h^{-1}(\mathcal{D})$ is the closure of the union of all rational curves in $T^*(\mathcal{SU}^s_C(r,d))$.*

Proof. – By Corollary 2.2 and Theorem 4.4, $h^{-1}(\mathcal{D})$ is covered by rational curves. By Proposition 4.2, there exists no rational curve in $T^*(\mathcal{SU}^s_C(r,d)) \setminus h^{-1}(\mathcal{D})$. □

Remark 4.7. – Note that for a general Hecke curve ℓ,

$$T\big(\mathcal{SU}^s_C(r,d)\big)\big|_\ell \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^{2r-2} \oplus \mathcal{O}^{(r^2-1)(g-1)-2r+1}$$

because ℓ is a minimal rational curve of $\mathcal{SU}^s_C(r,d)$ and $\ell \cdot K^{-1}_{\mathcal{SU}^s_C(r,d)} = 2r$. A section $\tilde{\ell}$ of $T^*(\mathcal{SU}^s_C(r,d))|_\ell$ gives a smooth rational curve in $\hat{S}$ in the notation of Corollary 2.2. On the other hand, for a general $s \in \mathcal{D}$, $h^{-1}(s)$ is an open subset of the compactified Jacobian of the spectral curve of s which has a unique node (cf. [10, Remark 1.7]). The normalization of the compactified Jacobian is a $\mathbf{P}_1$-bundle over an abelian variety which is the Jacobian of the normalization of the spectral curve. The curve $\tilde{\ell}$ is the image of a fiber of this $\mathbf{P}_1$-bundle. In [10, Remark 1.7], it was stated that the image of this $\mathbf{P}_1$-fiber has a node in the compactified Jacobian. This is inaccurate, as $\tilde{\ell}$ is a smooth rational curve. In fact, the morphism [10, (1.13)] does not exist because the pull-back of a torsion-free sheaf on the nodal curve to its normalization is not necessarily torsion-free. The normalization map of the compactified Jacobian identifies two points on different fibers of the $\mathbf{P}_1$-bundle, contrary to the claim in [10, Remark 1.7]. However this mistake in Remark 1.7 does not affect the rest of the argument in [10].

Remark 4.8. – It is possible to describe the section $\tilde{\ell}$ in Remark 4.7 more explicitly as a family of Hecke transformations of a Higgs field. Such description may give a more direct proof of Corollary 4.6 and more detailed information about the geometry of the Hitchin fibers. We will leave it for a future investigation.

Remark 4.9. – In the above manner, the sections of $T^*(\mathcal{SU}^s_C(r,d))$ over Hecke curves give a rank-1 foliation on an open subset of $h^{-1}(\mathcal{D})$. This foliation can be described in another way. The cotangent bundle $T^*(\mathcal{SU}^s_C(r,d))$ has a natural symplectic form. The restriction of the symplectic form on the hypersurface $h^{-1}(D)$ must be a holomorphic 2-form with a 1-dimensional kernel. It is not difficult to check that the foliation defined by this 1-dimensional kernel is precisely the foliation given by Hecke curves.

5. Applications

As an application of Theorem 4.4, we will give a proof of the non-abelian Torelli theorem, simplifying the proof in [10, Theorem E] for $g \geqslant 4$.

THEOREM 5.1. – *Let C and C' be two smooth projective curves of genus $g \geqslant 4$. Let $f:\mathcal{SU}^s_C(r,d) \to \mathcal{SU}^s_{C'}(r,d)$ be a biregular morphism. Then f induces a biregular morphism $C \cong C'$.*

The following is a direct consequence of Corollary 4.6.

LEMMA 5.2. – *In the situation of Theorem* 5.1*, let*

$$W := H^0(C,\omega_C^{\otimes 2}) \oplus \cdots \oplus H^0(C,\omega_C^{\otimes r}),$$
$$W' := H^0(C',\omega_{C'}^{\otimes 2}) \oplus \cdots \oplus H^0(C',\omega_{C'}^{\otimes r})$$

be the spaces of characteristic polynomials and

$$h : T^*\big(\mathcal{SU}^s_C(r,d)\big) \to W,$$
$$h' : T^*\big(\mathcal{SU}^s_{C'}(r,d)\big) \to W'$$

be the Hitchin maps. Let $\mathcal{D} \subset W$ (resp. $\mathcal{D}' \subset W'$) be the hypersurface consisting of characteristic polynomials with singular spectral curves and $\mathcal{S} \subset \mathbf{P}T^*(\mathcal{SU}^s_C(r,d))$ (resp. $\mathcal{S}' \subset \mathbf{P}T^*(\mathcal{SU}^s_{C'}(r,d))$) be the hypersurface corresponding to $h^{-1}(\mathcal{D})$ (resp. $(h')^{-1}(\mathcal{D}')$). Let $df^* : \mathbf{P}T^*(\mathcal{SU}^s_{C'}(r,d)) \to \mathbf{P}T^*(\mathcal{SU}^s_C(r,d))$ be the pull-back by f. Then $df^*(\mathcal{S}') = \mathcal{S}$.

Proof of Theorem 5.1. – Let E be a general point of $\mathcal{SU}^s_C(r,d)$ and $E' = f(E)$. By Lemma 5.2, $df^*_E(\mathcal{S}'_{E'}) = \mathcal{S}_E$. Thus by Theorem 4.4, f induces a biregular morphism $\mathcal{C}_E \cong \mathcal{C}_{E'}$. This induces a biregular morphism $\mathbf{P}\Omega_E \cong \mathbf{P}\Omega_{E'}$ by Theorem 3.1 and Theorem 3.7, and consequently a biregular morphism $C \cong C'$ because C (resp. C') is the Albanese image of $\mathbf{P}\Omega_E$ (resp. $\mathbf{P}\Omega_{E'}$). $\square$

A precise description of the automorphism group of $\mathcal{SU}^s_C(r,d)$, for $g \geqslant 3$, was given by Kouvidakis and Pantev. An essential part of their work was the following, which we will prove as an application of Theorem 4.4. This simplifies the proof in [10] for $g \geqslant 4$.

THEOREM 5.3. – *Let C be a smooth projective curve of genus $\geqslant 4$. The group of automorphisms of $\mathcal{SU}^s_C(r,d)$ is generated by automorphisms of the following two types when $r \nmid 2d$.*

(a) *$E \mapsto \gamma^* E$ where γ is an automorphism of the curve C, and*

(b) *$E \mapsto E \otimes \mu$ where μ is an r-torsion of the Picard group of C.*

When $r \mid 2d$, additional generators of the following type are needed.

(c) *$E \mapsto E^* \otimes \nu$ where ν is a line bundle of degree $\frac{2d}{r}$ on C whose rth power is isomorphic to the square of $\det(E)$.*

We need two simple lemmas.

LEMMA 5.4. – *Let E (resp. E') be a vector bundle of rank r on a smooth projective curve C of genus $\geqslant 4$ and Ω_E (resp. $\Omega_{E'}$) be the relative cotangent bundle on $\mathbf{P}E^*$ (resp. $\mathbf{P}(E')^*$) with respect to the natural projection $\varpi : \mathbf{P}E^* \to C$ (resp. $\varpi' : \mathbf{P}(E')^* \to C$). Suppose there exists a biregular morphism $G : \mathbf{P}\Omega_E \to \mathbf{P}\Omega_{E'}$. Then there exists a biregular automorphism $\gamma : C \to C$ making the following diagram commutative.*

$$\begin{array}{ccc} \mathbf{P}\Omega_E & \xrightarrow{G} & \mathbf{P}\Omega_{E'} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ C & \xrightarrow{\gamma} & C \end{array}$$

Moreover, either G descends to a biregular morphism $\mathbf{P}E^* \to \mathbf{P}(E')^*$ or it descends to a biregular morphism $\mathbf{P}E^* \to \mathbf{P}E'$.

Proof. – The existence of γ is obvious by considering Albanese map. Each fiber of π and π' is isomorphic to $\mathbf{P}T^*(\mathbf{P}_{r-1})$ which has exactly two Mori contractions (of extremal rays)

$$\mathbf{P}T^*(\mathbf{P}_{r-1}) \longrightarrow \mathbf{P}_{r-1} \quad \text{and} \quad \mathbf{P}T^*(\mathbf{P}_{r-1}) \longrightarrow \mathbf{P}^*_{r-1}.$$

Thus $\mathbf{P}\Omega_E$ (resp. $\mathbf{P}\Omega_{E'}$) has exactly two Mori contractions

$$\mathbf{P}\Omega_E \longrightarrow \mathbf{P}E^* \quad \text{and} \quad \mathbf{P}\Omega_E \longrightarrow \mathbf{P}E$$

$$\big(\text{resp.} \quad \mathbf{P}\Omega_{E'} \longrightarrow \mathbf{P}(E')^* \quad \text{and} \quad \mathbf{P}\Omega_{E'} \longrightarrow \mathbf{P}E'\big).$$

Thus G induces either $\mathbf{P}E^* \cong \mathbf{P}(E')^*$ or $\mathbf{P}E^* \cong \mathbf{P}E'$. □

LEMMA 5.5. – *In the situation of Lemma 5.4, assume that $\deg(E) = \deg(E') =: d$ and $\det(E) = \gamma^* \det(E')$. Then denoting by $\mathrm{Pic}^0(C)[r]$ the r-torsion subgroup of the Picard group, one of the following holds.*

(i) *If $r \nmid 2d$, there exists $\mu \in \mathrm{Pic}^0(C)[r]$ such that $E \cong \gamma^*(E' \otimes \mu)$.*

(ii) *If $r \mid 2d$, either there exists $\mu \in \mathrm{Pic}^0(C)[r]$ such that $E \cong \gamma^*(E' \otimes \mu)$, or there exists $\nu \in \mathrm{Pic}^{\frac{2d}{r}}(C)$ with $\nu^{\otimes r} = (\det(E'))^{\otimes 2}$ such that $E \cong \gamma^*((E')^* \otimes \nu)$.*

Proof. – From Lemma 5.4, it is obvious that either $E \cong \gamma^*(E' \otimes \mu)$ or $E \cong \gamma^*((E')^* \otimes \nu)$ for some line bundles μ, ν on C. The assumption $\det(E) = \gamma^* \det(E')$ can be easily translated to the properties of μ and ν described in (i) and (ii). □

Proof of Theorem 5.3. – Let σ be an automorphism of $\mathcal{SU}^s_C(r,d)$. Arguing as in the proof of Theorem 5.1, we see that σ induces a biregular morphism $G:\mathbf{P}\Omega_E \cong \mathbf{P}\Omega_{E'}$ for a general $E \in \mathcal{SU}^s_C(r,d)$ and $E' = \sigma(E)$. By Lemma 5.4, σ induces an automorphism $\gamma_E \in \mathrm{Aut}(C)$ for each general $E \in \mathcal{SU}^s_C(r,d)$. Since $\mathrm{Aut}(C)$ is finite, γ_E is independent of E for general E. Composing σ with an automorphism of type (a), we may assume that $\gamma_E = I_C$, the identity map of C. Then σ must agree with an automorphism of type (b) or (c) by Lemma 5.5. □

As a final application of Theorem 4.4, we prove the following result on morphisms between moduli spaces of bundles.

THEOREM 5.6. – *Let C and C' be two smooth projective curves of genus $g \geqslant 4$. Let $f:\mathcal{SU}^s_C(r,d) \to \mathcal{SU}^s_{C'}(r,d)$ be a surjective morphism. Then f is biregular.*

Proof. – The key point is to prove an analogue of Lemma 5.2. In other words, when $df^*: f^*T^*(\mathcal{SU}^s_{C'}(r,d)) \to T^*(\mathcal{SU}^s_C(r,d))$ is the natural morphism associated to f, we claim that $df^*(f^*\mathcal{S}') \subset \mathcal{S}$ in the notation of Lemma 5.2. Note that the proof of Lemma 5.2 does not work when f is *a priori* not biregular.

To prove the claim, let ℓ be a general Hecke curve on $\mathcal{SU}^s_{C'}(r,d)$ and $\hat{\ell} \subset \mathcal{SU}^s_C(r,d)$ be an irreducible component of $f^{-1}(\ell)$. An element $\sigma \in H^0(\ell, T^*(\mathcal{SU}^s_{C'}(r,d))$ defines an element $f^*\sigma \in H^0(\hat{\ell}, T^*(\mathcal{SU}^s_C(r,d)))$. Let $\ell^\flat$ be the image of $f^*\sigma$ in $T^*(\mathcal{SU}^s_C(r,d))$. Since W is affine, $h(\ell^\flat)$ is one point $s \in W$. Suppose $s \notin \mathcal{D}$. By Proposition 4.2, $h^{-1}(s)$ is an open subset A' of an abelian variety A. The natural projection $A' \to \mathcal{SU}^s_C(r,d)$ is dominant and so is its composition with f, which is denoted by $f': A' \to \mathcal{SU}^s_{C'}(r,d)$. But the complete curve $\ell^\flat$ satisfies $f'(\ell^\flat) = \ell$. This is a contradiction to Proposition 2.4, because the variety of Hecke tangents is not linear. It follows that $s \in \mathcal{D}$. Thus $f^*\sigma$ has its image in $h^{-1}(\mathcal{D})$. Since $(h')^{-1}(\mathcal{D}')$ is covered by images of σ by Corollary 2.2 and Theorem 4.4, this implies that $df^*((h')^{-1}(\mathcal{D}')) \subset h^{-1}(\mathcal{D})$, as claimed.

Choose an analytic open subset $U \subset \mathcal{SU}^s_C(r,d)$ such that $f|_U: U \to f(U)$ is biholomorphic. By the claim, for each $u \in U, (df_u)^*(\mathcal{S}'_{f(u)}) = \mathcal{S}_u$. By Theorem 4.4, $df_u(\mathcal{C}_u) = \mathcal{C}'_{f(u)}$. Then we can proceed as in the proof of Theorem 5.1 and Theorem 5.3 to show that $C \cong C'$ and $f|_U$ agrees with the restriction of an automorphism of $\mathcal{SU}^s_C(r,d)$ to U. Hence f is biregular. □

Remark 5.7. – The argument used in the proof of Theorem 5.6 shows that when $s \notin \mathcal{D}$, the projection $h^{-1}(s) \to \mathcal{SU}^s_C(r,d)$ cannot be proper over a Hecke curve. In fact, there exists no complete curve in $h^{-1}(s)$ which is mapped to a Hecke curve in $\mathcal{SU}^s_C(r,d)$. Since for any subvariety Z of codimension $\geqslant 2$ in $\mathcal{SU}^s_C(r,d)$, there exists a Hecke curve disjoint from Z, this means that the locus where the projection $h^{-1}(s) \to \mathcal{SU}^s_C(r,d)$ is not proper is of codimension 1 in $\mathcal{SU}^s_C(r,d)$.

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Jun-Muk HWANG
Korea Institute for Advanced Study,
207-43 Cheongryangri-dong,
Seoul 130-722, Korea
E-mail: jmhwang@kias.re.kr

S. RAMANAN
Institute of Mathematical Sciences,
Taramani, Chennai 600, India
Chennai Mathematical Institute,
G.N. Chetty Road,
Chennai 600 017, India
E-mail: sramanan@cmi.ac.in