A NEW FAMILY OF SURFACES
WITH $p_g = 0$ AND $K^2 = 3$

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ABSTRACT. – Let S be a minimal complex surface of general type with $p_g = 0$ such that the bicanonical map φ of S is not birational and let Z be the bicanonical image. In [M. Mendes Lopes, R. Pardini, Math. Z. 241 (4) (2002) 673–683] it is shown that either: (i) Z is a rational surface, or (ii) $K_S^2 = 3$, the map φ is a degree two morphism and Z is birational to an Enriques surface. Up to now no example of case (ii) was known. Here an explicit construction of all such surfaces is given. Furthermore it is shown that the corresponding subset of the moduli space of surfaces of general type is irreducible and uniruled of dimension 6.

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RÉSUMÉ. – Soit S une surface de type général avec $p_g = 0$ dont l'application bicanonique φ of S n'est pas birationnelle, et soit Z l'image bicanonique. Dans un précédent article, nous avons montré que soit Z est une surface rationnelle, soit $K_S^2 = 3$, φ est un morphisme de degré 2 et Z est (birationnellement) une surface d'Enriques. On ne savait pas si cette seconde possibilité pouvait être réalisée. Ici nous donnons une construction explicite de toutes ces surfaces, et nous montrons aussi que la partie correspondante de l'espace des modules des surfaces de type général est irréductible et unirationnelle de dimension 6.

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1. Introduction

The knowledge of surfaces of general type with $p_g = 0$ continues to be scarce in spite of much progress in surface theory. A minimal surface of general type with $p_g = 0$ satisfies $1 \leqslant K^2 \leqslant 9$ and examples for all possible values for K^2 are known (see, e.g., [1, Chapter VII, §11]).

In recent years we have undertaken to study surfaces with $p_g = 0$ by looking at their bicanonical map. When the bicanonical map is generically finite but not birational, this approach works and it has allowed us to obtain classification results and also, in some cases, information on the moduli space (see for instance [16,18,23]).

The first step in describing a class of surfaces with nonbirational bicanonical map is to analyze the bicanonical image. If S is a minimal surface with $p_g = 0$, then the bicanonical image Z of S is contained in $\mathbb{P}^{K^2}$, thus, in particular, for $K^2 = 1$ it is equal to $\mathbb{P}^1$. For $K^2 \geqslant 2$, Z is always a surface [27] and, if the bicanonical map φ is not birational, then Z is either rational or it is an Enriques surface [28]. In [17], it has been proven that the latter case can occur only if $K^2 = 3$ and φ is a morphism of degree 2. No such example appears in the literature.

Indeed, the known examples of minimal surfaces of general type with $p_g = 0$ and $K^2 = 3$ are the examples of Burniat and Inoue ([3,11], see also [24]), the examples due independently to J.H. Keum and D. Naie [12,21] and the recent examples due to F. Catanese [5]. The degree

of the bicanonical map of all these surfaces is equal to 4 (cf. [17,16]), although the Keum–
Naie examples are in fact double covers of nodal Enriques surfaces and their bicanonical map
factorizes through the covering map.

In this paper we prove that surfaces with $p_g = 0$ whose bicanonical image is an Enriques
surface do exist. More precisely, we give an explicit construction of all such surfaces and
we prove that the corresponding subset $\mathcal{E}$ of the moduli space of surfaces of general type is
irreducible and uniruled of dimension 6. Since the closure of $\mathcal{E}$ contains the Keum–Naie surfaces
(see Proposition 7.3), it follows that the fundamental group of our surfaces is isomorphic to
$\mathbb{Z}_2^2 \times \mathbb{Z}_4$ (see Corollary 7.2).

There is an analogy between the surfaces we study here and the so-called Todorov surfaces.
These are surfaces of general type with $p_g = 1$ and $q = 0$ that are mapped $2-$to-1 onto a K3
surface by the bicanonical map [26,19], and they are the only surfaces whose bicanonical image
is a K3 surface.

This analogy is accounted for by the fact that the surfaces that we construct can also be
obtained as free $\mathbb{Z}_2$-quotients of some Todorov surfaces with $K^2 = 6$. We remark that it is known
that, for fixed K^2, the Todorov surfaces form an irreducible subset of the moduli space [19]. For
$K^2 = 2$, in [6] the authors have described very precisely the irreducible component X of the
moduli space containing the Todorov surfaces, showing in particular that the bicanonical map is
birational for a general surface of X.

A natural question, that at the moment we are not able to answer (see Remark 7.4), is
whether the closure $\bar{\mathcal{E}}$ of the subset of the moduli space consisting of the surfaces studied
here is an irreducible component. If the answer were affirmative, then one would have a similar
phenomenon to the case of surfaces with $p_g = 0$, $K^2 = 6$ and bicanonical map of degree 4, which
form a component of the moduli space [16]. On the other hand, if the answer were negative, then
it would follow that Todorov surfaces with $K^2 = 6$ behave similarly to those with $K^2 = 2$.

Our description of surfaces whose bicanonical image Z is an Enriques surface is based on a
very detailed study of the normalization of Z. This is a polarized Enriques surface of degree 6
with 7 nodes, satisfying some additional conditions (see Proposition 2.1 and the setting of §3).
The analysis and construction of these Enriques surfaces form the bulk of this paper. The main
tools we use are the classification of linear systems on an Enriques surface, the analysis of the
configuration of singular fibres of certain elliptic pencils, the code associated to the nodes of the
surface and the corresponding Galois cover (cf. [9]).

The plan of the paper is as follows: Section 2 explains the relation between the surfaces we
are studying and a certain class of polarized Enriques surfaces (Σ, B) with 7 nodes; in Section 3
some properties of these Enriques surfaces are established and some examples are described; in
Section 4 we make a very detailed study of the singular fibres of the elliptic pencils of Σ and
we determine the code associated to the nodes of Σ; in Section 5 we describe a construction
yielding pairs (Σ, B) and prove that all such pairs are obtained in that way; in Section 6 we
introduce and study a quasi-projective variety parametrizing the isomorphism classes of pairs
(Σ, B) and finally in Section 7 we apply the previous results to describe the family of surfaces
whose bicanonical image is an Enriques surface.

Notation and conventions: We work over the complex numbers. A node of a surface is an
ordinary double point, namely a singularity analytically isomorphic to $x^2 + y^2 + z^2 = 0$. The
exceptional divisor of the minimal resolution of a node is a curve $C \simeq \mathbb{P}^1$ such that $C^2 = -2$.
A curve with these properties is called $-2-$curve, or nodal curve.

We say that a projective surface Σ with canonical singularities is minimal, of general type,
Enriques . . . if the minimal resolution of Σ is minimal, of general type, Enriques . . . . Our standard
reference for Enriques surfaces is [8], and we use freely its terminology.

Given an automorphism σ of a variety X, we say that a map $f \colon X \to Y$ is composed with σ if $f \circ \sigma = f$. If G is a finite group, a G-cover is a finite map $f \colon X \to Y$ of normal varieties together with a faithful G-action on X such that f is isomorphic to the quotient map $X \to X/G$. If $G = \mathbb{Z}_2$, then we say that f is a double cover. Contrary to what is often done (cf. for instance [22]), we do not require that f be flat.

We denote linear equivalence by $\equiv$ and numerical equivalence by $\sim_{\mathrm{num}}$. The group of line bundles modulo numerical equivalence on a variety Y is denoted by $\mathrm{Num}(Y)$.

2. **Surfaces with $p_g = 0$ and $K^2 = 3$ and Enriques surfaces**

As explained in the introduction, the bulk of this paper is a very detailed study of a class of polarized Enriques surfaces with 7 nodes. In this section we explain the relation between such Enriques surfaces and a class of minimal surfaces of general type with nonbirational bicanonical map. Let S be a minimal surface of general type with $p_g(S) = 0$ and $K_S^2 = 3$. We denote by $\varphi \colon S \to \mathbb{P}^3$ the bicanonical map of S and we assume that S has an involution σ such that:
  (a) φ is composed with σ;
  (b) the quotient surface $T := S/\sigma$ is birational to an Enriques surface.
We denote by X the canonical model of S. Abusing notation, we denote by the same letter the involution induced by σ on X.

PROPOSITION 2.1. – *In the above setting*:
  (i) *the quotient surface $\Sigma := X/\sigma$ is an Enriques surface with 7 nodes*;
  (ii) *the quotient map $\pi \colon X \to \Sigma$ is branched on the nodes of Σ and on a divisor B with negligible singularities, contained in the smooth part of Σ*;
  (iii) *B is ample and $B^2 = 6$*;
  (iv) *the bicanonical system $|2K_X| = \pi^*|B|$ is base point free*;
  (v) *the degree of the bicanonical map of S (and X) is either 2 or 4*.

Proof. – Since the bicanonical map φ of S factorizes through σ, by [18, Proposition 2.1] the isolated fixed points of σ are 7. The quotient surface $T := S/\sigma$ has 7 nodes, which are the images of the isolated fixed points of σ. The quotient map $S \to T$ is branched on the nodes and on a smooth divisor B_0 contained in the smooth part of T. By Lemma 7 of [28] and the remark following it, there exists a birational morphism $r \colon T \to \Sigma'$, where Σ' is an Enriques surface with 7 nodes, such that the exceptional curves of r are contained in the smooth part of T and the divisor $B := r(B_0)$ has negligible singularities. Let $S \to X' \xrightarrow{\pi'} \Sigma'$ be the Stein factorization of the induced map $S \to \Sigma'$. The map $\pi' \colon X' \to \Sigma'$ is a double cover branched on the nodes of Σ' and on the divisor B. The singularities of X' occur above the singularities of B, hence they are canonical and there is a birational morphism from X' to the canonical model X of S. More precisely, there is a commutative diagram:

$$
\begin{array}{ccc}
X' & \longrightarrow & X \\
{\scriptstyle \pi'} \downarrow & & \downarrow {\scriptstyle \pi} \\
\Sigma' & \longrightarrow & \Sigma
\end{array}
$$

where the horizontal arrows represent birational morphisms and π, π' are the quotient maps for the involutions induced by σ on X and X'.

By adjunction we have $2K_{X'} = \pi'^*(2K_{\Sigma'} + B) = \pi'^*B$, hence B is nef and $B^2 = 6$. Since the bicanonical map of X' factorizes through π', we actually have $|2K_{X'}| = \pi'^*|B|$. The same

argument as in the proof of [17, Theorem 5.1] shows that the system $|B|$ is free. Thus (Σ', B) is a pair as in the setting of §3 and we can apply Corollary 3.3, which is proven in §3, showing that B is ample. It follows that the horizontal maps in the above diagram are isomorphisms and we can identify X with X' and Σ with Σ'. The system $|B|$, being free, is not hyperelliptic, therefore by [7, Proposition 5.2.1] either it is birational or it has degree 2. Hence the degree of the bicanonical map of X and S has degree either 2 or 4. This completes the proof. $\square$

The previous proposition has a converse:

PROPOSITION 2.2. – *Let Σ be an Enriques surface with 7 nodes and let $\pi : X \to \Sigma$ be a double cover branched on the nodes of Σ and on a divisor B such that*:
  (a) *B is ample and $B^2 = 6$;*
  (b) *B is contained in the smooth part of Σ and it has negligible singularities.*
  Then X is the canonical model of a minimal surface S of general type with $p_g(S) = 0$ and $K_S^2 = 3$ and the bicanonical map of S factorizes through the map $S \to \Sigma$.
Condition (a) can be replaced by:
  (a$'$) *$|B|$ is free and $B^2 = 6$.*

Proof. – Assume that conditions (a) and (b) are satisfied.
The singular points of X lie above the singularities of B. Since B has negligible singularities, the singularities of X are canonical and one has $2K_X = \pi^* B$. Since B is ample, K_X is also ample and X is the canonical model of a surface S of general type. One has

$$K_S^2 = K_X^2 = \frac{1}{2}B^2 = 3.$$

To compute the birational invariants $\chi(S)$ and $p_g(S)$, we consider the minimal resolution of singularities $\eta : Y \to \Sigma$ and the flat double cover $\tilde{\pi} : \widetilde{X} \to Y$ obtained from π by taking base change with η. The surface $\widetilde{X}$ has canonical singularities and it is birational to X and S. The branch locus of $\tilde{\pi}$ consists of the inverse image $\widetilde{B}$ of B and of the -2-curves $N_1, \ldots, N_7$ that are exceptional for η. So $\tilde{\pi}$ is given by a relation $2L \equiv \widetilde{B} + N_1 + \cdots + N_7$, where L is a line bundle on Y. A standard computation gives $\chi(S) = \chi(\widetilde{X}) = \chi(Y) + \chi(L^{-1}) = 1$. By Kawamata–Viehweg vanishing one has $h^i(K_Y + L) = 0$ for $i > 0$, hence

$$p_g(S) = p_g(\widetilde{X}) = h^0(K_Y + L) = \chi(K_Y + L) = 0.$$

Since the singularities of the branch divisor B of π are negligible, it is well known that the smooth minimal model S of X can be obtained by repeatedly blowing up Σ at the singular points of B and taking base change and normalization. Hence the involution of S induced by σ has the same number of isolated base points as σ, that is 7, and the bicanonical map factorizes through it by [16, Proposition 2.1].
If (a$'$) holds, then B is ample by Corollary 3.3, and so condition (a$'$) implies condition (a). $\square$

3. Enriques surfaces with 7 nodes: examples

Recall that for Enriques surfaces we adopt the notation and the terminology of [8]. Our notation for the singular fibres of an elliptic pencil is the same as in [1], Chapter V, §7.
In this section we consider the following situation:
Set-up: Σ is a nodal Enriques surface with 7 nodes, $|B|$ is a base point free linear system of Σ such that $B^2 = 6$. We denote by $\eta : Y \to \Sigma$ the minimal desingularization and by $N_1, \ldots, N_7$ the

disjoint nodal curves contracted by η, and we set $\widetilde{B} := \eta^* B$. Furthermore, we assume that there exists $L \in \operatorname{Pic}(Y)$ such that $\widetilde{B} + N_1 + \cdots + N_7 = 2L$.

Remark 3.1. – The choice of this set-up is suggested by the results of the previous section. Indeed, the condition that the class $\widetilde{B} + N_1 + \cdots + N_7$ be divisible by 2 in $\operatorname{Pic}(Y)$ means that, given a curve $\widetilde{B} \in |\widetilde{B}|$, there exists a double cover $\tilde\pi \colon \widetilde{X} \to Y$ branched on the union of $\widetilde{B}$ and $N_1, \ldots, N_7$. If $\widetilde{B}$ is disjoint from $N_1, \ldots, N_7$ (recall that $\widetilde{B} N_i = 0$) and it has at most negligible singularities, then the surface $\widetilde{X}$ has canonical singularities, occurring above the singularities of $\widetilde{B}$. For $i = 1, \ldots, 7$, one has $\tilde\pi^* N_i = 2C_i$, where C_i is a -1-curve contained in the smooth part of $\widetilde{X}$. If we denote by X the surface obtained by contracting the C_i, then $\tilde\pi$ induces a double cover $\pi \colon X \to \Sigma$ branched over the image B of $\widetilde{B}$ and over the nodes of Σ. By Proposition 2.2, X is the canonical model of a minimal surface of general type with $p_g = 0$ and $K^2 = 3$ and the bicanonical map of X factorizes through π.

Notice that, since Y is an Enriques surface, the line bundle $L + K_Y$ also satisfies the relation $2(L + K_Y) \equiv \widetilde{B} + N_1 + \cdots + N_7$, so that a pair (Σ, B) as in the set-up determines two non-isomorphic double covers of Σ with the same branch locus.

In order to describe some examples, we need to prove first some general facts.

PROPOSITION 3.2. – *There exist three elliptic half-pencils $\widetilde{E}_1, \widetilde{E}_2, \widetilde{E}_3$ on Y such that*:
  (i) $\widetilde{E}_i \widetilde{E}_j = 1$ *for* $i \neq j$;
  (ii) $|\widetilde{B}| = |\widetilde{E}_1 + \widetilde{E}_2 + \widetilde{E}_3|$.

Proof. – Notice that the system $|\widetilde{B}|$, being base point free, is not hyperelliptic. Hence, if $|\widetilde{B}|$ is not as stated, then by Proposition 5.2.1 and Theorem 5.3.6 of [7] there are the following possibilities:
  (1) $|\widetilde{B}| = |2\widetilde{E}_0 + \widetilde{E}_1 + \theta_2|$, $\widetilde{E}_0 \widetilde{E}_1 = \widetilde{E}_0 \theta_2 = 1$, $\widetilde{E}_1 \theta_2 = 0$;
  (2) $|\widetilde{B}| = |3\widetilde{E}_0 + 2\theta_0 + \theta_1|$, $\widetilde{E}_0 \theta_0 = \theta_0 \theta_1 = 1$, $\widetilde{E}_0 \theta_1 = 0$,
where $\widetilde{E}_m$ are elliptic half-pencils and θ_m are nodal curves, $m = 0, 1, 2$. Consider the nodal curves $N_1, \ldots, N_7$ and recall that $\widetilde{B} + N_1 + \cdots + N_7$ is divisible by 2 in $\operatorname{Pic}(Y)$.

In case (1), suppose first that θ_2 is not one of the curves N_i. Since $N_i \widetilde{B} = 0$, necessarily the curves $\theta_2, N_1, \ldots, N_7$ are disjoint and so by Lemma 4.2 of [17] the divisor $\theta_2 + N_1 + \cdots + N_7$ is divisible by 2 in $\operatorname{Pic}(Y)$. Hence also $\widetilde{E}_1$ is divisible by 2 in $\operatorname{Pic}(Y)$, a contradiction. So θ_2 is one of the curves N_i, say $\theta_2 = N_7$. Then there exists $L \in \operatorname{Pic}(Y)$ such that $D = \widetilde{E}_1 + N_1 + \cdots + N_6 = 2L$ and we get $D^2 = -12$ and $L^2 = -3$, contradicting the fact that the intersection form on an Enriques surface is even.

Consider now case (2). As in case (1), if neither of the curves θ_0, θ_1 is one of the nodal curves $N_1, \ldots, N_7$, we conclude that the half-pencil $\widetilde{E}_0$ is divisible by 2 in $\operatorname{Pic}(Y)$, a contradiction. If θ_1 is one of the curves N_i, we also arrive at a contradiction as in case (1). Finally suppose that θ_0 is one of the curves N_i, say N_7. Then $\widetilde{E}_0(N_1 + \cdots + N_7) = 1$. Since $\widetilde{E}_0 \widetilde{B} = 2$, $\widetilde{B} + N_1 + \cdots + N_7$ is not divisible by 2 in $\operatorname{Pic}(Y)$, a contradiction.

So $|\widetilde{B}|$ is as stated. $\square$

COROLLARY 3.3. – *The divisor B is ample on Σ.*

Proof. – Denote by $\mathcal{R}_{\widetilde{B}}$ the set of irreducible curves C of Y with $\widetilde{B}C = 0$. By [8, Corollary 4.1.1], $\mathcal{R}_{\widetilde{B}}$ is a finite set and the corresponding classes are independent in $\operatorname{Num}(Y)$. By Proposition 3.2, we can write $\widetilde{B} = \widetilde{E}_1 + \widetilde{E}_2 + \widetilde{E}_3$, where the $\widetilde{E}_i$ are elliptic half-pencils. If we denote by V the subspace of $\operatorname{Num}(Y) \otimes \mathbb{Q}$ spanned by the classes of $\widetilde{E}_1, \widetilde{E}_2, \widetilde{E}_3$, then the classes of the curves of $\mathcal{R}_{\widetilde{B}}$ belong to $V^\perp$, which has dimension 7. So we have $\mathcal{R}_{\widetilde{B}} = \{N_1, \ldots, N_7\}$ and $B = \eta_* \widetilde{B}$ is ample on Σ. $\square$

The following is a partial converse to Proposition 3.2:

LEMMA 3.4. – *Let Y be a smooth Enriques surface containing 7 disjoint nodal curves $N_1, \ldots, N_7$. Assume that $\widetilde{E}_1, \widetilde{E}_2, \widetilde{E}_3$ are elliptic half pencils on Y such that $\widetilde{E}_i \widetilde{E}_j = 1$ if $i \ne j$ and $\widetilde{E}_i N_j = 0$ for every i, j. If we set $\widetilde{B} := \widetilde{E}_1 + \widetilde{E}_2 + \widetilde{E}_3$, then $\widetilde{B} + N_1 + \cdots + N_7$ is divisible by 2 in $\mathrm{Num}(Y)$.*

Proof. – Recall that $\mathrm{Num}(Y)$ is an even unimodular lattice of rank 10.

Let M be the sublattice of $\mathrm{Num}(Y)$ spanned by the classes of $\widetilde{E}_1 - \widetilde{E}_2$ and $\widetilde{E}_1 - \widetilde{E}_3$. The discriminant of M is equal to 3, hence M is primitive. Let M' be the sublattice spanned by the classes of the N_i and by the class of $\widetilde{B}$. The primitive closure of M' is $M^\perp$ and the code W associated to the set of classes $\widetilde{B}, N_1, \ldots, N_7$ is naturally isomorphic to the quotient group $M^\perp / M'$. Computing discriminants one gets:

$$2^8 \cdot 3 = \mathrm{disc}(M') = \mathrm{disc}(M^\perp) 2^{2 \dim W} = \mathrm{disc}(M) \cdot 2^{2 \dim W} = 3 \cdot 2^{2 \dim W},$$

namely $\dim W = 4$. Using the fact that the intersection form on Y is even, it is easy to check that the elements of W have weight divisible by 4. Since W has length 8, this implies that W is the extended Hamming code (see, e.g., [13]) and in particular it contains the vector of weight 8, i.e., $\widetilde{B} + N_1 + \cdots + N_7$ is divisible by 2 in $\mathrm{Num}(Y)$. $\quad\square$

For $i = 1, 2, 3$, we denote by $\widetilde{E}_i'$ the unique effective divisor in $|\widetilde{E}_i + K_Y|$ and we write $|\widetilde{F}_i| = |2\widetilde{E}_i| = |2\widetilde{E}_i'|$. Thus $|\widetilde{F}_i|$ is an elliptic pencil with double fibres $2\widetilde{E}_i$ and $2\widetilde{E}_i'$. The classes $\widetilde{E}_i$ are nef, hence $\widetilde{B} N_j = 0$ implies $\widetilde{E}_i N_j = 0$ for every i, j and therefore for $i = 1, 2, 3$ $|\widetilde{F}_i|$ induces an elliptic pencil $|F_i| = |2E_i| = |2E_i'|$ on Σ, where $E_i = \eta_* \widetilde{E}_i$, $E_i' = \eta_* \widetilde{E}_i'$ and $B = E_1 + E_2 + E_3$.

Example 3.5. – This example appears in [21] and in an unpublished paper by J. Keum [12]. One considers an Enriques surface $\overline{\Sigma}$ with 8 nodes as in Example 1 of [17]. The surface $\overline{\Sigma}$ has two isotrivial elliptic pencils $|F_1|$ and $|F_2|$ with $F_1 F_2 = 4$. The system $|F_1 + F_2|$ gives a degree 2 morphism onto a Del Pezzo quartic in $\mathbb{P}^4$ such that the nodes of $\overline{\Sigma}$ are mapped to smooth points (cf. [21, §2]). We take Σ to be the surface obtained by resolving one of the nodes of $\overline{\Sigma}$ and we denote by C the corresponding nodal curve. We denote by the same letter the pull-backs of $|F_1|$, $|F_2|$ on Σ and we set $B := F_1 + F_2 - C$. By the above discussion, the system $|B|$ is free and it gives a degree 2 map onto a Del Pezzo cubic in $\mathbb{P}^3$. By [17, Lemma 4.2], the class of $N_1 + \cdots + N_7 + C$ is divisible by 2 in $\mathrm{Pic}(Y)$, hence the class of $\widetilde{B} + N_1 + \cdots + N_7$ is also divisible by 2.

Let $2E_i$ be a double fibre of F_i, $i = 1, 2$. By Riemann–Roch there exists an effective divisor $E_3 \equiv E_1 + E_2 - C$. We have $E_3^2 = 0$ and $E_1 E_3 = E_2 E_3 = 1$. We claim that E_3 is an elliptic half-pencil, so that $B \equiv E_1 + E_2 + E_3$ as predicted by Proposition 3.2.

We now work on the nonsingular surface Y and, as usual, we denote by $\widetilde{D}$ the pull-back on Y of a divisor D of Σ. Let $G_i \in |\widetilde{F}_i|$, $i = 1, 2$, be the fibre containing C. By the description of $\overline{\Sigma}$ given in [17], G_1 and G_2 are fibres of type I_0^* and the divisor $G_3 := G_1 + G_2 - 2C$ is an elliptic configuration of type I_2^*. It follows that $|G_3|$ is an elliptic pencil and that $2\widetilde{E}_3$ is a double fibre of $|G_3|$.

Example 3.6. – Let $\mathcal{C} \subset \mathbb{P}^3$ be the Cayley cubic, defined by $x_1 x_2 x_3 + x_0 x_2 x_3 + x_0 x_1 x_3 + x_0 x_1 x_2 = 0$. The singularities of $\mathcal{C}$ are 4 nodes, that occur at the coordinate points and form an even set. The 6 lines joining the nodes are of course contained in $\mathcal{C}$. We label these lines by $e_1, e_1', e_2, e_2', e_3, e_3'$ in such a way that, for $i = 1, 2, 3$, e_i, e_i' is a pair of skew lines and e_1, e_2, e_3' are coplanar. The surface $\mathcal{C}$ contains 3 more lines l_1, l_2, l_3, contained in the plane $x_0 + x_1 + x_2 + x_3 = 0$. An elementary geometric argument shows that, up to a permutation

Fig. 1.

of the indices, we may assume that the line l_i meets e_i, e_i' and it does not meet e_j, e_j' for $i \neq j$. For $i = 1, 2, 3$ we denote by $|f_i|$ the moving part of the linear system cut out on $\mathcal{C}$ by the planes containing the line l_i. The general f_i is a smooth conic, hence one has $K_{\mathcal{C}} f_i = -2$, $f_i^2 = 0$ and $f_i f_j = 2$ for $i \neq j$. The singular fibres of $|f_i|$ are $2e_i$, $2e_i'$ and $l_j + l_k$, where i, j, k is a permutation of $1, 2, 3$.

Consider a curve $D \in |\mathcal{O}_{\mathcal{C}}(2)|$ such that D is contained in the smooth part of $\mathcal{C}$ and it has at most simple singularities. Let Σ be the double cover of $\mathcal{C}$ branched on D and on the four nodes of $\mathcal{C}$. The surface Σ has canonical singularities, occurring over the singular points of D. Standard computations (cf. the proof of Proposition 2.2) show that Σ is an Enriques surface and that the pull-back of the system of hyperplanes of $\mathbb{P}^3$ is a complete system $|B'|$ on Σ with $B'^2 = 6$. For $i = 1, 2, 3$, we consider on Σ the system $|F_i|$ obtained by pulling-back $|f_i|$. The system $|F_i|$ is an elliptic pencil, with double fibres $2E_i$ and $2E_i'$, where E_i, E_i' are the pull-backs of e_i, and e_i', respectively. For $i \neq j$ one has $E_i E_j = 1$. Furthermore, $B' \equiv E_1 + E_2 + E_3'$.

We consider now a special case of the above construction: we take D to be the union of the section H_0 of $\mathcal{C}$ with the plane $x_0 + x_1 + x_2 + x_3 = 0$ and of the section H with a general hyperplane tangent to $\mathcal{C}$. So H has an ordinary double point at the tangency point and is smooth elsewhere and H and H_0 intersect transversely at 3 points. The surface Σ thus obtained has 7 nodes, occurring above the singularities of $H_0 + H$. As usual we denote by $\eta : Y \to \Sigma$ the minimal resolution and by $N_1, \ldots, N_7$ the exceptional curves of η. The strict transform on Y of the line l_i is a nodal curve A_i. It is not difficult to check that one can relabel the curves $N_1, \ldots, N_7$ in such a way that N_7 corresponds to the singularity of Σ above the double point of H and the incidence relations of the set of curves $A_1, A_2, A_3, N_1, \ldots, N_6$ are as shown in the dual graph in Fig. 1.

As usual we denote by $\widetilde{E}_i$, $\widetilde{E}_i'$, $\widetilde{F}_i$, $\widetilde{B}'$ the pull-backs on Y of E_i, E_i', F_i, B'. The singular fibre G_i of $|\widetilde{F}_i|$ corresponding to the fibre $l_j + l_k$ of $|f_i|$ is of type I_2^*. More precisely, we have

$$G_1 = N_2 + N_3 + N_5 + N_6 + 2(A_3 + A_2 + N_1),$$
$$G_2 = N_1 + N_3 + N_4 + N_6 + 2(A_3 + A_1 + N_2),$$
$$G_3 = N_1 + N_2 + N_4 + N_5 + 2(A_1 + A_2 + N_3).$$

By Lemma 3.4, one of the classes $B' + N_1 + \cdots + N_7$ and $B' + K_Y + N_1 + \cdots + N_7$ is divisible by 2 in $\mathrm{Num}(Y)$. We will show later (Corollary 4.5) that the second case actually occurs. Hence we set $B := B' + K_\Sigma$.

Examples 3.5 and 3.6 share the common feature that either the system $|B|$ or the system $|B + K_\Sigma|$ is not birational. The next example shows that this does not happen in general.

Example 3.7. – By deforming Example 1 we show the existence of a pair (Σ, B) such that both $|B|$ and $|B + K_\Sigma|$ are birational.

We start with a pair (Σ_0, B_0) as in Example 3.5. The Kuranishi family $p \colon \mathcal{Y} \to U$ of the minimal resolution Y_0 of Σ_0 is smooth of dimension 10 by [1, Theorem VIII.19.3]. We may assume that U is contractible and that the family $\mathcal{Y}$ is differentiably trivial. Hence for every fibre $Y_t := p^{-1}(t)$ the inclusion $Y_t \to \mathcal{Y}$ induces an isomorphism $H^2(Y_t, \mathbb{Z}) \overset{\sim}{\longrightarrow} H^2(\mathcal{Y}, \mathbb{Z})$. The Leray spectral sequence gives $h^1(\mathcal{Y}, \mathcal{O}_\mathcal{Y}) = h^2(\mathcal{Y}, \mathcal{O}_\mathcal{Y}) = 0$, hence by the exponential sequence every integral cohomology class of $\mathcal{Y}$ comes from a unique holomorphic line bundle. In particular, there exist line bundles $\widetilde{\mathcal{E}}_1, \widetilde{\mathcal{E}}_2, \widetilde{\mathcal{E}}_3, \mathcal{N}_1, \ldots, \mathcal{N}_7, \mathcal{C}$ that restrict on the central fibre Y_0 to $\widetilde{E}_1, \widetilde{E}_2, \widetilde{E}_3$, $N_1, \ldots, N_7, C$, respectively. We set $\widetilde{\mathcal{B}} := \widetilde{\mathcal{E}}_1 + \widetilde{\mathcal{E}}_2 + \widetilde{\mathcal{E}}_3$ and for $t \in U$ we denote by $\widetilde{B}_t, \widetilde{E}_{i,t}, N_{i,t}$, C_t the restrictions to Y_t of the above bundles. Obviously, the class of $\widetilde{B}_t + N_{1,t} + \cdots + N_{7,t}$ is divisible by 2 in $H^2(Y_t, \mathbb{Z})$ for every $t \in U$.

By [4, Theorem 3.7], the subset U_1 of U where the classes $N_{i,t}$ are effective and irreducible is smooth of dimension 3, while the subset U_2 of U_1 where also C_t is effective is smooth of dimension 2. Since $C_t \equiv \widetilde{E}_{1,t} + \widetilde{E}_{2,t} - \widetilde{E}_{3,t}$, by [8, Theorem 4.7.2] the system $\widetilde{B}_t$ is birational for $t \in U_1 \setminus U_2$. On the other hand, by semicontinuity we may assume that $\widetilde{B}_t + K_{Y_t}$ is birational for every $t \in U$, since it is birational on the central fibre Y_0. So the required example can be obtained by taking Y_t with $t \in U_1 \setminus U_2$ and by blowing down the nodal curves $N_{1,t}, \ldots, N_{7,t}$.

4. Enriques surfaces with 7 nodes: codes and singular fibres

We keep the set-up and the notation of the previous section.
Here we make a detailed study of the code associated to the nodal curves $N_1, \ldots, N_7$ and of the singular fibres of the pencils $|\widetilde{F}_i|$. These results are needed in the following section, where we give a construction of all the pairs (Σ, B) as in the set-up of § 3.

We denote by V and V_{num}, respectively, the code and the numerical code associated to $N_1, \ldots, N_7$ (cf. [17, §2], [2]). Namely, V is the kernel of the map $\mathbb{Z}_2^7 \to \mathrm{Pic}(Y)/2\,\mathrm{Pic}(Y)$ that maps $(x_1, \ldots, x_7)$ to the class of $x_1 N_1 + \cdots + x_7 N_7$. The code V_{num} is defined in analogous way, replacing $\mathrm{Pic}(Y)$ by $\mathrm{Num}(Y)$. Clearly, V is a subcode of V_{num} of codimension $\leqslant 1$. We say that a divisor D is even if it is divisible by 2 in $\mathrm{Pic}(Y)$. In particular, if $D = \sum x_i N_i$ then D is even if and only if $(x_1, \ldots, x_7) \in V$ (we denote by the same letter the integer x_i and its class in $\mathbb{Z}_2$).

LEMMA 4.1. – $\dim V_{\mathrm{num}} = 3$.

Proof. – Since the determinant of the matrix $(\widetilde{E}_i \widetilde{E}_j)_{i,j=1,2,3}$ is equal to 2, the classes $\widetilde{E}_1, \widetilde{E}_2, \widetilde{E}_3$ span a primitive sublattice L of rank 3 of $\mathrm{Num}(Y)$. If L' is the sublattice spanned by the classes $N_1, \ldots N_7$, then V_{num} is isomorphic to the quotient group $L^\perp / L'$. So we have

$$2^7 = \mathrm{disc}(L') = 2^{2\dim V_{\mathrm{num}}} \mathrm{disc}(L^\perp) = 2^{2\dim V_{\mathrm{num}}} \mathrm{disc}(L),$$

namely $\dim V_{\mathrm{num}} = 3$. $\square$

LEMMA 4.2. – *The linear system $|B|$ separates the nodes of Σ.*

Proof. – This follows by [8, Lemma 4.6.3]. □

LEMMA 4.3. – *Denote by* $P_1, \ldots, P_7$ *the image points of* $N_1, \ldots, N_7$ *via the system* $|\widetilde{B}|$. *If* $N_1 + N_2 + N_3 + N_4$ *is an even divisor, then* P_5, P_6, P_7 *are collinear.*

Proof. – Notice first of all that the points $P_1, \ldots, P_7$ are distinct by Lemma 4.2.

Since $\widetilde{B} + N_1 + \cdots + N_7$ is even by assumption, there exists $M \in \mathrm{Pic}(Y)$ such that $2M \equiv \widetilde{B} - N_5 - N_6 - N_7$. Set $M' := M + K_Y$. Since $M^2 = M'^2 = 0$, there exist effective divisors $D \in |M|$ and $D' \in |M'|$. So the linear system $|\widetilde{B} - N_5 - N_6 - N_7|$ contains two distinct divisors $2D$ and $2D'$, hence it has positive dimension. This means that P_5, P_6, P_7 lie on a line. □

PROPOSITION 4.4. – *The code* V *has dimension* 2.

Proof. – By Lemma 4.1, to show that $\dim V = 2$ it is enough to show that $V \subsetneq V_{\mathrm{num}}$. So assume by contradiction that $V = V_{\mathrm{num}}$. Since $\dim V = 3$ and all the elements of V have weight 4, V is isomorphic to the Hamming code (see, e.g., [13]). By the definition of the Hamming code, the set of indices $\{1, \ldots, 7\}$ is in one-to-one correspondence with the nonzero vectors of $\mathbb{Z}_2^3$. The vectors corresponding to distinct indices i_1, i_2, i_3 span a plane of $\mathbb{Z}_2^3$ if and only if there is $v = (x_1, \ldots, x_7) \in V \setminus \{0\}$ such that $x_{i_1} = x_{i_2} = x_{i_3} = 0$. By Lemma 4.3, this happens if and only if the points $P_{i_1}, P_{i_2}, P_{i_3}$ lie on a line in $\mathbb{P}^3$. Hence the points $P_1, \ldots, P_7$ form a configuration isomorphic to the finite plane $\mathbb{P}^2(\mathbb{Z}_2)$. Since the line through two of the P_i contains a third point of the set, it is easy to check that $P_1, \ldots, P_7$ lie in a plane. On the other hand, it is well known that the plane $\mathbb{P}^2(\mathbb{Z}_2)$ cannot be embedded in $\mathbb{P}^2(\mathbb{C})$. So we have a contradiction and the proof is complete. □

We are now able to complete the description of Example 3.6:

COROLLARY 4.5. – *Let* Σ *be the surface of Example* 3.6. *Then* $\widetilde{B}' + K_Y + N_1 + \cdots + N_7$ *is divisible by* 2 *in* $\mathrm{Pic}(Y)$.

Proof. – In the notation of Example 3.6 we have

$$G_1 = 2(A_2 + A_3 + N_1) + N_2 + N_3 + N_5 + N_6 \equiv 2\widetilde{E}_1,$$

hence $N_2 + N_3 + N_5 + N_6$ is an even divisor. The same argument shows that the divisors $N_1 + N_2 + N_4 + N_5$ and $N_1 + N_3 + N_4 + N_6$ are also even. By Proposition 4.4 these are the only nonzero elements of V.

By Lemma 3.4 we know that one of the classes

$$\widetilde{B}' + N_1 + \cdots + N_7 \quad \text{and} \quad \widetilde{B}' + K_Y + N_1 + \cdots + N_7$$

is even. Assume by contradiction that $\widetilde{B}' + N_1 + \cdots + N_7$ is even. Pulling back to Y the section of $\mathcal{C}$ with H_0 we get $\widetilde{B}' \equiv 2(A_1 + A_2 + A_3 + N_1 + N_2 + N_3) + N_4 + N_5 + N_6$. Hence it follows that $N_1 + N_2 + N_3 + N_7$ is also an even divisor, a contradiction. □

The next result describes the possible configurations of singular fibres of the pencils $|\widetilde{F}_i|$ and relates them to the properties of the systems $|B|$ and $|B + K_\Sigma|$.

THEOREM 4.6. – *The possible configurations of fibres with singular support of the pencils* $|\widetilde{F}_i|$ *are the following*:
(1) *up to a permutation of the indices, the pencils* $|\widetilde{F}_1|$ *and* $|\widetilde{F}_2|$ *are isotrivial with* 2 *fibres of type* I_0^*, *while* $|\widetilde{F}_3|$ *has a fibre of type* I_2^* *and two fibres of type* I_2 *or* $_2I_2$.
In this case the system $|B|$ *has degree* 2 *and the system* $|B + K_\Sigma|$ *is birational*;

(2) *each of the pencils $|\widetilde{F}_i|$ has a fibre of type I_2^* and two fibres of type I_2 or $_2I_2$. The dual graph of the set of nodal curves that form the I_2^* fibres is the same as in Fig. 1.*
    In this case $|B|$ is birational and $|B + K_\Sigma|$ has degree 2;
(3) *each of the pencils $|\widetilde{F}_i|$ has a fibre of type I_0^* and three fibres of type I_2 or $_2I_2$.*
    In this case the systems $|B|$ and $|B + K_\Sigma|$ are both birational.

Remark 4.7. – The proof of Theorem 4.6 below actually shows more, namely that case (1) of Theorem 4.6 corresponds exactly to Example 3.5 (cf. Lemma 4.10) and that case (2) corresponds exactly to Example 3.6.

The proof of Theorem 4.6 is somewhat involved and requires some auxiliary lemmas.

LEMMA 4.8. – *Assume that for $i \neq j$, the pencils $|\widetilde{F}_i|$ and $|\widetilde{F}_j|$ on Y have singular fibres of type I_0^* or I_2^*, $G_i = 2C_i + N_{i_1} + \cdots + N_{i_4}$, respectively, $G_j = 2C_j + N_{j_1} + \cdots + N_{j_4}$.*
    Then $C_i C_j = 0$ and the set $\{i_1, \ldots, i_4\} \cap \{j_1, \ldots, j_4\}$ consists of two elements.

Proof. – The curve C_i is irreducible if G_i is of type I_0^* and it is a chain of 3 nodal curves if G_i is of type I_2^*. One has: $4 = \widetilde{F}_i \widetilde{F}_j = G_i G_j = 2C_i G_j$, namely $C_i(2C_j + N_{j_1} + \cdots + N_{j_4}) = 2$. We remark that $C_i N_{j_t}$ is equal to 1 if $j_t \in \{i_1, \ldots, i_4\}$, and it is equal to 0 otherwise. Since there are 7 of the N_i, one has $C_i(N_{j_1} + \cdots + N_{j_4}) > 0$. So either we have $C_i C_j = 0$ and

$$\{i_1, \ldots, i_4\} \cap \{j_1, \ldots, j_4\}$$

consists of two elements, or $C_i C_j = -1$ and $\{i_1, \ldots, i_4\} = \{j_1, \ldots, j_4\}$.
    Assume by contradiction that we are in the second case. This implies in particular that C_i and C_j are not both irreducible. Assume that C_i is irreducible. Then C_j is a chain of 3 nodal curves $C_j = A_1 + N + A_2$ such that each of the "end" curves A_1, A_2 meets exactly two of the curves $N_{i_1}, \ldots, N_{i_4}$ and the "central" curve N is one of the N_i. In fact, if N were not one of the N_i, then the classes of $N_1, \ldots, N_7, N, A_1, A_2$, being independent, would be a basis of $H^2(Y, \mathbb{Q})$, against the Index Theorem. Furthermore, from $C_i C_j = -1$ it follows that C_i is equal to A_1 or A_2, a contradiction, since C_i meets all the curves $N_{i_1}, \ldots, N_{i_4}$. So we have $C_j = A_1 + N + A_2$ as above and, with an analogous notation, $C_i = B_1 + N' + B_2$, where N' is again one of the N_i.
    Observe that $\theta C_i \geqslant -1$ for every irreducible curve θ. Since N' and N are different from $N_{i_1}, \ldots, N_{i_4}$, the relations $N' \widetilde{F}_j = 0$, $N \widetilde{F}_i = 0$ give: $N' C_j = N C_i = 0$ and

$$-1 = C_i C_j = (A_1 + A_2)C_i,$$

hence, say, $A_1 C_i = -1$, $A_2 C_i = 0$. So we can assume that $A_1 = B_1$, while A_2 is disjoint from C_i and B_2 is disjoint from C_j. Say that $A_1 = B_1$ meets the curves N_{i_1} and N_{i_2}. Then the connected divisor $\Delta = N_{i_1} + N_{i_2} + A_1 + N + N'$ is orthogonal to both $\widetilde{F}_i$ and $\widetilde{F}_j$, so its support is contained in both G_i and G_j. If $N \neq N'$, then the intersection form on the components of Δ is semidefinite, hence by Zariski's Lemma Δ is the support of both G_i and G_j, but this is impossible. Hence $N = N'$, but this contradicts the fact that A_2 and C_i are disjoint. $\square$

LEMMA 4.9. – (i) *The fibres with reducible support that occur in the pencils $|\widetilde{F}_i|$ can be of the following types: I_2, $_2I_2$, I_0^*, I_2^*;*
(ii) *each pencil $|\widetilde{F}_i|$ has at least a fibre of type I_0^* or I_2^*;*
(iii) *a fibre of type I_0^* of $|\widetilde{F}_i|$ contains 3 or 4 of the N_i, each with multiplicity 1, and a fibre of type I_2^* contains 4 of the N_i with multiplicity 1 and one with multiplicity 2.*

Proof. – We recall first of all that the multiple fibres of an elliptic pencil are of type mI_k, $k \geqslant 0$ ([1], Chapter V, §7) and that the multiple fibres of an elliptic pencil on an Enriques

surface are precisely two double fibres ([1], Chapter VIII). The nodal curves $N_1, \ldots, N_7$ are contained in fibres of $|\widetilde{F}_i|$ for $i = 1, 2, 3$. For every singular fibre F_s of $|\widetilde{F}_i|$, we denote by $r(F_s)$ the number of irreducible curves contained in F_s and different from $N_1, \ldots, N_7$. Since the subspace orthogonal to the class of $\widetilde{F}_i$ in $H^2(Y, \mathbb{Q})$ has dimension 9, Zariski's Lemma implies that $8 + \sum_{F_s}(r(F_s) - 1) \leqslant 9$, namely $r(F_s) \leqslant 2$ for every singular fibre F_s of $|\widetilde{F}_i|$ and there is at most one singular fibre F_s with $r(F_s) = 2$. This shows that the possible types are $_mI_2$, $_mI_3$, $_mI_4$, I_0^*, I_1^*, I_2^*, III, IV and that, except possibly one, the fibres with reducible support are of type $_mI_2$, III or I_0^*. On the other hand, we have $12 = c_2(Y) = \sum_{F_s \text{ singular}} e(F_s)$, hence the quantity $\sum_{F_s \text{ reducible}} e(F_s)$ is $\leqslant 12$. Using this remark and the fact that the 7 curves $N_1, \ldots, N_7$ are contained in fibres of $|\widetilde{F}_i|$ it is easy to show that types $_mI_3$, I_1^*, III and IV cannot occur and that the fibres cannot all be of type $_mI_2$ or $_mI_4$. This proves (ii).

Now assume that, say, $|\widetilde{F}_1|$ has a fibre G_1 of type I_4 (or $_2I_4$). Then G_1 contains two of the N_i, hence we can write the support of G_1 as $C_1 + C_2 + N_1 + N_2$, where $C_1C_2 = 0$, $C_iN_j = 1$. Assume that $C_1\widetilde{E}_2 = 0$. Then the connected fundamental cycle $C_1 + N_1 + N_2$ is contained in a reducible fibre G_2 of $|\widetilde{F}_2|$. Since $C_1N_i = 0$ for $i > 2$, the fibre G_2 is necessarily of type I_4 (or $_2I_4$). Since $r(G_1) = r(G_2) = 2$ by the above discussion and by (ii) it follows that both $|\widetilde{F}_1|$ and $|\widetilde{F}_2|$ have fibres G_1', respectively G_2', of type I_0^* and that the nodal curves appearing with multiplicity 1 in G_1' and G_2' are a subset of $\{N_3, \ldots, N_7\}$, contradicting Lemma 4.8. This shows that the intersection numbers $C_1\widetilde{E}_2, C_1\widetilde{E}_3, C_2\widetilde{E}_2, C_2\widetilde{E}_3$ are all strictly positive. Since $4 = (\widetilde{E}_2 + \widetilde{E}_3)\widetilde{F}_1 = (\widetilde{E}_2 + \widetilde{E}_3)G_1 \geqslant C_1\widetilde{E}_2 + C_1\widetilde{E}_3 + C_2\widetilde{E}_2 + C_2\widetilde{E}_3$, these numbers are all equal to 1. So the class of $C_1 - C_2$ is orthogonal to $N_1, \ldots, N_7, \widetilde{E}_1, \widetilde{E}_2, \widetilde{E}_3$. Since the classes of $N_1, \ldots, N_7, \widetilde{E}_1, \widetilde{E}_2, \widetilde{E}_3$ are a basis of $H^2(Y, \mathbb{Q})$, the class $C_1 - C_2$ is numerically equivalent to 0. On the other hand, we have $(C_1 - C_2)^2 = -4$, a contradiction. This finishes the proof of (i).

Statement (iii) follows by examining the admissible types of fibres, recalling that $r(F_s) \leqslant 2$ for every singular fibre F_s. $\quad\square$

LEMMA 4.10. – *Assume that there exists a nodal curve $C \subset Y$ such that*

$$\widetilde{E}_1C = N_1C = \cdots = N_7C = 0.$$

Then we have case (1) *of Theorem* 4.6.

Proof. – By [17, Lemma 4.2], the divisor $C + N_1 + \cdots + N_7$ is divisible by 2 in $\mathrm{Pic}(Y)$. Hence $C\widetilde{E}_i$ is even for $i = 1, 2, 3$. The curve C is contained in a fibre of $|\widetilde{F}_1|$, hence

$$C(\widetilde{E}_2 + \widetilde{E}_3) = C\widetilde{B} \leqslant \widetilde{B}\widetilde{F}_1 = 4.$$

On the other hand, since $\widetilde{B} + N_1 + \cdots + N_7$ is also even, the divisor $\widetilde{B} + C$ is even and so $(\widetilde{B} + C)^2 = 4 + 2\widetilde{B}C$ is divisible by 8. Hence we have $\widetilde{B}C = 2$. From $2 = \widetilde{B}C = C\widetilde{E}_2 + C\widetilde{E}_3$, it follows, say, $C\widetilde{E}_2 = 0$, $C\widetilde{E}_3 = 2$. Now, as in the proof of Lemma 4.9, we consider the contributions to $c_2(Y)$ and to the Picard number of Y of the various types of singular fibres. Since there are 8 disjoint nodal curves contained in the fibres of $|\widetilde{F}_1|$ and $|\widetilde{F}_2|$, one sees that the only possibility is that the fibres with singular support of both pencils are two fibres of type I_0^* and that each fibre of type I_0^* contains four of the curves $N_1, \ldots, N_7, C$, each with multiplicity 1. Recall that an elliptic pencil with 2 fibres of type I_0^* on an Enriques surface is isotrivial.

By Lemma 4.8 we can label the curves N_i in such a way that the singular fibres of $|\widetilde{F}_1|$ are $N_1 + N_2 + N_3 + N_4 + 2A_1$ and $N_5 + N_6 + N_7 + C + 2A_2$ and the singular fibres of $|\widetilde{F}_2|$ are $N_1 + N_2 + N_5 + N_6 + 2B_1$ and $N_3 + N_4 + N_7 + C + 2B_2$.

Computing intersection numbers, one sees that $\widetilde{E}_3 \sim_{\mathrm{num}} \widetilde{E}_1 + \widetilde{E}_2 - C$, namely

$$\widetilde{E}_3 \equiv \widetilde{E}_1 + \widetilde{E}_2 - C \quad \text{or} \quad \widetilde{E}_3 \equiv \widetilde{E}_1 + \widetilde{E}_2 - C + K_Y.$$

Since $\widetilde{B} + C$ is even, we conclude that $\widetilde{E}_3 \equiv \widetilde{E}_1 + \widetilde{E}_2 - C$. Hence the system $|\widetilde{B}|$ is equal to $|\widetilde{F}_1 + \widetilde{F}_2 - C|$ and it has degree 2 by [8, Theorem 4.7.2]. Looking at the adjunction sequence for C, one gets $0 = h^0(C + K_Y) = h^0(\widetilde{E}_1 + \widetilde{E}_2 - \widetilde{E}_3') = 0$, hence $|\widetilde{B} + K_Y|$ is birational again by [8, Theorem 4.7.2].

Now $\widetilde{F}_3 \equiv \widetilde{F}_1 + \widetilde{F}_2 - 2C \equiv N_5 + N_6 + N_7 + 2A_2 + N_3 + N_4 + N_7 + 2B_2$, hence $N_3 + N_4 + N_5 + N_6 + 2(A_2 + N_7 + B_2)$ is a fibre of $|\widetilde{F}_3|$ of type I_2^*. Thus Lemma 4.9 and the formula $12 = c_2(Y) = \sum_{F_s} e(F_s)$ imply that the remaining fibres with singular support are two fibres of type I_2 or $_2I_2$. □

Proof of Theorem 4.6. – By Lemma 4.10, we may assume that every nodal curve disjoint from $N_1, \ldots, N_7$ satisfies $C\widetilde{E}_i > 0$ for $i = 1, 2, 3$. Hence none of the pencils $|\widetilde{F}_i|$ can have two fibres of type I_0^*. By Lemma 4.9 and its proof one sees that in principle the possible configurations of fibres with reducible support are:

  (i) one fibre of type I_0^* and three fibres of type I_2 or $_2I_2$;
  (ii) one fibre of type I_2^* and two fibres of type I_2 or $_2I_2$.
Notice that in both cases all the fibres with singular support are reducible.

Assume that, say, $|\widetilde{F}_1|$ has a fibre G_1 of type I_2^*. By Lemma 4.9, (iii), G_1 contains five of the N_i and two more components A_2 and A_3. Each of the curves A_2 and A_3 meets 3 of the N_i and there is only one of the N_i that intersects both. We set $\lambda_2 = 2A_2 + \sum_i (A_2 N_i) N_i$ and $\lambda_3 = 2A_3 + \sum_i (A_3 N_i) N_i$. One has $\lambda_2^2 = \lambda_3^2 = -2$ and $G_1 = \lambda_2 + \lambda_3$. Since

$$2 = \widetilde{E}_2 G_1 = 2\widetilde{E}_2(A_2 + A_3),$$

we may assume $\widetilde{E}_2 A_2 = 1$, $\widetilde{E}_2 A_3 = 0$. Since by Corollary 3.3 the curves $N_1, \ldots, N_7$ are the only nodal curves orthogonal to $\widetilde{E}_1, \widetilde{E}_2, \widetilde{E}_3$, we have also $\widetilde{E}_3 A_3 = 1$, $\widetilde{E}_3 A_2 = 0$. The support of λ_3, being connected and orthogonal to $\widetilde{E}_2$, is contained in a fibre G_2 of $|\widetilde{F}_2|$. Since A_3 meets precisely three of the N_i, by Lemma 4.9 the fibre G_2 is also of type I_2^* and we can write as above $G_2 = \lambda_1 + \lambda_3$, where $\lambda_1 = 2A_1 + \sum_i (A_1 N_i) N_i$, with A_1 a nodal curve different from the N_i and such that $A_1 \widetilde{E}_3 = 0$, $A_1 \widetilde{E}_1 = 1$. Notice that the three nodal curves A_1, A_2 and A_3 are distinct. The same argument shows that λ_1 and λ_2 are contained in fibres of $|\widetilde{F}_3|$ of type I_2^*. By the proof of Lemma 4.9 each pencil $|\widetilde{F}_i|$ has at most one fibre of type I_2^*, hence λ_1 and λ_2 are contained in the same fibre G_3 and $G_3 = \lambda_1 + \lambda_2$. Assume that the curve N_i that appears with multiplicity 2 in G_1 and G_2 is the same, say N_7. Then N_7 is a component of λ_1, λ_2 and λ_3 and Lemma 4.8 implies that, up to a permutation of $1, \ldots, 6$ the incidence relations between the curves $N_1, \ldots, N_7, A_1, A_2, A_3$ are given by the dual graph of Fig. 2.

The divisor $A_1 + A_2 + A_3 + N_1 + \cdots + N_7$ is simply connected, hence its inverse image in the K3 cover of Y consists of two disjoint divisors isomorphic to it. It is easy to check that the intersection matrix of the components of these divisors is nondegenerate of type $(2, 18)$, but this contradicts the Index Theorem. So, up to a permutation of the indices, G_1 contains N_1 with multiplicity 2, G_2 contains N_2 with multiplicity 2 and G_3 contains N_3 with multiplicity 2. Using Lemma 4.8 again, one shows that, up to a permutation of the indices, N_7 is not contained in G_1, G_2, G_3 and the incidence relations between $A_1, A_2, A_3, N_1, \ldots, N_6$ are given by the dual graph in Fig. 1. Computing intersection numbers, one checks that $\widetilde{B}$ and the divisor $\lambda_1 + \lambda_2 + \lambda_3$ are numerically equivalent. The argument used to prove Corollary 4.5 shows that they are not
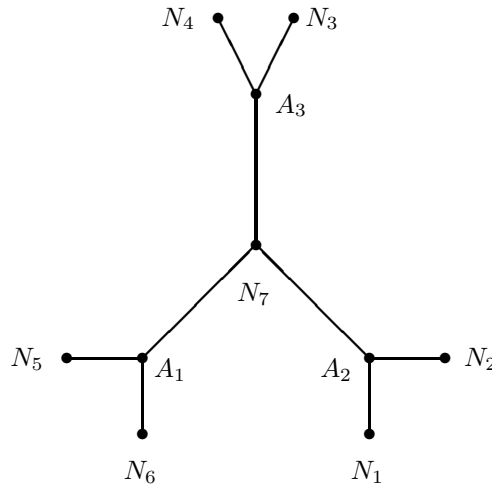
Fig. 2.

linearly equivalent, hence $\lambda_1 + \lambda_2 + \lambda_3 \equiv \widetilde{B} + K_Y$. Now the system $|\lambda_1 + \lambda_2 + \lambda_3|$ has degree 2 by Theorem 7.2 of [7] and $|\widetilde{B}|$ is birational. This settles case (2).

We are left with the case in which each of the pencils $|\widetilde{F}_i|$ has a fibre of type I_0^* and 3 fibres of type I_2 or $_2 I_2$. We recall that by [7, Proposition 5.2.1] and [8, Theorem 4.7.2] $|\widetilde{B}|$ has degree 2 if $|\widetilde{E}_1 + \widetilde{E}_2 - \widetilde{E}_3|$ is nonempty and it is birational otherwise. Assume that there is $\Delta \in |\widetilde{E}_1 + \widetilde{E}_2 - \widetilde{E}_3|$. Since $\widetilde{E}_1 \Delta = \widetilde{E}_2 \Delta = 0$ and $\widetilde{E}_3 \Delta = 2$, all the components of Δ are nodal curves contained in fibres of $|\widetilde{F}_1|$ and $|\widetilde{F}_2|$ and there is a component θ of Δ with $\theta \widetilde{E}_3 > 0$, but this cannot happen because of the configuration of reducible fibres of the pencils $|\widetilde{F}_i|$. Since this argument is purely numerical it shows also that $|\widetilde{B} + K_Y|$ is birational. $\square$

5. Enriques surfaces with 7 nodes: a general construction

Here we describe a construction giving pairs (Σ, B) as in the set-up of §3 and we prove that all such pairs can be obtained that way.

Construction 5.1. – Consider the following automorphisms of the projective line $\mathbb{P}^1$:

$$(x_0, x_1) \overset{e_1}{\mapsto} (x_0, -x_1); \qquad (x_0, x_1) \overset{e_2}{\mapsto} (x_1, x_0).$$

The subgroup Γ generated by e_1 and e_2 is isomorphic to $\mathbb{Z}_2^2$; we set $e_3 := e_1 + e_2$. The action of Γ can be lifted to the line bundle $\mathcal{O}_{\mathbb{P}^1}(2)$. Two such liftings differ by a character of Γ, hence for any chosen lifting a basis of eigenvectors of $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))$ is given by:

$$s(x_0, x_1) := x_0^2 + x_1^2, \quad d(x_0, x_1) := x_0^2 - x_1^2, \quad p(x_0, x_1) = x_0 x_1.$$

Denote by G the subgroup of automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ generated by the elements:

$$(e_1, e_1, 1), \quad (e_1, 1, e_1), \quad (e_2, e_2, e_2).$$

The group G is isomorphic to $\mathbb{Z}_2^3$. We denote by G_0 the subgroup of index 2 generated by $(e_1, e_1, 1)$ and $(e_1, 1, e_1)$. The fixed locus of the nonzero elements of G_0 has dimension 1, while the fixed locus of the elements of $G \setminus G_0$ has dimension 0.

Notice that, although the action of Γ on $\mathbb{P}^1$ does not lift to a linear representation on the space $H^0(\mathcal{O}_{\mathbb{P}^1}(1))$, the action of G on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is induced by a linear representation on $H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1}(1,1,1))$. Hence it is possible to lift the G-action to the line bundle $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1}(1,1,1)$ and, compatibly, to all its multiples. Notice also that the possible G-actions on $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1}(1,1,1)$ differ by a character of G, and thus they all induce the same action on $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1}(2,2,2)$. Denoting the homogeneous coordinates on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ by $x = (x_0, x_1)$, $y = (y_0, y_1)$, $z = (z_0, z_1)$, under this action the space $H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1}(2,2,2))$ decomposes into eigenspaces as follows:

$$T_0 := \big\langle s(x)s(y)s(z), s(x)d(y)d(z), d(x)s(y)d(z), d(x)d(y)s(z), p(x)p(y)p(z) \big\rangle;$$

$$T_1 := \big\langle s(x)s(y)d(z), s(x)d(y)s(z), d(x)s(y)s(z), d(x)d(y)d(z) \big\rangle;$$

$$T_2 := \big\langle s(x)s(y)p(z), p(x)p(y)s(z), d(x)d(y)p(z) \big\rangle;$$

$$T_3 := \big\langle s(x)p(y)p(z), p(x)s(y)s(z), p(x)d(y)d(z) \big\rangle;$$

$$T_4 := \big\langle p(x)s(y)d(z), p(x)d(y)s(z), d(x)p(y)p(z) \big\rangle;$$

$$T_5 := \big\langle s(x)p(y)s(z), p(x)s(y)p(z), d(x)p(y)d(z) \big\rangle;$$

$$T_6 := \big\langle s(x)d(y)p(z), p(x)p(y)d(z), d(x)s(y)p(z) \big\rangle;$$

$$T_7 := \big\langle s(x)p(y)d(z), p(x)d(y)p(z), d(x)p(y)s(z) \big\rangle.$$

The subspace T_0 corresponds to the trivial character and the subspace T_1 corresponds to the character orthogonal to G_0. The system $|T_0|$ is base point free, while for $i > 0$ the base locus of the system $|T_i|$ is nonempty and it contains the fixed locus of some element of $G \setminus G_0$.

Let $Z \in |T_0|$ be general. By Bertini's theorem Z is a smooth surface. The elements $(e_1, 1, e_1)$, $(1, e_1, e_1)$ and $(e_1, e_1, 1)$ act on Z fixing 8 points each and the remaining nonzero elements of G act freely on Z. By the adjunction formula, Z is a K3 surface, hence the quotient surface Z/G is a nodal Enriques surface with 6 nodes. For $i = 1, 2, 3$, the projection on the ith factor $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ induces an elliptic pencil on Z, which in turn gives an elliptic pencil $|F_i| = |2E_i|$ on Z/G. A standard argument shows that for $i \neq j$ one has $E_i E_j = 1$.

Assume now that Z has 8 nodes that form a G-orbit and no other singularities. Then the quotient surface Z/G has an extra node, which is the image of the 8 nodes of Z. By Lemma 3.4 either $E_1 + E_2 + E_3$ or $E_1 + E_2 + E_3 + K_{Z/G}$ is a divisor B as in the set-up of §3.

Set $\Sigma := Z/G$ and denote by $\pi : Z \to \Sigma$ the quotient map. Notice that $\pi^* B \cong \pi^*(B + K_\Sigma)$ is isomorphic to $\mathcal{O}_Z(2,2,2)$ and $\pi^* H^0(\Sigma, B)$ and $\pi^* H^0(\Sigma, B + K_\Sigma)$ are eigenspaces of $H^0(Z, \mathcal{O}_Z(2,2,2))$. So, considering the dimensions, they correspond to the restrictions to Z of T_0 and T_1. We will show later (Lemma 6.3) that the restriction of T_0 is equal to $\pi^* H^0(\Sigma, B + K_\Sigma)$ and the restriction of T_1 is equal to $\pi^* H^0(\Sigma, B)$.

The fact that Construction 5.1 can actually be performed, namely that there exists Z as required, is a consequence of the following theorem and of the examples given in §3.

THEOREM 5.2. – *Let (Σ, B) be a pair as in the set-up of* §3. *Then (Σ, B) can be obtained from Construction* 5.1.

Proof. – Let V be the code associated with the nodes of Σ, which is isomorphic to $\mathbb{Z}_2^2$ by Proposition 4.4. By [9], Proposition 2.1 and Remark 2, there is a Galois cover $\pi_0 : Z_0 \to \Sigma$ with Galois group $\mathrm{Hom}(V, \mathbb{C}^*) \cong \mathbb{Z}_2^2$ branched on the 6 nodes of Σ that appear in V. The map $Z_0 \to \Sigma$

can be factorized as $Z_0 \to Z_1 \to \Sigma$, where both maps are double covers branched on a set of 4 nodes. By [17, Proposition 3.1], Z_1 is a nodal Enriques surface with 6 nodes, hence, by ib., Z_0 is an Enriques surface with 4 nodes. Let $K \to \Sigma$ be the K3-cover of Σ and consider the following cartesian diagram:

(5.1)
$$
\begin{array}{ccc}
Z & \longrightarrow & Z_0 \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi_0} \\
K & \longrightarrow & \Sigma
\end{array}
$$

The surface Z is a K3 surface with 8 nodes and the map $Z \to Z_0$ is the K3-cover of Z_0. The composite map $\pi : Z \to \Sigma$ is a Galois cover with Galois group isomorphic to $\mathbb{Z}_2^3$. Notice (cf. [9], proof of Proposition 2.1 and Remark 2) that, although the cover $\pi_0 : Z_0 \to \Sigma$ is not uniquely determined (in fact there are four different possibilities), the cover $\pi : Z \to \Sigma$ does not depend on the choice of Z_0.

For the reader's convenience the proof is broken into steps.

Step 1: *For $i = 1, 2, 3$ there exist elliptic pencils $|C_i|$ on Z such that $\pi^* F_i \equiv 4C_i$.*

Since π is unramified in codimension 1, if $F_i \in |F_i|$ is general then $\pi^* F_i$ is a disjoint union of linearly equivalent elliptic curves. Hence, to prove the statement it is enough to show that $\pi^* F_i$ has 4 connected components. Let $\tilde{\pi} : \widetilde{Z} \to Y$ the Galois cover obtained from π by taking base change with the minimal desingularization $Y \to \Sigma$ and, as usual, denote by $|\widetilde{F_i}|$ the elliptic pencil of Y induced by $|F_i|$. By Theorem 4.6 the pencil $|\widetilde{F_i}|$ has a fibre G_i of type I_0^* or I_2^*. We write $G_i = 2A_i + N_1^i + \cdots + N_4^i$. By the results of §4, the nonzero elements of V correspond to the even sets $N_1^i + \cdots + N_4^i$, $i = 1, 2, 3$. So by the definition of Z and $\widetilde{Z}$ (cf. also [9, §2]) we have the following formula:

$$
\tilde{\pi}_* \mathcal{O}_{\widetilde{Z}} = \mathcal{O}_Y \oplus K_Y \oplus \left(\bigoplus_{i=1,2,3} \mathcal{O}_Y(A_i - \widetilde{E}_i) \right) \oplus \left(\bigoplus_{i=1,2,3} \mathcal{O}_Y(A_i - \widetilde{E}'_i) \right).
$$

The restriction of the line bundles $\mathcal{O}_Y(-K_Y)$, $\mathcal{O}_Y(A_i - \widetilde{E}_i)$ and $\mathcal{O}_Y(A_i - \widetilde{E}'_i)$ to a general $\widetilde{F}_i$ is trivial, hence $\tilde{\pi}^* \widetilde{F}_i$ has at least 4 connected components. So for $i = 1, 2, 3$ we can write

$$
\tilde{\pi}^* F_i \equiv m_i \widetilde{C}_i
$$

where $\widetilde{C}_i$ is a smooth connected elliptic curve and $m_i = 4$ or $m_i = 8$. Notice that $\widetilde{C}_i \widetilde{C}_j \geqslant 2$ for $i \neq j$, since otherwise the product of the pencils $|\widetilde{C}_i|$ and $|\widetilde{C}_j|$ would give a birational map $\widetilde{Z} \to \mathbb{P}^1 \times \mathbb{P}^1$. On the other hand, for $i \neq j$ we have $32 = \tilde{\pi}^* \widetilde{F}_i \tilde{\pi}^* \widetilde{F}_j = m_i m_j \widetilde{C}_i \widetilde{C}_j \geqslant 16 \widetilde{C}_i \widetilde{C}_j$, hence $m_i = m_j = 4$ and $\widetilde{C}_i \widetilde{C}_j = 2$. Finally, the pencils $|\widetilde{C}_i|$ induce pencils $|C_i|$ on Z such that $\pi^* F_i \equiv 4C_i$.

Step 2: *The product of the pencils $|C_1|$, $|C_2|$ and $|C_3|$ defines an embedding*

$$
\psi : Z \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1
$$

such that $\psi(Z)$ is a divisor of type $(2, 2, 2)$.

We remark that $D = C_1 + C_2 + C_3$ is ample on Z, since by Step 1 $2D = \pi^* B$ and B is ample on Σ by Corollary 3.3.

We let $\epsilon : W \to Z$ be the minimal desingularization, we denote by $N'_1, \ldots, N'_8$ the exceptional curves of ϵ and we set $\widetilde{D} := \epsilon^* D$, $\widetilde{C}_i := \epsilon^* C_i$. The divisor $\widetilde{D}$ is nef and big and the N'_i are the only irreducible curves that have intersection equal to 0 with $\widetilde{D}$. Since $\widetilde{D}^2 = 12$, by Reider's

Theorem and by the fact that the intersection form on a K3 surface is even, if two points $x, y \in W$ are not separated by $|\widetilde{D}|$, then there exists an effective connected divisor $A \ni x, y$ such that either: (a) $A^2 = -2$, $A\widetilde{D} = 0$, or (b) $A^2 = 0$, $A\widetilde{D} = 1$ or (c) $A^2 = 0$, $A\widetilde{D} = 2$. Possibility (a) corresponds to the case when both x and y belong to one of the curves N_i'. Recall that the pencils $|\widetilde{C}_i|$ have no multiple fibres, because the double fibres of the pencils $|\widetilde{F}_i|$ disappear when one takes the K3 cover (actually, it is not hard to prove that any elliptic fibration on a K3 has no multiple fibres). In case (b) one would have, say, $A\widetilde{C}_1 = A\widetilde{C}_2 = 0$, namely A would be a fibre of both $|\widetilde{C}_1|$ and $|\widetilde{C}_2|$, which is impossible. If $A^2 = 0$ and $A\widetilde{D} = 2$, then we have, say, $A\widetilde{C}_1 = 0$ and A is a fibre of $|\widetilde{C}_1|$. But in this case $A\widetilde{D} = 4$, a contradiction.

The above discussion shows that the map ψ is one-to-one onto its image and that the differential of ψ at every smooth point of Σ is nonsingular. In particular the image of ψ is an hypersurface with at most isolated singularities, hence it is normal. It follows that ψ is an isomorphism. The fact that the image is a divisor of type $(2, 2, 2)$ is a consequence of the fact that $C_i C_j = 2$ if $i \neq j$.

Step 3: *There exist coordinates on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ such that the surface $\psi(Z)$ is an element of the linear system $|T_0|$ defined in Construction 5.1 and the action of the Galois group of $Z \to \Sigma$ coincides with the group action defined there.*

Denote by G the Galois group of π. By the definition of the map ψ, the three copies of $\mathbb{P}^1$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ can be naturally identified with (the dual of) the linear systems $|C_1|$, $|C_2|$ and $|C_3|$, hence G acts on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and the embedding $\psi: Z \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is G-equivariant with respect to the given actions. We have seen in Step 1 that for every $i = 1, 2, 3$ there is a nonzero $g_i \in G$ such that g_i acts trivially on $|C_i|$. Since the fixed locus of g_i on $|C_1| \times |C_2| \times |C_3|$ has positive dimension and Z is ample, g_i has fixed points on Z. Since by construction the cover $\pi: Z \to \Sigma$ factorizes through the K3 cover $K \to \Sigma$, it follows that g_1, g_2, g_3 do not generate G. On the other hand g_i must act nontrivially on $|C_j|$ for $j \neq i$, since otherwise the fixed locus of g_i on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ would be a divisor and g_i would fix a curve of Z pointwise. Hence $G_0 := \{1, g_1, g_2, g_3\}$ is a subgroup of G isomorphic to $\mathbb{Z}_2^2$. Fix $h \in G \setminus G_0$. For every i we can choose homogeneous coordinates on $\mathbb{P}^1 = |C_i|$ such that, using the notation of Construction 5.1, the nonzero element of G_0/g_i acts as e_1 and h acts as e_2. With respect to these coordinates we have: $g_1 = (1, e_1, e_1)$, $g_2 = (e_1, 1, e_1)$, $g_3 = (e_1, e_1, 1)$, $h = (e_2, e_2, e_2)$, namely the G-action on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is the same as in Construction 5.1 and the surface Z, being G-invariant, belongs to one of the linear systems $|T_i|$, $i = 0, \ldots, 7$. In addition, each of the nonzero elements of G_0 fixes 8 points of Z and the elements of $G \setminus G_0$ act freely on Z. This is the same as saying that Z is in general position with respect to the fixed loci of all the elements, hence, as we have remarked in Construction 5.1, Z must be an element of $|T_0|$. $\square$

6. Enriques surfaces with 7 nodes: a parametrization

The aim of this section is to construct a quasi-projective variety parametrizing the isomorphism classes of pairs (Σ, B) as in the set up of §3 and to study the geometry of this space. In addition, we show the existence of a tautological family on a finite Galois cover of the parametre space. This tautological family admits a section and a simultaneous resolution.

These results are used in the next section to describe the subset of the moduli space of surfaces with $p_g = 0$ and $K^2 = 3$ consisting of the surfaces S that have an involution σ such that: (1) the quotient surface S/σ is birational to an Enriques surface; (2) the bicanonical map φ of S is composed with σ.

We use all the notation from the previous sections. For a pair (Σ, B) as in the set-up of §3 we denote as usual by $\eta: Y \to \Sigma$ the minimal desingularization and by $N_1, \ldots, N_7$ the exceptional

curves of η. In addition we assume that N_7 is the nodal curve that does not appear in the code V associated to $N_1, \ldots, N_7$ (cf. §4).

Denote by $\mathcal{N}$ the subset of $|T_0|$ consisting of the surfaces Z that satisfy the following conditions:

(a) Z is in general position with respect to the fixed loci of the elements of G;

(b) Z has 8 nodes that form a G-orbit and no other singularities.

The set $\mathcal{N}$ is clearly open in the set of singular surfaces of $|T_0|$ and it is nonempty by the results of the previous section, hence it is a quasi-projective variety of dimension 3. We denote by $\mathcal{I} \subset (\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) \times \mathcal{N}$ the incidence variety, consisting of the pairs (P, Z) such that P is a singular point of Z and we denote by p_1, p_2 the projections of $\mathcal{I}$ onto the two factors. There is a natural G-action on $\mathcal{I}$, which is free by the definition of $\mathcal{N}$, and the map $p_2 \colon \mathcal{I} \to \mathcal{N}$ is the quotient map with respect to this G-action.

The first goal of this section is to study the geometry of $\mathcal{N}$. We have the following:

THEOREM 6.1. – *The variety $\mathcal{N}$ is smooth, irreducible of dimension 3 and unirational.*

The proof that $\mathcal{N}$ is smooth is completely elementary (cf. Lemma 6.2 below), but proving the irreducibility requires a series of intermediate results.

LEMMA 6.2. – *The variety $\mathcal{N}$ is smooth of dimension 3.*

Proof. – Since the incidence variety $\mathcal{I}$ is a topological covering of $\mathcal{N}$, it is enough to prove that $\mathcal{I}$ is smooth of dimension 3. This can be easily seen by means of a local computation, using the fact that the linear system $|T_0|$ has no base points and the fact that for a pair (P, Z) in $\mathcal{I}$ the point P is an ordinary double point of Z. $\square$

LEMMA 6.3. – *Assume that the pair (Σ, B) is obtained from $Z \in \mathcal{N}$ using Construction 5.1 and let $\pi \colon Z \to \Sigma$ be the quotient map. Then*:

$$\pi^* H^0(\Sigma, B) = T_1|_Z; \qquad \pi^* H^0(\Sigma, B + K_\Sigma) = T_0|_Z.$$

Proof. – We have already remarked in Construction 5.1 that $H^0(\Sigma, B)$ and $H^0(\Sigma, B + K_\Sigma)$ pull back to $T_0|_Z$ and $T_1|_Z$. So we only need to decide which is which.

Let $s \in T_0|_Z$ be general, let D be the divisor of zeros of s and let $\overline{D}$ be the image of D in Σ. The divisor $\overline{D}$ is smooth and it is numerically equivalent to B. Let $f \colon X \to Z$ be the double cover branched on D. Denote by L the total space of the line bundle $\mathcal{O}_Z(1,1,1)$, by $p \colon L \to Z$ the projection and by z the tautological section of $p^* \mathcal{O}_Z(1,1,1)$. Then X is isomorphic to the hypersurface $\{z^2 - p^* s = 0\} \subset L$ and the G-action on L (cf. Construction 5.1) preserves X. Hence the G-action on Z lifts to X and we have a commutative diagram:

(6.1)
$$\begin{array}{ccc} X & \longrightarrow & Z \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle \pi} \\ \overline{X} & \longrightarrow & \Sigma \end{array}$$

where $q \colon X \to \overline{X} := X/G$ is the quotient map. By commutativity of the diagram, the map $\overline{X} \to \Sigma$ is a double cover branched on $\overline{D}$ and on a subset of the 6 nodes of Σ that are the images of the fixed points of the G-action on Z. As before, let $\eta \colon Y \to \Sigma$ be the minimal desingularization of Σ. Set $\widetilde{D} = \eta^* \overline{D}$ and denote by $N_1, \ldots, N_k$ the nodal curves of Y corresponding to the nodes of Σ where $\overline{X} \to \Sigma$ ramifies. The class of $\widetilde{D} + N_1 + \cdots + N_k$ is divisible by 2 in $\mathrm{Pic}(X)$, hence its self-intersection, which is equal to $6 - 2k$, is divisible by 8. Since $k \leqslant 6$, it follows that

$k = 3$. If $\widetilde{D}$ were linearly equivalent to $\widetilde{B}$, then $N_4 + N_5 + N_6 + N_7$ would be divisible by 2 in $\mathrm{Pic}(Y)$, contradicting the fact that N_7 does not appear in the code V associated with the curves $N_1, \ldots, N_7$. So we must have $\widetilde{D} \equiv \widetilde{B} + K_Y$, and thus $T_0|_Z = \pi^* H^0(\Sigma, B + K_\Sigma)$. $\square$

LEMMA 6.4. – *The curve N_7 is not contained in a double fibre of $|\widetilde{F}_i|$, for $i = 1, 2, 3$.*

Proof. – Assume by contradiction that N_7 is contained in a double fibre of, say, $|\widetilde{F}_1|$. Then, by Theorem 4.6, N_7 is contained in a fibre $2A$ of $|\widetilde{F}_1|$ with A of type I_2. The cover $\tilde{\pi} : \widetilde{Z} \to Y$ obtained from $\pi : Z \to \Sigma$ by taking base change with $\eta : Y \to \Sigma$ is étale over A. More precisely, by Step 1 of the proof of Theorem 5.2 the divisor $\tilde{\pi}^{-1}(A)$ is the disjoint union of 2 connected curves, each mapping to A with Galois group $\mathbb{Z}_2^2$, but this is impossible since the fundamental group of A is cyclic. $\square$

LEMMA 6.5. – *Assume that $|K_\Sigma + B|$ is birational and let $\psi : \Sigma \to \mathbb{P}^3$ be the corresponding morphism. Set $\Sigma' := \Sigma \setminus (E_1 \cup E_1' \cup \cdots \cup E_3')$.*
Then the restricted map $\psi|_{\Sigma'} : \Sigma' \to \psi(\Sigma')$ is an isomorphism.

Proof. – The map ψ is a morphism onto a sextic of $\mathbb{P}^3$. The divisor B is ample by Corollary 3.3, hence $K_\Sigma + B$ is also ample and $\psi : \Sigma \to \psi(\Sigma)$ is the normalization map. For $i = 1, 2, 3$, the supports E_i, E_i' of the double fibres of $|F_i|$ are mapped 2-to-1 onto distinct lines L_i, L_i' which are double for $\psi(\Sigma)$. The general curve of $|K_\Sigma + B|$ is smooth of genus 4, hence the general section C of $\psi(\Sigma)$ has geometric genus 4. Since C has arithmetic genus 10 and it has at least 6 singular points $C \cap L_1, \cdots, C \cap L_3'$, it follows that $L_1, \ldots, L_3'$ are the only 1−dimensional components of the singular locus of $\psi(\Sigma)$. Since $K_\Sigma + B \equiv E_1 + E_2 + E_3' \equiv \cdots \equiv E_1' + E_2' + E_3'$, the inverse image of $\psi(\Sigma) \setminus (L_1 \cup \cdots \cup L_3')$ is Σ'. The surface $\psi(\Sigma') = \psi(\Sigma) \setminus (L_1 \cup \cdots \cup L_3')$ is normal, since it is an hypersurface and it is smooth in codimension 1. It follows that the map $\psi|_{\Sigma'} : \Sigma' \to \psi(\Sigma')$ is an isomorphism. $\square$

We denote by $\mathcal{N}_0 \subset \mathcal{N}$ the set of surfaces Z such that $|T_0|$ induces a birational map $Z/G \to \mathbb{P}^3$. By Lemma 6.3 $Z \in \mathcal{N}_0$ if and only if the system $|K_\Sigma + B|$ is birational, where (Σ, B) is the pair obtained from Z by Construction 5.1. The set $\mathcal{N}_0$ is open in $\mathcal{N}$.

PROPOSITION 6.6. – *The set $\mathcal{N}_0$ is dense in $\mathcal{N}$.*

Proof. – We only give an outline of the proof, which otherwise would be quite lengthy. The reader can easily fill in the missing details.

Let $Z \in \mathcal{N}$ be a point. Denote by (Σ, B) the pair obtained from Z by Construction 5.1 and denote by $Z \to \Sigma$ the G-cover defined in the proof of Theorem 5.2. Let Y be the minimal desingularization of Σ, let $N_1, \ldots, N_7$ be the corresponding nodal curves on Y and let $\widetilde{Z} \to Y$ be the G-cover obtained from $\pi : Z \to \Sigma$ by taking base change with η. As in Example 3.7, consider the restriction $\mathcal{Y}_1 \to (U_1, 0)$ of the Kuranishi family of Y to the subset where the classes of $N_1, \ldots, N_7$ stay effective. We recall that the base of the Kuranishi family of Y is smooth of dimension 10 by [1, Theorem VIII.19.3]. Thus the set U_1 is smooth of dimension 3 by [4, Theorem 2.14].

After possibly shrinking U_1, one can construct by standard arguments a G-cover $\widetilde{\mathcal{Z}} \to \mathcal{Y}_1$ that specializes to $\widetilde{Z} \to Y$ over the point $0 \in U_1$. The induced family of surfaces $\widetilde{\mathcal{Z}} \to U_1$ is smooth. Furthermore, after possibly shrinking U_1 again, the proof of Theorem 5.1 can be easily modified to show the existence of a G-equivariant map $\Psi \; \widetilde{\mathcal{Z}} \to U_1 \times (\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$ over U_1 that extends the birational map $\widetilde{Z} \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ obtained by composing $\widetilde{Z} \to Z$ with the embedding $\psi : Z \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ defined in Step 2 of the proof of Theorem 5.1. For every $t \in U_1$ the surface $\Psi(\widetilde{\mathcal{Z}}_t)$ belongs to the linear system $|T_0|$, and hence, for small $t \in U_1$, to $\mathcal{N}$.

Assume now that $Z \notin \mathcal{N}_0$. By Lemma 6.3, the system $|B + K_\Sigma|$ is not birational. The configuration of reducible fibres of the pencils $|\widetilde{F}_i|$ on Y is described in Theorem 4.6(2). By the proof of Theorem 4.6, the components of the reducible fibres of the elliptic pencils can be labelled in such a way that their dual diagram is the one given in Fig. 1 and $F_i A_j = 2\delta_{i,j}$. In the notation of the proof of Theorem 4.6, we have

$$\lambda_1 + \lambda_2 + \lambda_3 \equiv \widetilde{B} + K_Y \quad \text{and} \quad \lambda_1 + \lambda_2 \equiv \widetilde{F}_3.$$

Then we have

$$N_1 + N_2 + N_6 + 2A_3 = \lambda_3 \equiv \widetilde{B} + K_Y - \widetilde{F}_3 \equiv \widetilde{E}_1 + \widetilde{E}_2 - \widetilde{E}_3 + K_Y.$$

We denote by $\widetilde{\mathcal{E}}_i$ the line bundle on $\mathcal{Y}_1$ that extends $\widetilde{E}_i$, $i = 1, 2, 3$, by $\widetilde{\mathcal{B}} = \widetilde{\mathcal{E}}_1 + \widetilde{\mathcal{E}}_2 + \widetilde{\mathcal{E}}_3$ the line bundle that extends B and by $\mathcal{N}_i$ the divisor that extends N_i, $i = 1, \ldots, 7$. By [7, Proposition 5.2.1] and [8, Theorem 4.7.2], the system $|K_{Y_t} + \widetilde{\mathcal{B}}_t|$ is not birational if and only if $\widetilde{\mathcal{E}}_{1,t} + \widetilde{\mathcal{E}}_{2,t} - \widetilde{\mathcal{E}}_{3,t} + K_{Y_t}$ is effective. Again by [4, Theorem 3.7], the subset U_2 of the base U of the Kuranishi family of Y where the divisors

$$\widetilde{\mathcal{E}}_{1,t} + \widetilde{\mathcal{E}}_{2,t} - \widetilde{\mathcal{E}}_{3,t} + K_{Y_t}, \quad \mathcal{N}_{3,t}, \quad \mathcal{N}_{4,t}, \quad \mathcal{N}_{5,t}, \quad \mathcal{N}_{7,t}$$

are effective is smooth of dimension 2 in a neighbourhood of $t = 0$. Hence $U_1 \setminus U_2$ is nonempty and for $t \in U_1 \setminus U_2$ the surface $\Psi(\widetilde{\mathcal{Z}}_t)$ is an element of $\mathcal{N}_0$. This proves that $\mathcal{N}_0$ is dense in $\mathcal{N}$. □

Proof of Theorem 6.1. – The variety $\mathcal{N}$ is smooth of dimension 3 by Lemma 6.2. By Lemma 6.6, to complete the proof it suffices to show that $\mathcal{N}_0$ is irreducible and unirational.

Denote by $\mathcal{I}_0$ the restriction to $\mathcal{N}_0$ of the incidence variety $\mathcal{I}$ and denote again by

$$p_1 : \mathcal{I}_0 \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \quad \text{and} \quad p_2 : \mathcal{I}_0 \to \mathcal{N}_0$$

the projections. The map p_2 is an étale G-cover by construction. We prove the theorem by showing that p_1 is injective.

Let $P \in p_1(\mathcal{I}_0)$ be a point. The fibre $p_1^{-1}(P)$ is a (nonempty) open subset of the linear subsystem of $|T_0|$ consisting of the surfaces which are singular at P. Assume that $p_1^{-1}(P)$ has positive dimension and let Z, Z' be two distinct surfaces in $p_1^{-1}(P)$, let (Σ, B) be the pair obtained from Z by Construction 5.1 and let $\eta : Y \to \Sigma$ be the minimal resolution. Recall that the image of P is the node of Σ corresponding to the nodal curve N_7 of Y that does not appear in the code V. Denote by D the divisor on $\Sigma = Z/G$ induced by the restriction of Z' to Z. By Lemma 6.3, $D \equiv B + K_\Sigma$. It is not difficult to check that the pull back $\widetilde{D}$ of D to Y vanishes on N_7 of order at least 2. Hence $h^0(Y, K_Y + \widetilde{B} - 2N_7) > 0$ and the restriction map

$$H^0(Y, K_Y + \widetilde{B} - N_7) \to H^0(N_7, \mathcal{O}_{N_7}(2))$$

is not surjective. This is a contradiction to Lemma 6.5, since Z is an element of $\mathcal{N}_0$. □

Let $\mathcal{Z} \subset (\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) \times \mathcal{N}$ be the universal family. The group G acts on $\mathcal{Z}$ preserving the fibres of $\mathcal{Z} \to \mathcal{N}$, hence we can take the quotient and obtain a family $\mathfrak{S} := \mathcal{Z}/G \to \mathcal{N}$, which is easily seen to be flat. We can also define a polarization $\mathcal{B}$ on $\mathfrak{S}$ as follows: we modify the G-action on the line bundle $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1}(2, 2, 2)$ considered in Construction 5.1 by multiplying it with the nontrivial character of G orthogonal to the subgroup G_0. The effect of this choice is

that the G-invariant sections now correspond to the subspace T_1. Denote by $Bs \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ the base locus of the system $|T_1|$. The restriction of $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1}(2,2,2)$ to $(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) \setminus Bs$ is generated at every point by global sections which are invariant for the chosen linearization, hence it is the pull back of a line bundle $\overline{\mathcal{B}}$ from the quotient $((\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) \setminus Bs)/G$. One can check that, by the definition of $\mathcal{N}$, the family $\mathcal{Z}$ is contained in $(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \setminus Bs) \times \mathcal{N}$. Hence the projection onto the first factor induces a map

$$\mathfrak{S} \to (\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \setminus Bs)/G.$$

We let $\mathcal{B}$ be the pull back of $\overline{\mathcal{B}}$ via this map. For every $t \in \mathcal{N}$ the elements of T_1 give global sections of the restriction B_t of $\mathcal{B}$ to the fibre Σ_t of $\mathfrak{S}$ at t. By Construction 5.1 and Lemma 6.3, (Σ_t, B_t) is a pair as in the set-up of §3 and by Theorem 5.2 all pairs (Σ, B) occur as (Σ_t, B_t) for some $t \in N$.

A *simultaneous resolution* of a flat family $\mathcal{S} \to U$ of normal projective surfaces is a flat family $\mathcal{Y} \to U$ with a map $\mathcal{Y} \to \mathcal{S}$ over U such that for every $t \in U$ the restricted map $Y_t \to S_t$ is the minimal resolution of the singularities of S_t.

PROPOSITION 6.7. – *The family $\mathfrak{S} \to \mathcal{N}$ admits a simultaneous resolution $\mathcal{Y} \to \mathcal{N}$.*

Proof. – To construct a resolution one can proceed as follows. First one blows up inside the family

$$\mathcal{Z} \subset \big((\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) \setminus Bs\big) \times \mathcal{N}$$

the fixed loci of the nonzero elements of G and the set of the singular points of the fibres, thus obtaining a smooth family $\mathcal{Z}'$ such that the fixed loci of the elements of G on $\mathcal{Z}'$ are divisors. Then one takes as the simultaneous resolution of $\mathfrak{S}$ the quotient $\mathcal{Y} := \mathcal{Z}'/G$. □

LEMMA 6.8. – *The family of smooth surfaces $\mathcal{Y} \to \mathcal{N}$ admits a section $Sec \subset \mathcal{Y}$.*

Proof. – Consider the family $\mathcal{Z} \subset (\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) \times \mathcal{N}$. For every $t \in \mathcal{N}$ the curve

$$C := \big\{(1,1,1,1)\big\} \times \mathbb{P}^1$$

meets Z_t transversally at two smooth points, hence it induces an étale bisection of $\mathcal{Z} \to \mathcal{N}$. The image of this bisection in $\mathfrak{S} = \mathcal{Z}/G$ is a section that intersects every fibre Σ_t at a smooth point, and its inverse image in $\mathcal{Y}$ is the required section Sec. □

We recall briefly from [20, Chapter 0, §5] (see also [10]) the main facts about relative Picard schemes.

Given a family $\mathcal{X} \to T$, one defines the relative Picard functor from the category of schemes over T to the category of sets. Given a scheme $T' \to T$, the relative Picard functor associates to T' the quotient of the group of isomorphism classes of line bundles on $\mathcal{X} \times_T T'$ by the subgroup of the classes of line bundles pulled back from T'. If $\mathcal{X} \to T$ admits a section, one can define the relative Picard functor also by taking the isomorphism classes of "normalized" line bundles, namely of the line bundles whose restriction to the pull back over T' of the given section is trivial. If $\mathcal{X} \to T$ is flat and projective with reduced irreducible fibres and it admits a section, then the relative Picard functor is represented by a group scheme $\mathrm{Pic}_{\mathcal{X}/T} \to T$. Therefore by Lemma 6.8 we can consider the scheme $\mathrm{Pic}_{\mathcal{Y}/\mathcal{N}} \to \mathcal{N}$, where $\mathcal{Y}$ is the family defined in Proposition 6.7. Denote by $\widetilde{\mathcal{B}}$ the pull back to $\mathcal{Y}$ of the line bundle $\mathcal{B}$ that we have previously defined on $\mathfrak{S}$ and denote by Ex the exceptional divisor of the simultaneous resolution $\mathcal{Y} \to \mathfrak{S}$. The line bundle $\widetilde{\mathcal{B}} \otimes \mathcal{O}_{\mathcal{Y}}(Ex)$ defines a section $b : \mathcal{N} \to \mathrm{Pic}_{\mathcal{Y}/\mathcal{N}}$. We define $\widetilde{\mathcal{N}} \subset \mathrm{Pic}_{\mathcal{Y}/\mathcal{N}}$ to be the inverse image

of $b(\mathcal{N})$ via the multiplication by 2 map $\mathrm{Pic}_{\mathcal{Y}/\mathcal{N}} \to \mathrm{Pic}_{\mathcal{Y}/\mathcal{N}}$. So $\widetilde{\mathcal{N}}$ is closed in $\mathrm{Pic}_{\mathcal{Y}/\mathcal{N}}$ and the natural map $\widetilde{\mathcal{N}} \to \mathcal{N}$ is an étale double cover. A point of $\widetilde{\mathcal{N}}$ determines a pair (Σ, B) together with a solution $L \in \mathrm{Pic}(Y)$ of the linear equivalence

$$2L \equiv \widetilde{B} + N_1 + \cdots + N_7.$$

We are finally ready to construct varieties that parametrize the isomorphism classes of pairs (Σ, B) and the isomorphism classes of triples (Σ, B, L) as above. We do this by taking the quotient of $\mathcal{N}$ and $\widetilde{\mathcal{N}}$ by a suitable finite group.

Let $St(G)$ be the subgroup of $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$ consisting of the elements γ such that $\gamma G \gamma^{-1} = G$. It is easy to verify that $St(G)$ is a finite group. The group $St(G)$ permutes the G-eigenspaces $T_0, \ldots, T_7$ of $H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1}(2, 2, 2))$. Since T_0 is the only eigenspace of dimension 5 and T_1 is the only eigenspace of dimension 4, it follows that $St(G)$ preserves T_0 and T_1. In view of this observation, it follows from the definitions given so far that $St(G)$ acts on $\mathcal{N}$, on the families $\mathcal{Z}$, $\mathfrak{S}$ and $\mathcal{Y}$ and on the line bundle $\mathcal{B}$ on $\mathfrak{S}$, and that all these actions are compatible. Clearly, the action of $St(G)$ on $\mathcal{Y}$ maps to itself the exceptional divisor Ex and therefore we also have an action of $St(G)$ on $\widetilde{\mathcal{N}}$.

THEOREM 6.9. –
(i) *The set of isomorphism classes of pairs (Σ, B) as in the set up of §3 is in one-to-one correspondence with the quasiprojective variety $\mathcal{N}/St(G)$;*
(ii) *The set of isomorphism classes of triples (Σ, B, L), where (Σ, B) is a pair as above and $L \in \mathrm{Pic}(Y)$ satisfies $2L \equiv \widetilde{B} + N_1 + \cdots + N_7$, is in one-to-one correspondence with the quasi-projective variety $\widetilde{\mathcal{N}}/St(G)$.*

Proof of Theorem 6.9. – The proof is an easy consequence of the previous results.

Consider for instance statement (i) and let (Σ, B), (Σ', B') be isomorphic pairs. Then one observes the following facts: (1) up to the order, an isomorphism of the pairs (Σ, B) and (Σ', B') necessarily maps the pencils $|F_i|$ on Σ to the pencils $|F_i'|$ on Σ', since these pencils can be recovered from the geometry of the image of the map associated to the system $|B|$, respectively $|B'|$; (2) any isomorphism $\Sigma \to \Sigma'$ lifts to an isomorphism $Z \to Z'$ of the corresponding covers, since these are defined intrinsically; (3) by (1) the isomorphism $Z \to Z'$ is given by an element of $St(G)$.

The second statement can be proven in a similar way. $\quad\square$

Since $\mathcal{N}$ is irreducible by Theorem 6.1, the variety $\mathcal{N}/St(G)$ is also irreducible. The variety $\widetilde{\mathcal{N}}$, being an étale double cover of $\mathcal{N}$, either is irreducible or it is the disjoint union of two components isomorphic to $\mathcal{N}$. We close this section by showing that, in any case, taking the quotient of $\widetilde{\mathcal{N}}$ by $St(G)$ we get an irreducible variety.

PROPOSITION 6.10. – *The variety $\widetilde{\mathcal{N}}/St(G)$ is irreducible.*

Proof. – The variety $\widetilde{\mathcal{N}}$, being an étale cover of $\mathcal{N}$, is smooth by Lemma 6.2 and thus $\widetilde{\mathcal{N}}/St(G)$ is normal. So, to prove that $\widetilde{\mathcal{N}}/St(G)$ is irreducible it suffices to show that it is connected. We do this by showing that there exist a point t in $\mathcal{N}$ and an automorphism $\gamma \in St(G)$ such that $\gamma t = t$ but γ exchanges the two points of $\widetilde{\mathcal{N}}$ lying over t. This amounts to finding a pair (Σ, B) such that there exists an automorphism h of Σ with $h^* B \equiv B$ and such that the induced automorphism of Y exchanges the two solutions in $\mathrm{Pic}(Y)$ of the relation

$$2L \equiv \widetilde{B} + N_1 + \cdots + N_7.$$

Indeed by Theorem 5.2 such a pair is isomorphic to (Σ_t, B_t) for some t. Moreover, h induces an automorphism h' of Z, since Z is defined intrinsically. The set of pencils $|F_1|, |F_2|, |F_3|$ is determined uniquely by B, hence it is preserved by h. It follows that h' is compatible with the embedding $Z \to |C_1| \times |C_2| \times |C_3|$ (cf. proof of Theorem 5.2, Step 2). In other words, if we identify $|C_1| \times |C_2| \times |C_3|$ with $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ as in the proof of Theorem 5.2, then h' is induced by an element γ of $St(G)$. The pair (Σ, B) that we construct is a special instance of Example 3.5 (cf. also [17]).

So we let $\mathbb{Z}_2^2 = \{1, e_1, e_2, e_3\}$ act on a product $D_1 \times D_2$ of elliptic curves by

$$(x, y) \overset{e_1}{\mapsto} (-x, y+b), \qquad (x, y) \overset{e_2}{\mapsto} (x+a, -y),$$

where $a \in D_1$ and $b \in D_2$ are nonzero elements of order 2. The quotient surface $\overline{\Sigma}$ is an Enriques surface with 8 nodes and has two elliptic pencils $|F_1|, |F_2|$ such that $F_1 F_2 = 4$, induced by the projections of $D_1 \times D_2$ onto the two factors. One of the double fibres of $|F_1|$ occurs over the image in $\mathbb{P}^1 = D_1/\mathbb{Z}_2^2$ of the points 0 and a and the other one occurs over the image of the remaining 2-torsion points a_1 and a_2. The fibres over the image in $\mathbb{P}^1$ of the fixed points of $x \mapsto -x + a$ map contain 4 nodes each and they give rise to two fibres of type I_0^* on the resolution Y of $\overline{\Sigma}$. Now we assume in addition that D_1 admits an automorphism τ of order 4 fixing the origin 0. The fixed locus of τ consists of the origin and of another point of order 2. Hence we may take a in the above construction to be a fixed point of τ. We observe that τ exchanges the points a_1 and a_2. Consider the automorphism

$$h_0 : D_1 \times D_2 \to D_1 \times D_2$$

defined by $(x, y) \mapsto (\tau x + a_1, y)$. The automorphism h_0 commutes with the elements of $\mathbb{Z}_2^2$, hence it induces an automorphism $\bar{h}$ of the quotient surface $\overline{\Sigma}$, that clearly maps each fibre of $|F_2|$ to itself. The square of the map $x \to \tau x + a_1$ is equal to the map $x \mapsto -x + a$. Thus $x \mapsto \tau x + a_1$ has order 4 and it fixes 2 points, that are necessarily also fixed points of $x \to -x + a$. Hence $\bar{h}$ maps to itself each of the fibres with 4 nodes of $|F_1|$ and it induces the identity on one of them. On the other hand, $\bar{h}$ exchanges the two double fibres of $|F_1|$. We let Σ be the surface obtained by resolving one of the singular points of $\overline{\Sigma}$ that are fixed by $\bar{h}$, we denote by C the exceptional curve of $\Sigma \to \overline{\Sigma}$ and we set $B := |F_1 + F_2 - C|$, where we omit to denote pull backs. Clearly, $\bar{h}$ induces an automorphism h of Σ and an automorphism of Y that we also denote by h. As usual we denote by $|\widetilde{F}_i|$, $i = 1, 2$, the pull back of $|F_i|$ to Y and by $2\widetilde{E}_i$, $2\widetilde{E}_i'$ the double fibres of $|\widetilde{F}_i|$. Furthermore we let C_1 and C_2 be the multiple components of the two fibres of type I_0^* of $|\widetilde{F}_1|$. Then the solutions in $\operatorname{Pic}(Y)$ of the relation

$$2L \equiv \widetilde{B} + N_1 + \cdots + N_7$$

are the linear equivalence classes of $3\widetilde{E}_1 + \widetilde{E}_2 - C_1 - C_2 - C$ and of $3\widetilde{E}_1' + \widetilde{E}_2 - C_1 - C_2 - C$. It is clear by the above description that these classes are exchanged by h. $\square$

7. A new family of surfaces with $p_g = 0$ and $K^2 = 3$

In this section we apply the previous results to the study of the moduli of surfaces of general type with $p_g = 0$ and $K^2 = 3$. We refer the reader to M. Manetti's Ph.D. thesis [14] for an excellent survey of the known results on this moduli space.

We keep the notation from the previous sections. Also we let $\mathcal{M}$ be the moduli space of (canonical models) of surfaces of general type with $p_g = 0$ and $K^2 = 3$, and we denote by $\mathcal{E}$ the

subset of $\mathcal{M}$ consisting of the canonical surfaces whose bicanonical map is composed with an involution σ such that the quotient surface X/σ is birational to an Enriques surface. Notice that, if X belongs to $\mathcal{E}$, then, by Theorem 2.1, X/σ is in fact a nodal Enriques surface with 7 nodes.

THEOREM 7.1. – *The set $\mathcal{E}$ is constructible.*
The closure $\overline{\mathcal{E}}$ of $\mathcal{E}$ in $\mathcal{M}$ is irreducible and uniruled of dimension 6.

Proof. – The claims follow from the results of Section 6 by standard constructions.
Let $\mathcal{N}, \widetilde{\mathcal{N}}$ be the spaces introduced in §6 and let $\tilde{q} : \widetilde{\mathcal{Y}} \to \widetilde{\mathcal{N}}$ be the family obtained by pulling back the family $\mathcal{Y} \to \mathcal{N}$ defined in Proposition 6.7. We denote again by $\widetilde{\mathcal{B}}$ and Ex the pullbacks on $\widetilde{\mathcal{Y}}$ of the corresponding objects of $\mathcal{Y}$. By Lemma 6.8, the family $\mathcal{Y} \to \mathcal{N}$ has a section Sec, that induces a section of $\tilde{q}$ that we denote again by Sec. Up to tensoring with a line bundle pulled back from $\widetilde{\mathcal{N}}$, we may also assume that the line bundle $\widetilde{\mathcal{B}}$ is normalized with respect to the section Sec, namely that its restriction to Sec is trivial. Then, if we denote by $\mathcal{L}$ the pull back to $\widetilde{\mathcal{Y}}$ of the normalized Poincaré line bundle on $\mathcal{Y} \times_{\mathcal{N}} \operatorname{Pic}_{\mathcal{Y}/\mathcal{N}}$, we have the equivalence relation

$$2\mathcal{L} \equiv \widetilde{\mathcal{B}} + Ex.$$

Using these data one can construct a family $p : \mathcal{X} \to \mathcal{V}$ of surfaces of general type such that: (1) the base $\mathcal{V}$ of the family is a nonempty open set in a $\mathbb{P}^3$-bundle over $\widetilde{\mathcal{N}}$, hence it is irreducible and ruled of dimension 6; (2) the image of the induced map $\mathcal{V} \to \mathcal{M}$ is precisely $\mathcal{E}$; (3) the map $\mathcal{V} \to \mathcal{M}$ has finite fibres. $\square$

COROLLARY 7.2. – *Let S be a smooth surface such that the canonical model of S is in $\mathcal{E}$. Then*:

$$\pi_1(S) \simeq \mathbb{Z}_2^2 \times \mathbb{Z}_4.$$

Proof. – Since blowing up does not change the fundamental group of a smooth surface, we may assume that S is minimal. By Theorem 7.1, all the minimal surfaces whose canonical model is in $\mathcal{E}$ have the same fundamental group, so the statement follows by [21, Theorem 3.1]. $\square$

PROPOSITION 7.3. – *If $X \in \mathcal{E}$, then the bicanonical map φ of X is a morphism of degree either* 2 *or* 4. *The subset $\mathcal{E}_{d4}$ consisting of the surfaces for which $\deg \varphi = 4$ is a closed subset of $\mathcal{E}$ of codimension* 1 *and its closure $\overline{\mathcal{E}_{d4}}$ is irreducible.*

Proof. – The fact that φ is a morphism of degree 2 or 4 is immediate by Proposition 2.1. Since $\deg \varphi$ is a semicontinuous function of $X \in \mathcal{M}$, the set $\mathcal{E}_{d4}$ is clearly closed in $\mathcal{E}$ and it is a proper subset of $\mathcal{E}$ by Examples 3.6 and 3.7. To show the last part of the statement one proceeds as in the proof of Theorem 7.1 by constructing a 5-dimensional family of surfaces that maps onto $\mathcal{E}_{d4}$ with finite fibres. By Theorem 4.6, Propositions 2.1 and 2.2, the fibres of this family are the double covers $X \to \Sigma$, with Σ an Enriques surface with 7 nodes, branched on the nodes and on a divisor B with negligible singularities, and such that the pair (Σ, B) is as in Example 3.5. We omit the explicit construction of this family, which is standard by the classification of Enriques surfaces with 8 nodes given in [17]. $\square$

Remark 7.4. – The main question left open by Theorem 7.1 is whether $\overline{\mathcal{E}}$ is an irreducible component of $\mathcal{M}$. To answer this question one has to consider for $X \in \mathcal{E}$ the natural map of functors $Def(X, \sigma) \to Def(X)$, where $Def(X)$ denotes deformations of X and $Def(X, \sigma)$ denotes deformations of X with an involution extending σ. One needs to decide whether this map is surjective for a general X. To show that this is indeed the case, it is enough to exhibit one surface $X \in \mathcal{E}$ such that the map $Def(X, \sigma) \to Def(X)$ is smooth, and smoothness can in turn be checked by means of an infinitesimal computation. Unfortunately, although we can show

that for a smooth $X \in \mathcal{E}$ the functor $Def(X, \sigma)$ is smooth, we have not been able to prove the smoothness of $Def(X, \sigma) \to Def(X)$. Notice that, since the expected dimension of $Def(X)$ is equal to 4, Theorem 7.1 implies that the obstruction space T_X^2 of $Def(X)$ has dimension $\geqslant 2$ at every point of $\mathcal{E}$.

Acknowledgements

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