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Existence of discretely self-similar solutions to the Navier–Stokes equations for initial value in $L^2_{loc}(\mathbb{R}^3)$

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Abstract

We prove the existence of a forward discretely self-similar solutions to the Navier–Stokes equations in $\mathbb{R}^3 \times (0, +\infty)$ for a discretely self-similar initial velocity belonging to $L^2_{loc}(\mathbb{R}^3)$.

MSC: 76B03; 35Q31

Keywords: Navier–Stokes equations; Existence; Discretely self-similar solutions

1. Introduction

In this paper we study the existence of forward discretely self-similar (DSS) solutions to the Navier–Stokes equations in $Q = \mathbb{R}^3 \times (0, +\infty)$

$$\nabla \cdot u = 0, \tag{1.1}$$

$$\partial_t u + (u \cdot \nabla)u - \Delta u = -\nabla\pi, \tag{1.2}$$

with the initial condition

$$u = u_0 \quad \text{on} \quad \mathbb{R}^3 \times \{0\}. \tag{1.3}$$

Here $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ denotes the velocity of the fluid, and $u_0(x) = (u_{0,1}(x), u_{0,2}(x), u_{0,3}(x))$, while π stands for the pressure. In case $u_0 \in L^2(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$ in the sense of distributions the global in time existence of weak solutions to (1.1)–(1.3), which satisfy the global energy inequality for almost all $t \in (0, +\infty)$

$$\frac{1}{2}\|u(t)\|_2^2 + \int_0^t \|\nabla u(s)\|_2^2 ds \le \frac{1}{2}\|u_0\|_2^2 \tag{1.4}$$

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has been proved by Leray [9]. On the other hand, the important questions of regularity and uniqueness of solutions to (1.1)–(1.3) are still open. The first significant results in this direction have been established by Scheffer [10] and later by Caffarelli, Kohn, Nirenberg [2] for solutions (u, π) that also satisfy the following local energy inequality for almost all $t \in (0, +\infty)$ and for all nonnegative $\phi \in C^\infty_c(Q)$

$$\frac{1}{2}\int\limits_{\mathbb{R}^3} |u(t)|^2\phi(x,t)dx + \int\limits_0^t\int\limits_{\mathbb{R}^3} |\nabla u|^2\phi dxds$$
$$\leq \frac{1}{2}\int\limits_0^t\int\limits_{\mathbb{R}^3} |u|^2\Big(\frac{\partial}{\partial t}+\Delta\Big)\phi dxds + \frac{1}{2}\int\limits_0^t\int\limits_{\mathbb{R}^3} (|u|^2+2\pi)u\cdot\nabla\phi dxds. \tag{1.5}$$

On the other hand, the space $L^2(\mathbb{R}^3)$ excludes homogeneous spaces of degree -1 belonging to the scaling invariant class. In fact we observe that $u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$ solves the Navier–Stokes equations with initial velocity $u_{0,\lambda}(x) = \lambda u_0(\lambda x)$, for any $\lambda > 0$. This suggests to study of the Navier–Stokes system for initial velocities in a homogeneous space X of degree -1, which means that $\|v\|_X = \|v_\lambda\|_X$ for all $v \in X$. Koch and Tataru proved in [7] that $X = BMO^{-1}$ is the largest possible space with scaling invariant norm which guarantees well-posedness under smallness condition. On the contrary, for self-similar (SS) initial data fulfilling $u_{0,\lambda} = u$ for all $\lambda > 0$ a natural space seems to be $X = L^{3,\infty}(\mathbb{R}^3)$. This space is embedded into the space $L^2_{uloc}(\mathbb{R}^3)$, which contains uniformly local square integrable functions. Obviously, possible solutions to the Navier–Stokes equations with $u_0 \in L^2_{uloc}(\mathbb{R}^3)$ do not satisfy the global energy equality, rather the local energy inequality in the sense of Caffarelli–Kohn–Nirenberg. Such solutions are called local Leray solutions. The existence of global in time local Leray solutions has been proved by Lemariè-Rieusset in [8] (see also in [6] for more details). This concept has been used by Bradshaw and Tsai [1] for the construction of a discretely self-similar (λ-DSS, $\lambda > 1$) local Leray solution for a λ-DSS initial velocity $u_0 \in L^{3,\infty}(\mathbb{R}^3)$. This result generalizes the previous results of Jia and Šverák [5] concerning the existence of SS local Leray solution, and the result by Tsai in [11], which proves the existence of a λ-DSS Leray solution for λ near 1. However, for the λ-DSS initial data it would be more natural to assume $u_0 \in L^2_{loc}(\mathbb{R}^3)$ instead $L^{3,\infty}(\mathbb{R}^3)$. In general, such initial value does not belong to $L^2_{uloc}(\mathbb{R}^3)$ and therefore it does not belong to the Morrey class $M^{2,1}$, rather to the weighted space $L^2_k(\mathbb{R}^3)$ of all $v \in L^2_{loc}(\mathbb{R}^3)$ such that $\frac{v}{(1+|x|^k)} \in L^2(\mathbb{R}^3)$ for all $\frac{1}{2} < k < +\infty$.

Since the authors in [1] work on the existence of periodic solutions to the time dependent Leray equation a certain spatial decay is necessary which can be ensured for initial data in $L^{3,\infty}(\mathbb{R}^3)$. On the other hand, applying the local L^2 theory it would be more natural to assume $u_0 \in L^2(B_\lambda \setminus B_1)$ only. As explained in [1] their method even breaks down for initial data in the Morrey class $M^{2,1}(\mathbb{R}^3)$, which is a much smaller subspace of $L^2_{loc}(\mathbb{R}^3)$. By using an entirely different method we are able to construct a global weak solutions for such DSS initial data.

In the present paper we introduce a new notion of a local Leray solution satisfying a local energy inequality with projected pressure. To the end, we provide the notations of function spaces which will be used in the sequel. By $L^s(G)$, $1 \leq s \leq \infty$, we denote the usual Lebesgue spaces. The usual Sobolev spaces are denoted by $W^{k,s}(G)$ and $W^{k,s}_0(G)$, $1 \leq s \leq +\infty$, $k \in \mathbb{N}$. The dual of $W^{k,s}_0(G)$ will be denoted by $W^{-k,s'}(G)$, where $s' = \frac{s}{s-1}$, $1 < s < +\infty$. For a general space of vector fields X the subspace of solenoidal fields will be denoted by X_σ. In particular, the space of solenoidal smooth fields with compact support is denoted by $C^\infty_{c,\sigma}(\mathbb{R}^3)$. In addition we define the energy space

$$V^2(G\times(0,T)) = L^\infty(0,T;L^2(G)) \cap L^2(0,T;W^{1,2}(G)), \quad 0 < T \leq +\infty.$$

We now recall the definition of the local pressure projection $E^*_G : W^{-1,s}(G) \to W^{-1,s}(G)$ for a given bounded C^2-domain $G \subset \mathbb{R}^3$, introduced in [13] based on the unique solvability of the steady Stokes system (cf. [4]). More precisely, for any $F \in W^{-1,s}(G)$ there exists a unique pair $(v, p) \in W^{1,s}_{0,\sigma}(G) \times L^s_0(G)$ which solves weakly the steady Stokes system

$$\begin{cases} \nabla\cdot v = 0 \quad \text{in} \quad G, \quad -\Delta v + \nabla p = F \quad \text{in} \quad G, \\ v = 0 \text{ on } \quad \partial G. \end{cases} \tag{1.6}$$

Here $W^{1,s}_{0,\sigma}(G)$ stands for closure of $C^\infty_{c,\sigma}(\mathbb{R}^3)$ with respect to the norm in $W^{1,s}(G)$, while $L^s_0(G)$ denotes the subspace of $L^s(G)$ with vanishing average. Then we set $E^*_G(F) := \nabla p$, where ∇p denotes the gradient function in $W^{-1,s}(G)$ defined as

$$\langle \nabla p, \varphi\rangle = -\int_G p\nabla\cdot\varphi dx, \quad \varphi \in W^{1,s'}_0(G).$$

Remark 1.1. From the existence and uniqueness of weak solutions (v, p) to (1.6) for given $F \in W^{-1,s}(G)$ it follows that

$$\|\nabla v\|_{s,G} + \|p\|_{s,G} \le c\|F\|_{-1,s,G}, \tag{1.7}$$

where $c =$ const depending on s and the geometric properties of G, and depending only on s if G equals a ball or an annulus, which holds due to the scaling properties of the Stokes equation. In case F is given by $\nabla\cdot f$ for $f \in L^s(\mathbb{R}^3)^9$ then (1.7) gives

$$\|p\|_{s,G} \le c\|f\|_{s,G}. \tag{1.8}$$

According to the estimate $\|\nabla p\|_{-1,s,G} \le \|p\|_{s,G}$, and using (1.8), we see that the operator E^*_G is bounded in $W^{-1,s}(G)$. Furthermore, as $E^*_G(\nabla p) = \nabla p$ for all $p \in L^s_0(G)$ we see that E^*_G defines a projection.

2. In case $F \in L^s(G)$, using the canonical embedding $L^s(G) \hookrightarrow W^{-1,s}(G)$, by the aid of elliptic regularity we get $E^*_G(F) = \nabla p \in L^s(G)$ together with the estimate

$$\|\nabla p\|_{s,G} \le c\|F\|_{s,G}, \tag{1.9}$$

where the constant in (1.9) depends only on s and G. In case G equals a ball or an annulus this constant depends only on s (cf. [4] for more details). Accordingly the restriction of E^*_G to the Lebesgue space $L^s(G)$ appears to be a projection in $L^s(G)$. This projection will be denoted still by E^*_G.

Definition 1.2 *(Local Leray solution with projected pressure).* Let $u_0 \in L^2_{loc}(\mathbb{R}^3)$. A vector function $u \in L^2_{loc,\sigma}(\mathbb{R}^3 \times [0, +\infty))$ is called a *local Leray solution to* (1.1)–(1.3) *with projected pressure*, if for any bounded C^2 domain $G \subset \mathbb{R}^3$ and $0 < T < +\infty$

1. $u \in V^2_\sigma(G \times (0, T)) \cap C_w([0, T]; L^2(G))$.
2. u is a distributional solution to (1.2), i.e. for every $\varphi \in C^\infty_c(Q)$ with $\nabla\cdot\varphi = 0$

$$\iint_Q -u\cdot\frac{\partial\varphi}{\partial t} - u\otimes u : \nabla\varphi + \nabla u : \nabla\varphi dxdt = 0. \tag{1.10}$$

3. $u(t) \to u_0$ in $L^2(G)$ as $t \to 0^+$.
4. The following local energy inequality with projected pressure holds for every nonnegative $\phi \in C^\infty_c(G \times (0, +\infty))$, and for almost every $t \in (0, +\infty)$

$$\begin{aligned}
&\frac{1}{2}\int_G |v_G(t)|^2\phi dx + \int_0^t\int_G |\nabla v_G|^2\phi dxds \\
&\quad\le \frac{1}{2}\int_0^t\int_G |v_G|^2\Big(\Delta + \frac{\partial}{\partial t}\Big)\phi + |v_G|^2 u\cdot\nabla\phi) dxds \\
&\qquad + \int_0^t\int_G (u\otimes v_G) : \nabla^2 p_{h,G}\phi dxdt + \int_0^t\int_G p_{1,G} v_G\cdot\nabla\phi dxds \\
&\qquad\qquad + \int_0^t\int_G p_{2,G} v_G\cdot\nabla\phi dxds,
\end{aligned} \tag{1.11}$$

where $v_G = u + \nabla p_{h,G}$, and

$$\nabla p_{h,G} = -E^*_G(u),$$
$$\nabla p_{1,G} = -E^*_G((u\cdot\nabla)u), \quad \nabla p_{2,G} = E^*_G(\Delta u).$$

Remark 1.3. 1. Note that due to $\nabla\cdot u = 0$ the pressure $p_{h,G}$ is harmonic, and thus smooth in x. Furthermore, as it has been proved in [13] the pressure gradient $\nabla p_{h,G}$ is continuous in $G\times[0,+\infty)$.

2. The notion of local suitable weak solution to the Navier–Stokes equations satisfying the local energy inequality (1.11) has been introduced in [12]. One can show without difficulty that any suitable weak solution in the sense of [2] is a local suitable weak solution in the above sense, satisfying in particular the inequality (1.11) (see Appendix of [3] for a complete proof). As it has been shown there such solutions enjoy the same partial regularity properties as the usual suitable weak solutions in the Caffarelli–Kohn–Nirenberg theorem.

Our main result is the following

Theorem 1.4. *For any λ-DSS initial data $u_0\in L^2_{loc,\sigma}(\mathbb{R}^3)$ there exists at least one local Leray solution with projected pressure $u\in L^2_{loc,\sigma}(\mathbb{R}^3\times[0,+\infty))$ to the Navier–Stokes equations (1.1)–(1.3) in the sense of Definition 1.2, which is discretely self-similar.*

We close this section by describing the structure of the paper. In Section 2 we consider a linearized problem of the Navier–Stokes equations, where the convection term of (1.2) is replaced by $(b\cdot\nabla)u$ with a given λ-DSS function b. For a λ-DSS solution of such linearized equations we derive various a priori estimates, which will be used later for construction of the desired solution of the original problem. In Section 3 based on the a priori estimates of Section 2, combined with the Schauder fixed point theorem, we complete the proof of Theorem 1.4. In Appendix we prove several important properties of the λ-DSS solutions.

2. Solutions of the linearized problem with initial velocity in $L^2_{\lambda-DSS}$

Let $1<\lambda<+\infty$ be fixed. For $f:\mathbb{R}^3\to\mathbb{R}^3$ we denote $f_\lambda(x) := \lambda f(\lambda x)$, $x\in\mathbb{R}^3$. For a time dependent function $f: Q\to\mathbb{R}^3$ we denote $f_\lambda(x,t) := \lambda f(\lambda x, \lambda^2 t)$, $(x,t)\in\mathbb{R}^3\times(0,+\infty)$. We now define for $1\le s\le+\infty$

$$L^s_{\lambda-DSS}(\mathbb{R}^3) := \left\{u\in L^1_{loc}(\mathbb{R}^3)\,\middle|\, u\in L^s(B_\lambda\setminus B_1), u_\lambda = u \text{ a.e. in } \mathbb{R}^3\right\},$$
$$L^s_{\lambda-DSS}(Q) := \left\{u\in L^1_{loc}(Q)\,\middle|\, u\in L^s(Q_\lambda\setminus Q_1), u_\lambda = u \text{ a.e. in } Q\right\}.$$

Here B_r stands for the usual ball in $\mathbb{R}^3$ with center 0 and radius $r>0$, while $Q_r = B_r\times(0,r^2)$.

In the present section we consider the following linearized problem in Q

$$\nabla\cdot u = 0, \tag{2.1}$$
$$\partial_t u + (b\cdot\nabla)u - \Delta u = -\nabla\pi \tag{2.2}$$

with the initial condition

$$u = u_0 \quad \text{on} \quad \mathbb{R}^3\times\{0\}, \tag{2.3}$$

where u_0 belongs to $L^2_{\lambda-DSS}(\mathbb{R}^3)$ with $\nabla\cdot u_0 = 0$, and $b\in L^s_{\lambda-DSS}(Q)$, $3\le s\le 5$, with $\nabla\cdot b = 0$ both in the sense of distributions. We give the following notion of a local solution with projected pressure for the linear system (2.1), (2.2).

Definition 2.1 *(Local solution with projected pressure to the linearized problem).* Let $u_0\in L^2_{loc,\sigma}(\mathbb{R}^3)$ and let $b\in L^3_{loc,\sigma}(\mathbb{R}^3\times[0,+\infty))$. A vector function $u\in L^2_{loc,\sigma}(\mathbb{R}^3\times[0,+\infty))$ is called a *local solution to* (2.1)–(2.3) *with projected pressure*, if for any bounded C^2 domain $G\subset\mathbb{R}^3$ and $0<T<+\infty$ the following conditions are satisfied

1. $u\in V^2(G\times(0,T))\cap C_w([0,T];L^2(G))$.

2. u is a distributional solution to (2.2), i.e. for every $\varphi \in C_c^\infty(Q)$ with $\nabla \cdot \varphi = 0$

$$\iint\limits_Q -u \cdot \frac{\partial \varphi}{\partial t} - b \otimes u : \nabla \varphi + \nabla u : \nabla \varphi dxdt = 0. \tag{2.4}$$

3. $u(t) \to u_0$ in $L^2(G)$ as $t \to 0^+$.
4. The following local energy inequality with projected pressure holds for every nonnegative $\phi \in C_c^\infty(G \times (0, +\infty))$, and for almost every $t \in (0, +\infty)$

$$\begin{aligned} &\frac{1}{2} \int\limits_G |v_G(t)|^2 \phi dx + \int\limits_0^t \int\limits_G |\nabla v_G|^2 \phi dxds \\ &\quad \le \frac{1}{2} \int\limits_0^t \int\limits_G |v_G|^2 \Big(\Delta + \frac{\partial}{\partial t}\Big)\phi + |v_G|^2 b \cdot \nabla \phi) dxds \\ &\qquad + \int\limits_0^t \int\limits_G (b \otimes v_G) : \nabla^2 p_{h,G} \phi dxdt + \int\limits_0^t \int\limits_G p_{1,G} v_G \cdot \nabla \phi dxds \\ &\qquad\qquad + \int\limits_0^t \int\limits_G p_{2,G} v_G \cdot \nabla \phi dxds \end{aligned} \tag{2.5}$$

where $v_G = u + \nabla p_{h,G}$, and

$$\nabla p_{h,G} = -E_G^*(u),$$
$$\nabla p_{1,G} = -E_G^*((b \cdot \nabla)u), \quad \nabla p_{2,G} = E_G^*(\Delta u).$$

Theorem 2.2. *Let $b \in L^3_{\lambda-DSS}(Q) \cap L^{\frac{18}{5}}(0, T; L^3(B_1))$, $0 < T < +\infty$, with $\nabla \cdot b = 0$ in the sense of distributions. Suppose that $b \in L^3_{loc}(0, \infty; L^\infty(\mathbb{R}^3))$. For every $u_0 \in L^2_{\lambda-DSS}(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$ in the sense of distributions, there exists a unique local solution with projected pressure $u \in L^2_{loc,\sigma}(\mathbb{R}^3 \times [0, +\infty))$ to (2.1)–(2.3) according to Definition 2.1 such that for any $0 < \rho < +\infty$ and $0 < T < +\infty$ it holds*

$$u \in L^3_{\lambda-DSS}(Q), \tag{2.6}$$
$$u \in C([0, T]; L^2(B_\rho)), \tag{2.7}$$
$$\|u\|_{L^\infty(0,T;L^2(B_{\rho^{\frac{1}{4}}}))} + \|\nabla u\|_{L^2(B_{\rho^{\frac{3}{5}}} \times (0,T))} \le C_0 K_0 \Big(\rho^{\frac{1}{2}} + |||b|||^3 \max\{T^{\frac{13}{18}}, T^{\frac{1}{2}}\}\Big), \tag{2.8}$$
$$\|u\|_{L^4(0,T;L^3(B_1))} \le C_0 K_0 \Big(1 + |||b|||^3 \max\{T^{\frac{13}{18}}, T^{\frac{1}{2}}\}\Big), \tag{2.9}$$

where $K_0 := \|u_0\|_{L^2(B_1)}$ and $|||b||| = \|b\|_{L^{\frac{18}{5}}(0,T;L^3(B_1))}$, while $C_0 > 0$ denotes a constant depending on λ only.

Before turning to the proof of Theorem 2.1, we show the existence and uniqueness of weak solutions to the linear system (2.1)–(2.3) for L^2_σ initial data.

Lemma 2.3. *Let $b \in L^3_{\lambda-DSS}(Q) \cap L^{\frac{18}{5}}(0, T; L^3(B_1))$, $0 < T < +\infty$ with $\nabla \cdot b = 0$ in the sense of distributions. Suppose that $b \in L^3_{loc}(0, \infty; L^\infty(B_1))$. For every $u_0 \in L^2_\sigma(\mathbb{R}^3)$ there exists a unique weak solution $u \in V^2_\sigma(Q) \cap C([0, +\infty); L^2(\mathbb{R}^3))$ to (2.1)–(2.3), which satisfies the global energy equality for all $t \in [0, +\infty)$*

$$\frac{1}{2}\|u(t)\|_2^2 + \int\limits_0^t \int\limits_{\mathbb{R}^3} |\nabla u|^2 dxds = \frac{1}{2}\|u_0\|_2^2. \tag{2.10}$$

Proof. *1. Existence*: By using standard linear theory of parabolic systems we easily get the existence of a weak solution $u \in V^2(Q) \cap C_w([0,+\infty); L^2(\mathbb{R}^3))$ to (2.1)–(2.3) which satisfies the global energy inequality for almost all $t \in (0,+\infty)$

$$\frac{1}{2}\|u(t)\|_2^2 + \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 dxds \le \frac{1}{2}\|u_0\|_2^2. \tag{2.11}$$

It is well known that such solutions have the property

$$u(t) \to u_0 \quad in \quad L^2(\mathbb{R}^3) \quad as \quad t \to 0^+. \tag{2.12}$$

On the other hand, from the assumption of the Lemma it follows that for all $t_0 \in (0,T)$

$$\|bu\|_{L^2(\mathbb{R}^3\times(t_0,T))} \le \|b\|_{L^2(t_0,T;L^\infty(\mathbb{R}^3))}\|u_0\|_2.$$

Accordingly, $u \in C((0,T]; L^2(\mathbb{R}^3))$, and for all $t_0 \in (0,T]$ and $t \in [t_0,T]$ the following energy equality holds true

$$\frac{1}{2}\|u(t)\|_2^2 + \int_{t_0}^t \int_{\mathbb{R}^3} |\nabla u|^2 dxds = \frac{1}{2}\|u(t_0)\|_2^2. \tag{2.13}$$

Now letting $t_0 \to 0$ in (2.13), and observing (2.12), we are led to (2.10).

By a similar argument, making use of (2.12) we easily prove the local energy inequality (2.5).

2. Uniqueness: Let $v \in V^2_\sigma(Q)$ be a second solution to (2.1)–(2.3) satisfying the global energy equality. As we have seen above this solution belongs to $C([0,+\infty); L^2(\mathbb{R}^3))$. Setting $w = u - v$, by our assumption on b it follows that $b \otimes w \in L^2(\mathbb{R}^3 \times (t_0,T])$ for any $t_0 \in (0,T]$. Accordingly, as above we get the following energy equality

$$\frac{1}{2}\|w(t)\|_2^2 + \int_{t_0}^t \int_{\mathbb{R}^3} |\nabla w|^2 dxds = \frac{1}{2}\|w(t_0)\|_2^2. \tag{2.14}$$

Verifying that $w(t_0) \to 0$ in $L^2(\mathbb{R}^3)$ as $t_0 \to 0^+$ from (2.14) letting $t_0 \to 0^+$ it follows that $\|w(t)\|_2 = 0$ for all $t \in [0,T]$. This completes the proof of the uniqueness. □

Proof of Theorem 2.2. Since u_0 is λ-DSS we have $\lambda u_0(\lambda x) = u_0(x)$ for all $x \in \mathbb{R}^3$. We define the extended annulus $\tilde{A}_k = B_{\lambda^k} \setminus B_{\lambda^{k-3}}$, $k \in \mathbb{N}$. Clearly, $B_1 \cup (\cup_{k=1}^\infty \tilde{A}_k) = \mathbb{R}^3$. There exists a partition of unity $\{\psi_k\}$ such that $\operatorname{supp}\psi_k \subset \tilde{A}_k$ for $k \in \mathbb{N}$ and $\operatorname{supp}\psi_0 \subset B_1$, and $0 \le \psi_k \le 1$, $|\nabla^2\psi_k| + |\nabla\psi_k|^2 \le c\lambda^{-2k}$, $k \in \mathbb{N} \cup \{0\}$. We set $u_{0,k} = \mathbb{P}(u_0\psi_k)$, $k \in \mathbb{N} \cup \{0\}$, where $\mathbb{P}$ denotes the Leray–Helmholtz projection. Clearly,

$$u_0 = \sum_{k=0}^\infty u_{0,k}, \tag{2.15}$$

where the limit in (2.15) is taken in the sense of $L^2_{loc}(\mathbb{R}^3)$.

Let $k \in \mathbb{N} \cup \{0\}$ be fixed. Thanks to Lemma 2.3 we get a unique weak solution $u_k \in V^2_\sigma(Q)$ to the problem

$$\nabla \cdot u_k = 0 \quad \text{in} \quad Q, \tag{2.16}$$

$$\partial_t u_k + (b\cdot\nabla)u_k - \Delta u_k = -\nabla\pi_k \quad \text{in} \quad Q, \tag{2.17}$$

$$u_k = u_{0,k} \quad \text{on} \quad \mathbb{R}^3 \times \{0\}, \tag{2.18}$$

satisfying the following global energy equality for all $t \in [0,+\infty)$

$$\frac{1}{2}\|u_k(t)\|_2^2 + \int_0^t \int_{\mathbb{R}^3} |\nabla u_k|^2 dxds = \frac{1}{2}\|u_{0,k}\|_2^2. \tag{2.19}$$

By using the transformation formula, we get

$$\|u_{0,k}\|_2^2 \le \int\limits_{\mathbb{R}^3} |u_0\psi_k|^2 dx \le \int\limits_{\tilde{A}_k} |u_0|^2 dx = \lambda^{3k} \int\limits_{\tilde{A}_1} |u_0(\lambda^k x)|^2 dx$$
$$= \lambda^k \int\limits_{\tilde{A}_1} |\lambda^k u_0(\lambda^k x)|^2 dx = \lambda^k \int\limits_{\tilde{A}_1} |u_0(x)|^2 dx \le cK_0^2 \lambda^k. \tag{2.20}$$

Combining (2.19) and (2.20), we are led to

$$\|u_k\|^2_{L^\infty(0,T;L^2)} + \|\nabla u_k\|^2_{L^2(0,T;L^2)} \le cK_0^2\lambda^k. \tag{2.21}$$

Next, let $\lambda^{\frac{3}{5}k} \le r < \rho \le \lambda^{\frac{3}{5}(k+1)}$ be arbitrarily chosen, but fixed. By introducing the local pressure we have

$$\frac{\partial v_{k,\rho}}{\partial t} + (b\cdot\nabla)u_k - \Delta v_{k,\rho} = -\nabla p_{1,k,\rho} - \nabla p_{2,k,\rho},$$

where $v_{k,\rho} = u_k + \nabla p_{h,k,\rho}$, and

$$\nabla p_{h,k,\rho} = -E^*_{B_\rho}(u_k),$$
$$\nabla p_{1,k,\rho} = -E^*_{B_\rho}((b\cdot\nabla)u_k), \quad \nabla p_{2,k,\rho} = E^*_{B_\rho}(\Delta u_k).$$

The following local energy equality holds true for all $\phi \in C^\infty_c(B_\rho)$ and for all $t \in [0,T]$,

$$\frac{1}{2}\int\limits_{B_\rho} |v_{k,\rho}(t)|^2\phi^6 dx + \int\limits_0^t \int\limits_{B_\rho} |\nabla v_{k,\rho}|^2 \phi^6 dxds$$
$$= \frac{1}{2}\int\limits_0^t \int\limits_{B_\rho} |v_{k,\rho}|^2 \Delta\phi^6 dxds + \frac{1}{2}\int\limits_0^t \int\limits_{B_\rho} |v_{k,\rho}|^2 b\cdot\nabla\phi^6 dxds$$
$$+ \int\limits_0^t \int\limits_{B_\rho} (b\otimes v_{k,\rho}) : \nabla^2 p_{h,k,\rho}\phi^6 dxds + \int\limits_0^t \int\limits_{B_\rho} p_{1,k,\rho} v_{k,\rho}\cdot\nabla\phi^6 dxds$$
$$+ \int\limits_0^t \int\limits_{B_\rho} p_{2,k,\rho} v_{k,\rho}\cdot\nabla\phi^6 dxds + \frac{1}{2}\int\limits_{B_\rho} |v_{0,k}|^2\phi^6 dx$$
$$= I + II + III + IV + V + VI. \tag{2.22}$$

Let $\phi \in C^\infty_c(\mathbb{R}^3)$ denote a cut off function such that $0 \le \phi \le 1$ in $\mathbb{R}^3$, $\phi \equiv 1$ on B_r, $\phi \equiv 0$ in $\mathbb{R}^3 \setminus B_\rho$, and $|\nabla^2\phi| + |\nabla\phi|^2 \le c(\rho - r)^{-2}$ in $\mathbb{R}^3$.

Let $m \in \mathbb{N}$ be chosen so that $\lambda^{m-1} \le \rho < \lambda^m$. Then we estimate

$$\|b\|^3_{L^3(B_\rho\times(0,T))} = \lambda^{5m} \int\limits_0^{T\lambda^{-2m}} \int\limits_{B_{\rho\lambda^{-m}}} |b(\lambda^{-m}x, \lambda^{-2m}t)|^3 dxdt$$
$$= \lambda^{2m} \int\limits_0^{T\lambda^{-2m}} \int\limits_{B_{\rho\lambda^{-m}}} |b(x,t)|^3 dxdt$$
$$\le c\lambda^{2m-\frac{1}{3}m} T^{\frac{1}{6}} \|b\|^3_{L^{\frac{18}{5}}(0,T;L^3(B_1))} \le c|||b|||^3 \rho^{\frac{5}{3}} T^{\frac{1}{6}},$$

where and hereafter the constants appearing in the estimates may depend on λ. The above estimate together with $\rho^{\frac{5}{3}} \le \lambda^{k+1}$ yields

$$\|b\|_{L^3(B_\rho\times(0,T))} \le c|||b|||\lambda^{\frac{1}{3}k}T^{\frac{1}{18}}. \tag{2.23}$$

In what follows we extensively make use of the estimate for almost all $t\in(0,T)$

$$\|\nabla p_{h,k,\rho}(t)\|_{L^2(B_\rho)} \lesssim \|u_k(t)\|_{L^2(B_\rho)}, \tag{2.24}$$

which is an immediate consequence of (1.9). In addition, we easily verify the inequality

$$\|\nabla^2 p_{h,k,\rho}(t)\|_{L^2(B_\rho)} \lesssim \|\nabla u_k(t)\|_{L^2(B_\rho)}. \tag{2.25}$$

Indeed, observing that

$$\nabla^2 p_{h,k,\rho}(t) = \nabla(\nabla p_{h,k,\rho}(t) - u(t)_{B_\rho}) = -\nabla E^*_{B_\rho}(u_k(t) - u_k(t)_{B_\rho})$$

by means of elliptic regularity along with the Poincaré inequality we get

$$\begin{aligned}\|\nabla^2 p_{h,k,\rho}(t)\|^2_{L^2(B_\rho)} &\le c\rho^{-2}\|u_k(t) - u_k(t)_{B_\rho}\|^2_{L^2(B_\rho)} + c\|\nabla u_k(t)\|^2_{L^2(B_\rho)}\\ &\le c\|\nabla u_k(t)\|^2_{L^2(B_\rho)}.\end{aligned}$$

Whence, (2.25).

(i) With the help of (2.21) we easily deduce that

$$I \le c(\rho-r)^{-2}\int_0^t\int_{B_\rho} |u_k|^2 dxds \le cK_0^2(\rho-r)^{-2}\lambda^k T.$$

(ii) Next, using Hölder's inequality and Young's inequality together with (2.21), (2.23), (2.24) and (2.25), we estimate

$$\begin{aligned}II &\le (\rho-r)^{-1}\int_0^t\int_{B_\rho} |b||v_{k,\rho}|^2\phi^5 dxds\\
&\le c(\rho-r)^{-1}T^{\frac{1}{6}}\|b\|_{L^3(B_\rho\times(0,T))}\|v_{k,\rho}\phi^3\|_{L^\infty(0,T;L^2)}\|v_{k,\rho}\phi^2\|_{L^2(0,T;L^6)}\\
&\le c(\rho-r)^{-2}T^{\frac{2}{3}}\|b\|_{L^3(B_\rho\times(0,T))}\|v_{k,\rho}\phi^3\|_{L^\infty(0,T;L^2)}\|u_k\|_{L^\infty(0,T;L^2)}\\
&\qquad + c(\rho-r)^{-1}T^{\frac{1}{6}}\|b\|_{L^3(B_\rho\times(0,T))}\|v_{k,\rho}\phi^3\|_{L^\infty(0,T;L^2)}\|\nabla v_{k,\rho}\phi^2\|_{L^2(0,T;L^2)}\\
&\le c|||b|||K_0(\rho-r)^{-2}\lambda^{\frac{5}{6}k}T^{\frac{13}{18}}\|v_{k,\rho}\phi^3\|_{L^\infty(0,T;L^2)}\\
&\qquad + c|||b|||(\rho-r)^{-1}\lambda^{\frac{1}{3}k}T^{\frac{2}{9}}\|v_{k,\rho}\phi^3\|_{L^\infty(0,T;L^2)}\|\nabla v_{k,\rho}\phi^3\|^{\frac{2}{3}}_{L^2(0,T;L^2)}\|\nabla v_{k,\rho}\|^{\frac{1}{3}}_{L^2(0,T;L^2(B_\rho))}\\
&\le c|||b|||K_0(\rho-r)^{-2}\lambda^{\frac{5}{6}k}T^{\frac{13}{18}}\|v_{k,\rho}\phi^3\|_{L^\infty(0,T;L^2)}\\
&\qquad + c|||b|||K_0^{\frac{1}{3}}(\rho-r)^{-1}\lambda^{\frac{1}{2}k}T^{\frac{2}{9}}\|v_{k,\rho}\phi^3\|_{L^\infty(0,T;L^2)}\|\nabla v_{k,\rho}\phi^3\|^{\frac{2}{3}}_{L^2(0,T;L^2)}\\
&\le c|||b|||^2K_0^2(\rho-r)^{-4}\lambda^{\frac{5}{3}k}T^{\frac{13}{9}}\\
&\qquad + c|||b|||^6K_0^2(\rho-r)^{-6}\lambda^{3k}T^{\frac{4}{3}} + \frac{1}{8}\|v_{k,\rho}\phi^3\|^2_{L^\infty(0,T;L^2)} + \frac{1}{4}\|\nabla v_{k,\rho}\phi^3\|^2_{L^2(0,T;L^2)}\\
&\le cK_0^2(\rho-r)^{-3}\lambda^k\max\{T^{\frac{13}{9}},T\} + c|||b|||^6K_0^2(\rho-r)^{-6}\lambda^{3k}\max\{T^{\frac{13}{9}},T\}\\
&\qquad + \frac{1}{8}\|v_{k,\rho}\phi^3\|^2_{L^\infty(0,T;L^2)} + \frac{1}{4}\|\nabla v_{k,\rho}\phi^3\|^2_{L^2(0,T;L^2)}.\end{aligned}$$

(iii) In what follows we make use the following estimates using the fact that $p_{h,k,\rho}$ is harmonic. By using the identity

$$\int\limits_{\mathbb{R}^3} |\nabla h|^2 \phi^2 dx = \frac{1}{2} \int\limits_{\mathbb{R}^3} h^2 \Delta \phi^2 dx$$

for any harmonic function h on B_ρ, and cut off function $\phi \in C_c^\infty(B_\rho)$, we get

$$\|\nabla^3 p_{h,k,\rho}(t)\phi^3\|_2 \le c(\rho - r)^{-1} \|\nabla^2 p_{h,k,\rho}(t)\phi^2\|_2 \le (\rho - r)^{-2} \|\nabla p_{h,k,\rho}(t)\|_{2,B_\rho}. \tag{2.26}$$

By the aid of Sobolev's inequality, together with (2.26), we get for almost every $t \in (0,T)$

$$\begin{aligned}
\|\nabla^2 p_{h,k,\rho}(t)\phi^3\|_6 &\le c(\rho - r)^{-1} \|\nabla^2 p_{h,k,\rho}(t)\phi^2\|_{2,B_\rho} + c\|\nabla^3 p_{h,k,\rho}(t)\phi^3\|_2 \\
&\le c(\rho - r)^{-1} \|\nabla^2 p_{h,k,\rho}(t)\phi^2\|_{2,B_\rho} \\
&\le c(\rho - r)^{-2} \|\nabla p_{h,k,\rho}(t)\|_{2,B_\rho} \\
&\le c(\rho - r)^{-2} \|u_k(t)\|_{2,B_\rho}.
\end{aligned}$$

Integrating both sides of the above estimate, and estimating the right-hand side of the resultant inequality by (2.21), we arrive at

$$\|\nabla^2 p_{h,k,\rho}\phi^3\|_{L^2(0,T;L^6)} \le c(\rho - r)^{-2} T^{\frac{1}{2}} K_0 \lambda^{\frac{1}{2}k}. \tag{2.27}$$

Arguing as above, and using (2.27), we find

$$\begin{aligned}
III &\le cT^{\frac{1}{6}} \|b\|_{L^3(0,TL^3(B_\rho))} \|v_k\phi^3\|_{L^\infty(0,T;L^2)} \|\nabla^2 p_{h,k,\rho}\phi^3\|_{L^2(0,T;L^6)} \\
&\le cK_0(\rho - r)^{-2} T^{\frac{2}{3}} \lambda^{\frac{1}{2}k} \|b\|_{L^3(0,TL^3(B_\rho))} \|v_k\phi^3\|_{L^\infty(0,T;L^2)} \\
&\le c|||b|||K_0(\rho - r)^{-2} \lambda^{\frac{1}{2}k} T^{\frac{13}{18}} \|v_k\phi^3\|_{L^\infty(0,T;L^2)} \\
&\le c|||b|||^2 K_0^2 (\rho - r)^{-4} \lambda^k T^{\frac{13}{9}} + \frac{1}{8} \|v_k\phi^3\|^2_{L^\infty(0,T;L^2)}.
\end{aligned}$$

(iv) We now going to estimate IV. Using (1.8), and arguing similar as before, we estimate

$$\begin{aligned}
IV &\le c(\rho - r)^{-1} \|p_{1,k,\rho}\|_{L^{\frac{6}{5}}(0,T;L^2(B_\rho))} \|v_{k,\rho}\phi^3\|_{L^6(0,t;L^2)} \\
&\le c(\rho - r)^{-1} T^{\frac{1}{6}} \|bu_k\|_{L^{\frac{6}{5}}(0,T;L^2(B_\rho))} \|v_{k,\rho}\phi^3\|_{L^\infty(0,T;L^2)} \\
&\le c(\rho - r)^{-1} T^{\frac{1}{6}} \|b\|_{L^3(0,T;L^3(B_\rho))} \|u_k\|_{L^2(0,T;L^6(B_\rho))} \|v_{k,\rho}\phi^3\|_{L^\infty(0,T;L^2)} \\
&\le c|||b|||(\rho - r)^{-1} \lambda^{\frac{1}{3}k} T^{\frac{2}{9}} \|u_k\|_{L^2(0,T;L^6(B_\rho))} \|v_{k,\rho}\phi^3\|_{L^\infty(0,T;L^2)} \\
&\le c|||b|||K_0(\rho - r)^{-1} \rho^{-1} \lambda^{\frac{1}{3}k} T^{\frac{13}{18}} \|u_k\|_{L^\infty(0,T;L^2)} \|v_{k,\rho}\phi^3\|_{L^\infty(0,T;L^2)} \\
&\quad + c|||b|||(\rho - r)^{-1} \lambda^{\frac{1}{3}k} T^{\frac{2}{9}} \|v_{k,\rho}\phi^3\|_{L^\infty(0,T;L^2)} \|\nabla u_k\|^{\frac{1}{3}}_{L^2(0,T;L^2(B_\rho))} \|\nabla u_k\|^{\frac{2}{3}}_{L^2(0,T;L^2(B_\rho))} \\
&\le c|||b|||K_0(\rho - r)^{-1} \lambda^{\frac{7}{30}k} T^{\frac{13}{18}} \|v_{k,\rho}\phi^3\|_{L^\infty(0,T;L^2)} \\
&\quad + c|||b|||K_0^{\frac{1}{3}} (\rho - r)^{-1} \lambda^{\frac{1}{2}k} T^{\frac{2}{9}} \|v_{k,\rho}\phi^3\|_{L^\infty(0,T;L^2)} \|\nabla u_k\|^{\frac{2}{3}}_{L^2(0,T;L^2(B_\rho))} \\
&\le c|||b|||^2 K_0^2 (\rho - r)^{-2} \lambda^{\frac{7}{15}k} T^{\frac{13}{9}} + c|||b|||^6 K_0^2 (\rho - r)^{-6} \lambda^{3k} T^{\frac{4}{3}} \\
&\quad + \frac{1}{8} \|v_{k,\rho}\phi^3\|^2_{L^\infty(0,T;L^2)} + \frac{1}{4} \|\nabla u_k\|^2_{L^2(0,T;L^2(B_\rho))} \\
&\le (1 + |||b|||^6) K_0^2 (\rho - r)^{-6} \lambda^{\frac{17}{5}k} \max\{T^{\frac{13}{9}}, T\} \\
&\qquad + \frac{1}{8} \|v_{k,\rho}\phi^3\|^2_{L^\infty(0,T;L^2)} + \frac{1}{4} \|\nabla u_k\|^2_{L^2(0,T;L^2(B_\rho))}.
\end{aligned}$$

(v) Recalling the definition of $p_{2,k,\rho}$, using (1.8), (2.21) and Young's inequality, we get

$$\begin{aligned} V &\le c(\rho-r)^{-1}\|p_{2,k,\rho}\|_{L^2(0,T;L^2(B_\rho))}\|v_{k,\rho}\phi^3\|_{L^2(0,T;L^2)} \\ &\le c(\rho-r)^{-1}T^{\frac{1}{2}}\bigg(\int_0^T\int_{B_\rho}|\nabla u_k|^2dxdt\bigg)^{\frac{1}{2}}\|v_{k,\rho}\phi^3\|_{L^\infty(0,T;L^2)} \\ &\le cK_0^2(\rho-r)^{-2}\lambda^k T+\frac{1}{8}\|v_{k,\rho}\phi^3\|^2_{L^\infty(0,T;L^2)} \\ &\le cK_0^2(\rho-r)^{-6}\lambda^{\frac{17}{5}k}T+\frac{1}{8}\|v_{k,\rho}\phi^3\|^2_{L^\infty(0,T;L^2)}. \end{aligned}$$

(vi) It only remains to evaluate VI. Let $k\ge 9$. Then $\frac{3}{5}(k+1)\le k-3$. Thus, $\operatorname{supp}(\psi_k)\cap B_\rho=\emptyset$. In particular, $\psi_k u_0=0$ in B_ρ. This shows that, almost everywhere in B_ρ it holds

$$u_{0,k}=\mathbb{P}(\psi_k u_0)-\psi_k u_0$$

which is a gradient field. Accordingly, almost everywhere in B_ρ

$$v_{0,k}=u_{0,k}-E^*_{B_\rho}(u_{0,k})=u_{0,k}-u_{0,k}=0.$$

Hence

$$VI=0.$$

For $k\le 8$ we find

$$VI\le\|u_{0,k}\|^2_{L^2(B_\rho)}\le c\sum_{k=0}^{8}\|u_0\psi_k\|^2_{L^2}\le c\|u_0\|^2_{L^2(B_{\lambda^8})}\le cK_0^2.$$

We now insert the above estimates of $I,\dots,VI$ into the right-hand side of (2.22). This gives

$$\begin{aligned} &\operatorname*{ess\,sup}_{t\in(0,T)}\int_{B_\rho}|v_{k,\rho}(t)|^2\phi^6dx+\int_0^T\int_{B_\rho}|\nabla v_{k,\rho}|^2\phi^6dxdt \\ &\quad\le cK_0^2\max\{8-k,0\}+c(1+|||b|||^6)K_0^2\max\{T^{\frac{13}{9}},T\}(\rho-r)^{-6}\lambda^{\frac{17}{5}k} \\ &\qquad+\frac{1}{4}\int_0^T\int_{B_\rho}|\nabla u_k|^2dxdt. \end{aligned}\tag{2.28}$$

On the other hand, employing (2.26) and (2.21)

$$\int_{B_\rho}|\nabla^2 p_{h,k,\rho}|^2\phi^6dxdt\le cK_0^2(\rho-r)^{-2}\lambda^kT,$$

we estimate

$$\begin{aligned} &\int_0^T\int_{B_r}|\nabla u_k|^2dxdt \\ &\quad\le 2\int_0^T\int_{B_\rho}|\nabla v_{k,\rho}|^2\phi^6dxdt+2\int_0^T\int_{B_\rho}|\nabla^2 p_{h,k,\rho}|^2\phi^6dxdt \\ &\quad\le 2\int_0^T\int_{B_\rho}|\nabla v_{k,\rho}|^2\phi^6dxdt+cK_0^2(\rho-r)^{-2}\lambda^kT. \end{aligned}\tag{2.29}$$

Combining (2.28) and (2.29), we are led to

$$\int_0^T \int_{B_r} |\nabla u_k|^2 dx dt$$
$$\leq cK_0^2 \max\{8-k, 0\} + c(1+|||b|||^6)K_0^2 \max\{T^{\frac{13}{9}}, T\}(\rho - r)^{-6}\lambda^{\frac{17}{5}k}$$
$$+ \frac{1}{2}\int_0^T \int_{B_\rho} |\nabla u_k|^2 dx dt. \tag{2.30}$$

By virtue of a routine iteration argument from (2.30) we get for all $\rho \in [\lambda^{\frac{3}{5}k}, 2\lambda^{\frac{3}{5}k}]$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \int_{B_{\rho/2}} |v_{k,\rho}(t)|^2 dx + \int_0^T \int_{B_{\rho/2}} |\nabla u_k|^2 dx dt$$
$$\leq cK_0^2 \max\{8-k, 0\} + c(1+|||b|||^6)K_0^2 \max\{T^{\frac{13}{9}}, T\}\rho^{-6}\lambda^{\frac{17}{5}k}$$
$$\leq cK_0^2 \max\{8-k, 0\} + c(1+|||b|||^6)K_0^2 \max\{T^{\frac{13}{9}}, T\}\lambda^{-\frac{1}{5}k}. \tag{2.31}$$

In addition, by using the mean value property of harmonic functions along with (2.21), we estimate for almost all $t \in (0, T)$

$$\|\nabla p_{h,k,\rho}(t)\|^2_{L^2(B_{\lambda^{\frac{1}{4}k}})} \leq c\lambda^{\frac{3}{4}k}\|\nabla p_{h,k,\rho}(t)\|^2_{L^\infty(B_{\rho/2})}$$
$$\leq c\lambda^{-\frac{21}{20}k}\|\nabla p_{h,k,\rho}(t)\|^2_{L^2(B_\rho)}$$
$$\leq c\lambda^{-\frac{21}{20}k}\|u_k\|^2_{L^\infty(0,T;L^2(B_\rho))} \leq cK_0^2\lambda^{-\frac{1}{20}k}.$$

Combining this estimate with (2.31), we obtain

$$\operatorname*{ess\,sup}_{t\in(0,T)} \int_{B_{\lambda^{\frac{1}{4}k}}} |u_k(t)|^2 dx + \int_0^T \int_{B_{\lambda^{\frac{3}{5}k}}} |\nabla u_k|^2 dx dt$$
$$\leq cK_0^2\Big(1 + |||b|||^6 \max\{T^{\frac{13}{9}}, T\}\Big)\lambda^{-\frac{1}{20}k}. \tag{2.32}$$

Next, let $l \in \mathbb{N}$ be fixed. Then (2.32) implies for all $k \geq l$

$$\|u_k\|_{L^\infty(0,T;L^2(B_{\lambda^{\frac{1}{4}l}}))} + \|\nabla u_k\|_{L^2(B_{\lambda^{\frac{3}{5}l}}\times(0,T))}$$
$$\leq \|u_k\|_{L^\infty(0,T;L^2(B_{\lambda^{\frac{1}{4}k}}))} + \|\nabla u_k\|_{L^2(B_{\lambda^{\frac{3}{5}k}}\times(0,T))}$$
$$\leq cK_0\Big(1 + |||b|||^3 \max\{T^{\frac{13}{18}}, T^{\frac{1}{2}}\}\Big)\lambda^{-\frac{1}{40}k}. \tag{2.33}$$

Thus, by means of triangular inequality we find for each $N \in \mathbb{N}$, $N > l$

$$\Big\|\sum_{k=0}^N u_k\Big\|_{L^\infty(0,T;L^2(B_{\lambda^{\frac{1}{4}l}}))} + \Big\|\sum_{k=0}^N \nabla u_k\Big\|_{L^2(B_{\lambda^{\frac{3}{5}l}}\times(0,T))}$$
$$\leq \sum_{k=0}^{l-1}\|u_k\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} + \sum_{k=0}^{l-1}\|\nabla u_k\|_{L^2(\mathbb{R}^3\times(0,T))}$$
$$+ \sum_{k=l}^{N}\|u_k\|_{L^\infty(0,T;L^2(B_{\lambda^{\frac{1}{4}l}}))} + \sum_{k=0}^{N}\|\nabla u_k\|_{L^2(B_{\lambda^{\frac{3}{5}l}}\times(0,T))}$$

$$\leq cK_0\lambda^{\frac{1}{2}l} + cK_0\Big(1 + |||b|||^3 \max\{T^{\frac{13}{18}}, T^{\frac{1}{2}}\}\Big)$$
$$\leq cK_0\Big(\lambda^{\frac{1}{2}l} + |||b|||^3 \max\{T^{\frac{13}{18}}, T^{\frac{1}{2}}\}\Big).$$

Therefore, $u^N = \sum_{k=0}^N u_k \to u$ in $V^2_{\rm loc}(\mathbb{R}^3 \times [0,T])$ as $N \to \infty$. It is readily seen that u is a weak solution to (1.1)–(1.3), and by virtue of the above estimate we see that for every $1 \leq \rho < \infty$

$$\|u\|_{L^\infty(0,T;L^2(B_{\rho^{\frac{1}{4}}}))} + \|\nabla u\|_{L^2(B_{\rho^{\frac{3}{5}}}\times(0,T))} \leq cK_0\Big(\rho^{\frac{1}{2}} + |||b|||^3 \max\{T^{\frac{13}{18}}, T^{\frac{1}{2}}\}\Big). \tag{2.34}$$

In particular, in (2.34) taking $\rho = 1$, and using Sobolev's embedding theorem, we get

$$\|u\|_{L^4(0,T;L^3(B_1))} + \|u\|_{V^2(B_1\times(0,T))} \leq C_0K_0\Big(1 + |||b|||^3 \max\{T^{\frac{13}{18}}, T^{\frac{1}{2}}\}\Big) \tag{2.35}$$

with a constant $C_0 > 0$ depending only on λ. Furthermore, by means of the assumption on b we see that u satisfies (2.5) with the equality(=) replaced by the inequality($\leq$) and this belongs to $C([0,T]; L^2(B_R))$ for all $0 < R < +\infty$, and therefore it is unique. It remains to show that $u_\lambda = u$. Let $N \in \mathbb{N}$, $N \geq 4$. We set $w^N = u^N - u^N_\lambda$. Recalling that $b = b_\lambda$, it follows that w^N solves the system

$$\nabla \cdot w^N = 0 \quad \text{in} \quad Q_{\lambda^{-2}T}, \tag{2.36}$$
$$\partial_t w^N + (b\cdot\nabla)w^N - \Delta w^N = -\nabla\pi^N \quad \text{in} \quad Q_{\lambda^{-2}T}, \tag{2.37}$$
$$w^N = w^N_0 \quad \text{on} \quad \mathbb{R}^3 \times \{0\}, \tag{2.38}$$

where

$$\begin{aligned} w_0^N &= \sum_{k=0}^N u_{0,k} - (u_{0,k})_\lambda = \sum_{k=0}^N \mathbb{P}(u_0\psi_k) - (\mathbb{P}(u_0\psi_k))_\lambda \\ &= u_0\sum_{k=0}^N \psi_k - \Big(u_0\sum_{k=0}^N\psi_k\Big)_\lambda + \nabla\mathcal{N} * (u_0\cdot\nabla\sum_{k=0}^N\psi_k) - \Big(\nabla\mathcal{N} * (u_0\cdot\nabla\sum_{k=0}^N\psi_k)\Big)_\lambda \\ &= u_0\Big(\sum_{k=0}^N\psi_k - \Big(\sum_{k=0}^N\psi_k\Big)(\lambda\cdot)\Big) + \nabla\mathcal{N} * (u_0\cdot\nabla\sum_{k=0}^N\psi_k) - \Big(\nabla\mathcal{N} * (u_0\cdot\nabla\sum_{k=0}^N\psi_k)\Big)_\lambda, \end{aligned}$$

where $\mathcal{N} = \frac{1}{4\pi|x|}$ stands for the Newton potential. For obtaining the third line in the above equalities we used the fact that $(u_0)_\lambda = u_0$. Owing to $\sum_{k=0}^N \psi_k = 1$ in $B_{\lambda^{N-3}}$ we have

$$\Big(\sum_{k=0}^N\psi_k - \Big(\sum_{k=0}^N\psi_k\Big)(\lambda\cdot)\Big) = 0 \quad \text{in} \quad B_{\lambda^{N-4}}. \tag{2.39}$$

Let $\lambda^{\frac{3}{5}N} \leq r < \rho \leq \lambda^{\frac{3}{5}(N+1)}$ be arbitrarily chosen, but fixed. Let $\phi \in C^\infty_c(\mathbb{R}^3)$ denote a cut off function such that $0 \leq \phi \leq 1$ in $\mathbb{R}^3$, $\phi \equiv 1$ on B_r, $\phi \equiv 0$ in $\mathbb{R}^3 \setminus B_\rho$, and $|\nabla^2\phi| + |\nabla\phi|^2 \leq c(\rho - r)^{-2}$ in $\mathbb{R}^3$. Without loss of generality we may assume that $\lambda^{\frac{3}{5}(N+1)} \leq \lambda^{N-4}$. Thus, in view of (2.39) we infer that w^N_0 is a gradient field in B_ρ, and therefore

$$w^N_0 - E^*_{B_\rho}(w^N_0) = 0 \quad \text{a.e. in } B_\rho. \tag{2.40}$$

By a similar reasoning we have used to prove (2.30) we get the estimate

$$\|w^N\|^2_{L^2(0,\lambda^{-2}T;L^6(B_r))} + \int_0^{\lambda^{-2}T}\int_{B_r}|\nabla w^N|^2dxdt$$
$$\leq cK_0^2(1 + |||b|||^6)\max\{T^{\frac{13}{9}}, T\}(\rho - r)^{-6}\lambda^{\frac{17}{5}N} + \frac{1}{2}\int_0^{\lambda^{-2}T}\int_{B_\rho}|\nabla w^N|^2dxdt. \tag{2.41}$$

Once more applying an iteration argument, together with the latter estimate, we deduce from (2.41)

$$\|w^N\|^2_{L^2(0,\lambda^{-2}T;L^6(B_{\lambda^{\frac{3}{5}N}}))} \le cK_0^2(1+|||b|||^6)\max\{T^{\frac{13}{9}},T\}\lambda^{-\frac{1}{5}N}. \tag{2.42}$$

Accordingly, for all $0<\rho<\infty$,

$$w^N \to 0 \quad in \quad L^2(0,\lambda^{-2}T;L^6(B_\rho)) \quad as \quad N\to+\infty.$$

On the other hand, observing that $w^N = u^N - (u^N)_\lambda \to u - u_\lambda$ in $L^2(0,\lambda^{-2}T;L^6(B_\rho))$ as $N\to\infty$, we conclude that $u=u_\lambda$. This completes the proof of the theorem. □

3. Proof of Theorem 1.4

We divide the proof in three steps. Firstly, given a λ-DSS function $b\in L^{\frac{18}{5}}_{loc}([0,\infty);L^3_{\rm loc}(\mathbb{R}^3))$ we get the existence of a unique λ-DSS local solution with projected pressure u to the linearized system (2.1)–(2.3), replacing b by $R_\varepsilon b$ therein (cf. appendix for the notion of the mollification R_ε). Secondly, based on the first step we may construct a mapping $\mathcal{T}: M\to M$, which is continuous and compact. Application of Schauder's fixed point theorem gives a local suitable solution with projected pressure to the approximated Navier–Stokes equation. Thirdly, letting $\varepsilon\to 0^+$ in the weak formulation and in the local energy inequality (2.5), we obtain the existence of the desired local Leray solution with projected pressure to (1.1)–(1.3).

We set

$$T := \min\left\{\frac{1}{64C_0^6K_0^6\lambda^{\frac{10}{3}}}, \left(\frac{1}{64C_0^6K_0^6\lambda^{\frac{10}{3}}}\right)^{\frac{9}{13}}\right\}. \tag{3.1}$$

Furthermore, set $X = L^3_{\lambda-DSS}(Q)\cap L^{\frac{18}{5}}(0,T;L^3_{loc,\sigma}(\mathbb{R}^3))$ equipped with the norm

$$|||v||| := \|v\|_{L^{\frac{18}{5}}(0,T;L^3(B_1))}, \qquad v\in X.$$

Then we define,

$$M = \Big\{ b\in X \,\Big|\, |||b||| \le 2C_0K_0 \Big\}.$$

We now fix $0<\varepsilon<\lambda-1$. For $b\in M$ we set

$$b_\varepsilon := R_\varepsilon b,$$

where R_ε stands for the mollification operator defined in the appendix below. According to Theorem 2.2 there exists a unique λ-DSS solution $u\in X$ to (2.1)–(2.3) with b_ε in place of b. Observing (2.35), it follows that

$$\|u\|_{L^4(0,T;L^3(B_1))} + \|u\|_{V^2(B_1\times(0,T))} \le C_0K_0\Big(1+|||b_\epsilon|||^3\max\{T^{\frac{13}{18}},T^{\frac{1}{2}}\}\Big). \tag{3.2}$$

In view of (A.2) having $|||b_\varepsilon|||^3 \le \lambda^{\frac{5}{3}}|||b|||^3$, (3.2) together with (3.1) implies that

$$|||u||| \le 2C_0K_0,$$

and thus $u\in M$. By setting $\mathcal{T}_\varepsilon(b):=u$ defines a mapping $\mathcal{T}_\varepsilon: M\to M$.

$\mathcal{T}_\varepsilon$ is closed. In fact, let $\{b_k\}$ be a sequence in M such that $b_k\to b$ in X as $k\to\infty$, and let $u_k:=\mathcal{T}_\varepsilon(b_k)$, $k\in\mathbb{N}$, such that $u_k\to u$ in X as $k\to\infty$. From (3.2) it follows that $\{u_k\}$ is bounded in $V^2_\sigma(B_1\times(0,T))$, and thus, eventually passing to a subsequence, we find that $u_k\to u$ weakly in $V^2_\sigma(B_1\times(0,T))$ as $k\to\infty$. Since u_k solves (2.1)–(2.3) with $b_{k,\varepsilon}=R_\varepsilon b_k$ in place of b, from the above convergence properties we deduce that $u\in M\cap V^2_\sigma(B_1\times(0,T))$ solves (2.1)–(2.3). Accordingly, $u=\mathcal{T}_\varepsilon(b)$.

$\mathcal{T}_\varepsilon(M)$ is relatively compact in X. To see this, let $\{u_k=\mathcal{T}_\varepsilon(b_k)\}\subset\mathcal{T}_\varepsilon(M)$ be any sequence. Then $u_k\in L^2_{loc,\sigma}(\mathbb{R}^3\times[0,\infty))$ is a λ-DSS local suitable weak solution with projected pressure to

$$\nabla\cdot u_k = 0 \quad \text{in} \quad Q, \tag{3.3}$$

$$\partial_t u_k + (b_{k,\varepsilon}\cdot\nabla)u_k - \Delta u_k = -\nabla\pi_k \quad \text{in} \quad Q, \tag{3.4}$$

$$u_k = u_0 \quad \text{on} \quad \mathbb{R}^3\times\{0\}. \tag{3.5}$$

Introducing the local pressure, we have

$$\partial_t v_k + (b_{k,\varepsilon}\cdot\nabla)u_k - \Delta u_k = -\nabla p_{1,k} - \nabla p_{2,k} \quad \text{in} \quad B_2\times(0,T), \tag{3.6}$$

where $v_k = u_k + \nabla p_{h,k}$, and

$$\nabla p_{h,k} = -E^*_{B_2}(u_k),$$
$$\nabla p_{1,k} = -E^*_{B_2}((b_{k,\varepsilon}\cdot\nabla)u_k), \quad \nabla p_{2,k} = E^*_{B_2}(\Delta u_k).$$

Thus, (3.4) implies that $v'_k = \nabla\cdot(-b_{k,\varepsilon}\otimes u_k + \nabla u_k - p_{1,k}I - p_{2,k}I)$ in $B_2\times(0,T)$. Since $b_k, u_k \in M$ we get the estimate

$$\| - b_{k,\varepsilon}\otimes u_k + \nabla u_k - p_{1,k}I - p_{2,k}I\|_{L^{\frac{9}{5}}(0,T;L^{\frac{3}{2}}(B_2))} \le c(1 + C_0^2K_0^2).$$

Furthermore, by means of the reflexivity of $L^2(0,T;W^{1,2}(B_2))$, and using Banach–Alaoglu's theorem we get a subsequence $\{u_{k_j}\}$ and a function $u\in M\cap V^2_{loc,\sigma}(\mathbb{R}^3\times[0,T])$ such that

$$u_{k_j}\to u \quad \textit{weakly in} \quad L^2(0,T;W^{1,2}(B_2)),$$
$$u_{k_j}\to u \quad \textit{weakly}^* \textit{ in} \quad L^\infty(0,T;L^2(B_2)) \quad \textit{as} \quad j\to\infty.$$

In particular, we have for almost every $t\in(0,T)$

$$u_{k_j}(t)\to u(t) \quad \textit{weakly in} \quad L^2(B_2) \quad \textit{as} \quad j\to\infty. \tag{3.7}$$

In addition, verifying that $\{v_{k_j}\}$ is bounded in $V^2(B_2\times(0,T))$, by Lions–Aubin's compactness lemma we see that

$$v_{k_j}\to v \quad \textit{in} \quad L^2(B_2\times(0,T)) \quad \textit{as} \quad j\to+\infty, \tag{3.8}$$

where $v = u + \nabla p_h$, and $\nabla p_h = -E^*(u)$. Now, let $t\in(0,T)$ be fixed such that (3.7) is satisfied. Then

$$\nabla p_{h,k_j}(t)\to\nabla p_h(t) \quad \textit{weakly in} \quad L^2(B_2) \quad \textit{as} \quad j\to\infty. \tag{3.9}$$

Since $p_{h,k}$ is harmonic in B_2, from (3.9) we deduce that

$$\nabla p_{h,k_j}(t)\to\nabla p_h(t) \quad \textit{a. e. in} \quad B_2 \quad \textit{as} \quad j\to\infty. \tag{3.10}$$

On the other hand, using the mean value property of harmonic functions, we see that $\{\nabla p_{h,k}\}$ is bounded in $L^\infty(B_1\times(0,T))$. Appealing to Lebesgue's theorem of dominated convergence, we infer from (3.10) that

$$\nabla p_{h,k_j}\to\nabla p_h \quad \textit{in} \quad L^2(B_1\times(0,T)) \quad \textit{as} \quad j\to\infty. \tag{3.11}$$

Now combining (3.8) and (3.11), we obtain $u_{k_j}\to u$ in $L^2(B_1\times(0,T))$. Recalling that $\{u_{k_j}\}$ is bounded in $V^2(B_1\times(0,T))$, we get the desired convergence property $u_{k_j}\to u$ in X as $j\to\infty$. To see this we argue as follows. Eventually passing to a subsequence, we may assume that $u_{k_j}\to u$ almost everywhere in $B_1\times(0,T)$. Let $\varepsilon>0$ be arbitrarily chosen. We denote $A_m = \{(x,t)\in B_1\times(0,T)\,|\,\exists\, j\ge m : |u_{k_j}(x,t)-u(x,t)|>\varepsilon\}$. Clearly, $\cap_{m=1}^\infty A_m$ is a set of Lebesgue measure zero. Thus meas $A_m\to 0$ as $m\to\infty$. We now get the following estimate

$$\begin{aligned}
&\|u_{k_j}-u\|_{L^{\frac{18}{5}}(0,T;L^3(B_1))} = \\
&\le \|(u_{k_j}-u)\chi_{A_m}\|_{L^{\frac{18}{5}}(0,T;L^3(B_1))} + \|(u_{k_j}-u)\chi_{A_m^c}\|_{L^{\frac{18}{5}}(0,T;L^3(B_1))} \\
&\le \|u_{k_j}-u\|_{L^{\frac{168}{45}}(0,T;L^{\frac{28}{9}}(B_1))}\|\chi_{A_m}\|_{L^{\frac{504}{5}}(0,T;L^{84}(B_1))} + \|(u_{k_j}-u)\chi_{A_m^c}\|_{L^{\frac{18}{5}}(0,T;L^3(B_1))}. \\
&\le c(\text{meas}\, A_m)^{\frac{5}{504}} + c\varepsilon.
\end{aligned}$$

This shows that $|||u_{k_j}-u|||\to 0$ as $j\to\infty$. Applying Schauder's fixed point theorem, we get a function $u_\varepsilon\in M$ such that $u_\varepsilon = \mathcal{T}_\varepsilon(u_\varepsilon)$. Thus, u_ε is a local suitable weak solution with projected pressure to

$$\nabla \cdot u_\varepsilon = 0 \quad \text{in} \quad Q, \tag{3.12}$$

$$\partial_t u_\varepsilon + (R_\varepsilon u_\varepsilon \cdot \nabla) u_\varepsilon - \Delta u_\varepsilon = -\nabla \pi_\varepsilon \quad \text{in} \quad Q, \tag{3.13}$$

$$u_\varepsilon = u_0 \quad \text{on} \quad \mathbb{R}^3 \times \{0\}. \tag{3.14}$$

In particular, we have the a-priori estimate

$$\|u_\varepsilon\|_{L^4(0,T;L^3(B_1))} + \|u_\varepsilon\|_{V^2(B_1\times(0,T))} \le 2C_0 K_0. \tag{3.15}$$

Let $\{\varepsilon_j\}$ be a sequence of positive numbers in $(0, \lambda - 1)$. Since u_{ε_j} is λ-DSS we may apply Lemma B.5 which shows that, after redefining u_{ε_j} on a set in $[0, +\infty)$ of measure zero, it holds $u_\varepsilon \in C_w([0, +\infty), L^2_{loc}(\mathbb{R}^3)))$ together with

$$M(u_{\varepsilon_j}) = [0, +\infty),$$

where $M(u_{\varepsilon_j})$ denotes the set of all $t \in [0, +\infty)$ such that for all $k \in \mathbb{Z}$ and almost every $x \in \mathbb{R}^3$

$$u(x,t) = \lambda^k u_{\varepsilon_j}(\lambda^k x, \lambda^{2k} t).$$

We now define for $t \in [0, +\infty)$ and $j \in \mathbb{N}$ the set $P_j(t) \subset \mathbb{R}^3$ such that

$$u_{\varepsilon_j}(x,t) = \lambda^k u_{\varepsilon_j}(\lambda^k x, \lambda^{2k} t) \quad \forall x \in P_j(t), \quad \forall k \in \mathbb{Z}.$$

Since $t \in M(u_{\varepsilon_j})$ it holds $\operatorname{meas} \mathbb{R}^3 \setminus P_j(t) = 0$. Accordingly, $\operatorname{meas} \mathbb{R}^3 \setminus P(t) = 0$, where $P(t) = \cap_{j=1}^\infty P_j(t)$. In other words, it holds

$$u_{\varepsilon_j}(x,t) = \lambda^k u_{\varepsilon_j}(\lambda^k x, \lambda^{2k} t) \quad \forall x \in P(t), \quad \forall k \in \mathbb{Z}, \forall j \in \mathbb{N}.$$

By means of the reflexivity we get a sequence $\varepsilon_j \to 0^+$ as $j \to \infty$ and $u \in V^2_{loc,\sigma}(\mathbb{R}^3 \times [0, T])$ such that

$$u_{\varepsilon_j} \to u \quad \textit{weakly in} \quad L^2(0,T; W^{1,2}(B_1)) \quad \textit{as} \quad j \to +\infty,$$

$$u_{\varepsilon_j} \to u \quad \textit{weakly* in} \quad L^\infty(0,T; L^2(B_1)) \quad \textit{as} \quad j \to +\infty.$$

Arguing as in the proof the compactness of $\mathcal{T}_\varepsilon$, we infer

$$u_{\varepsilon_j} \to u \quad \textit{in} \quad L^{\frac{18}{5}}(0,T; L^3(B_1)) \quad \textit{as} \quad j \to 0^+.$$

Note that u is λ-DSS, since u is obtained as a limit of sequence of λ-DSS functions.

Together with Lemma A.3 we see that

$$R_{\varepsilon_j} u_{\varepsilon_j} \to u \quad \textit{in} \quad L^{\frac{18}{5}}(0,T; L^3(B_1)) \quad \textit{as} \quad j \to 0^+. \tag{3.16}$$

This shows that $u \in L^2_{loc,\sigma}(\mathbb{R}^3 \times [0, +\infty))$ is a local Leray solution with projected pressure to (1.1)–(1.3). □

Conflict of interest statement

The authors declare there exists no conflict of interest with respect to the content of this paper.

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Appendix A. Mollification for DSS functions

Let $1 < \lambda < +\infty$. Let $u \in L^s_{\lambda-DSS}(\mathbb{R}^3)$. Let $\rho \in C^\infty_c(B_1)$ denote the standard mollifying kernel such that $\int_{\mathbb{R}^3} \rho dx = 1$. For $0 < \varepsilon < \lambda - 1$ we define

$$(R_\varepsilon u)(x,t) = \frac{1}{(\sqrt{t}\varepsilon)^3} \int\limits_{B_{\sqrt{t}\varepsilon}} u(x-y,t) \rho\Big(\frac{y}{\sqrt{t}\varepsilon}\Big) dy, \quad (x,t) \in Q.$$

We have the following

Lemma A.1. *R_ε defines a bounded operator from $L^s_{\lambda-DSS}(Q)$ into itself. Furthermore, for all $u \in L^s_{\lambda-DSS}(Q)$ it holds for all $(x.t) \in Q$*

$$|(R_\varepsilon u)(x,t)| \leq c\{\sqrt{t\varepsilon}\}^{-\frac{3}{s}}\|u(\cdot,t)\|_{L^s(B_{\sqrt{t\varepsilon}}(x))} \tag{A.1}$$

with a constant $c > 0$ depending on s only.

Proof. Let $u \in L^s_{\lambda-DSS}(Q)$. First we will verify that $R_\varepsilon u$ is λ-DSS. Indeed, using the transformation formula of the Lebesgue integral, we calculate for any $(x,t) \in Q$,

$$\begin{aligned}
\lambda(R_\varepsilon u)(\lambda x, \lambda^2 t) &= \frac{1}{\lambda^2(\sqrt{t\varepsilon})^3}\int\limits_{B_{\lambda\sqrt{t\varepsilon}}} u(\lambda x - y, \lambda^2 t)\rho\Big(\frac{y}{\lambda\sqrt{t\varepsilon}}\Big)dy,\\
&= \frac{1}{(\sqrt{t\varepsilon})^3}\int\limits_{\mathbb{R}^3} \lambda u(\lambda(x-y), \lambda^2 t)\rho\Big(\frac{y}{\sqrt{t\varepsilon}}\Big)dy\\
&= \frac{1}{(\sqrt{t\varepsilon})^3}\int\limits_{\mathbb{R}^3} u(x-y,t)\rho\Big(\frac{y}{\sqrt{t\varepsilon}}\Big)dy = (R_\varepsilon u)(x,t).
\end{aligned}$$

Firstly, let $\lambda^{-2} < t \leq 1$. Noting that $(R_\varepsilon u)(\cdot,t) = u(\cdot,t) * \rho_{\sqrt{t\varepsilon}}$, where $\rho_{\sqrt{t\varepsilon}}(y) = \frac{1}{(\sqrt{t\varepsilon})^3}\rho\Big(\frac{y}{\sqrt{t\varepsilon}}\Big)$, recalling that $\varepsilon < \lambda - 1$, by means of Young's inequality we find

$$\|(R_\varepsilon u)(\cdot,t)\|^s_{L^s(B_1)} \leq \|u(\cdot,t)\|^s_{L^s(B_{1+\varepsilon})}\|\rho_{\sqrt{t\varepsilon}}\|^s_{L^1} = \|u(\cdot,t)\|^s_{L^s(B_\lambda)}.$$

Integrating the above inequality over $(\lambda^{-2}, 1)$, and using a suitable change of coordinates, we obtain

$$\begin{aligned}
\|R_\varepsilon u\|_{L^s(B_1\times(\lambda^{-2},1))} &\leq \|u\|_{L^s(B_\lambda\times(\lambda^{-2},1))}\\
&= \|u\|_{L^s(B_1\times(\lambda^{-2},1))} + \|u\|_{L^s(B_\lambda\setminus B_1\times(\lambda^{-2},1))}\\
&= \|u\|_{L^s(B_1\times(\lambda^{-2},1))} + \lambda^{\frac{5-s}{s}}\|u\|_{L^s(B_1\setminus B_{\lambda^{-1}}\times(\lambda^{-4},\lambda^{-2}))}.
\end{aligned}$$

Secondly, for $0 < t < \lambda^{-2}$ we estimate

$$\|(R_\varepsilon u)(\cdot,t)\|^s_{L^s(B_1\setminus B_{\lambda^{-1}})} \leq \|u(\cdot,t)\|^s_{L^s(B_\lambda\setminus B_{\lambda^{-1}})}\|\rho_{\sqrt{t\varepsilon}}\|^s_{L^1} = \|u(\cdot,t)\|^s_{L^s(B_\lambda\setminus B_{\lambda^{-1}})}.$$

Integration over $(0, \lambda^{-2})$ in time yields

$$\begin{aligned}
\|R_\varepsilon u\|_{L^s(B_1\setminus B_{\lambda^{-1}}\times(0,\lambda^{-2}))} &\leq \|u\|_{L^s(B_\lambda\setminus B_{\lambda^{-1}}\times(0,\lambda^{-2}))}\\
&\leq \|u\|_{L^s(B_1\setminus B_{\lambda^{-1}}\times(0,\lambda^{-2}))} + \|u\|_{L^s(B_\lambda\setminus B_1\times(0,\lambda^{-2}))}\\
&= \|u\|_{L^s(B_1\setminus B_{\lambda^{-1}}\times(0,\lambda^{-2}))} + \lambda^{\frac{5-s}{s}}\|u\|_{L^s(B_1\setminus B_{\lambda^{-1}}\times(0,\lambda^{-4})}.
\end{aligned}$$

Combining the last two estimates, we get

$$\|R_\varepsilon u\|_{L^s(Q_1\setminus Q_{\lambda^{-1}})} \leq (1+\lambda^{\frac{5-s}{s}})\|u\|_{L^s(Q_1\setminus Q_{\lambda^{-1}})}.$$

This shows that $R_\varepsilon : L^s_{\lambda-DSS}(Q) \to L^s_{\lambda-DSS}(Q)$ is bounded.

The inequality (A.1) follows immediately from the definition of $R_\varepsilon u$ with the help of Hölder's inequality. □

Remark A.2. Arguing as in the proof of Lemma A.1, we get for any $u \in L^3_{\lambda-DSS}(Q) \cap L^{\frac{18}{5}}(0,T; L^3(B_1))$, $0 < T < 1$

$$\|R_\varepsilon u\|_{L^{\frac{18}{5}}(0,T;L^3(B_1))} \leq \lambda^{\frac{5}{9}}\|u\|_{L^{\frac{18}{5}}(0,T;L^3(B_1))}. \tag{A.2}$$

Lemma A.3. *Let $u \in L^3_{\lambda-DSS}(Q) \cap L^{\frac{18}{5}}(0,T; L^3(B_1))$, $0 < T \leq 1$. Then*

$$R_\varepsilon u \to u \quad \text{in} \quad L^{\frac{18}{5}}(0,T; L^3(B_1)) \quad \text{as} \quad \varepsilon \to 0^+. \tag{A.3}$$

Proof. First by the absolutely continuity of the Lebesgue integral we see that for almost all $t \in (0,T)$

$$(R_\varepsilon u)(\cdot,t) \to u(\cdot,t) \quad in \quad L^3(B_1) \quad as \quad \varepsilon \to 0^+.$$

Let $A \subset (0,T)$ be any Lebesgue measurable set. By Young's inequality of convolutions we get for almost all $t \in (0,T)$

$$\int\limits_A \|(R_\varepsilon u)(\cdot,t)\|_{L^3(B_1)}^{\frac{18}{5}} dt \le \int\limits_A \|u(\cdot,t)\|_{L^3(B_\lambda)}^{\frac{18}{5}} dt.$$

Since $u \in L^{\frac{18}{5}}(0,T;L^3(B_\lambda))$, the assertion (A.3) follows by the aid of Vitali's convergence lemma. □

Appendix B. Weak trace for time dependent λ-DSS functions

Let $1<\lambda<+\infty$. A measurable function $u : Q \to \mathbb{R}^3$ is said to be λ-DSS, if for almost every $(x,t) \in Q$

$$u(x,t) = \lambda u(\lambda x, \lambda^2 t). \tag{B.1}$$

We denote by $M(u)$ the set of all $t \in [0,+\infty)$ such that for all $k \in \mathbb{Z}$

$$u(x,t) = \lambda^k u(\lambda^k x, \lambda^{2k} t) \quad \text{for a. e.} \quad x \in \mathbb{R}^3. \tag{B.2}$$

Lemma B.1. *The set* $[0,+\infty) \setminus M(u)$ *is a set of Lebesgue measure zero.*

Proof. For $m \in \mathbb{N}$ and $k \in \mathbb{Z}$ by $A_{m,k}$ we denote the set of all $t \in [0,+\infty)$ such that

$$\text{meas}\left\{x \in \mathbb{R}^3 \,\middle|\, u(x,t) \neq \lambda^k u(\lambda^k x, \lambda^{2k} t)\right\} \ge \frac{1}{m}.$$

Since u is discretely self-similar, we must have $\text{meas}(A_{m,k}) = 0$. Since $M(u) \setminus [0,+\infty) = \cup_{k\in\mathbb{Z}} \cup_{m=1}^{\infty} A_{m,k}$ the assertion follows. □

Lemma B.2. *For every* $t \in [0,+\infty)$ *it holds* $t \in M(u)$ *iff* $\lambda^2 t \in M(u)$.

Proof. Let $t \in M(u)$. There exists a set $P \subset \mathbb{R}^3$ with $\text{meas}(\mathbb{R}^3 \setminus P) = 0$ such that (B.2) holds for all $x \in P$. Define $P_k = \{y = \lambda^k x \,|\, x \in P\}$, $k \in \mathbb{Z}$. Clearly, $\text{meas}(\mathbb{R}^3 \setminus \cap_{k\in\mathbb{Z}} P_k) = 0$. Let $x \in \cap_{k\in\mathbb{Z}} P_k$. Then $x, \lambda^{-1}x \in P$, and therefore for all $k \in \mathbb{Z}$ we get $u(\lambda^{-1}x,t) = \lambda u(x,\lambda^2 t) = \lambda^{k+1} u(\lambda^k x, \lambda^{2+2k} t)$, which is equivalent to

$$u(x,\lambda^2 t) = \lambda^k u(\lambda^k x, \lambda^{2k}\lambda^2 t).$$

This shows that $\lambda^2 t \in M(u)$. Similarly, we get the opposite direction. □

As an immediate consequence of Lemma B.2 we see that

$$t \in M(u) \quad \Longleftrightarrow \quad \lambda^{2k} t \in M(u) \quad \forall k \in \mathbb{Z}. \tag{B.3}$$

Let $\{v_j\}$ be a sequence in $L^2_{loc}(\mathbb{R}^3)$. We say

$$v_j \to v \quad weakly\ in \quad L^2_{loc}(\mathbb{R}^3) \quad as \quad j \to +\infty$$

if for every $0 < R < +\infty$

$$v_j \to v \quad weakly\ in \quad L^2(B_R) \quad as \quad j \to +\infty.$$

Lemma B.3. *Let* $\{v_j\}$ *be a sequence in* $L^2_{loc}(\mathbb{R}^3)$ *such that for all* $0 < R < +\infty$

$$\sup_{j\in\mathbb{N}} \|v_j\|_{L^2(B_R)} < +\infty. \tag{B.4}$$

Then there exists a subsequence $\{v_{j_m}\}$ *and* $v \in L^2_{loc}(\mathbb{R}^3)$ *such that*

$$v_{j_m} \to v \quad weakly\ in \quad L^2_{loc}(\mathbb{R}^3) \quad as \quad m \to +\infty.$$

Proof. By induction and the reflexivity of $L^2(B_m)$ we construct a sequence of subsequences $\{v_{j_k^{(m)}}\} \subset \{v_{j_k^{(m-1)}}\}$ and $\{v_{j_k^0}\} = \{v_j\}$ such that for some $v_m \in L^2(B_k)$ it holds

$$v_{j_k^{(m)}} \to v_m \quad in \quad L^2(B_m) \quad as \quad k \to +\infty$$

($m \in \mathbb{N}$). Clearly, $v_m|_{B_{m-1}} = v_{m-1}$. This allows us to define $v : \mathbb{R}^3 \to \mathbb{R}$ be setting $v = v_m$ on B_m. Then by Cantor's diagonalization principle the subsequence $v_{j_m} = v_{j_m^{(m)}}$ meets the requirements. □

We denote $\mathcal{V} = L^\infty_{loc}([0, +\infty); L^2_{loc}(\mathbb{R}^3))$ the space of all measurable functions $u : Q \to \mathbb{R}$ such that $u \in L^\infty(0, R^2, L^2(B_R))$ for all $0 < R < +\infty$. By $\mathcal{V}_{\lambda-DSS}$ we denote the space of all λ-DSS functions $u \in \mathcal{V}$.

Lemma B.4. *Let $u \in \mathcal{V}_{\lambda-DSS}$. We assume that $\|u(t)\|_{L^2(B_R)} \le \|u\|_{L^\infty(0,R^2;L^2(B_R))}$ for all $t \in (0, R^2)$, $0 < R < +\infty$. There exists a constant $C > 0$ such that for every $t \in M(u)$*

$$\|u(t)\|_{L^2(B_R)} \le C \max\left\{ R^{1/2} \|u\|_{L^\infty(0,1;L^2(B_1))}, \|u(t)\|_{L^2(B_{\sqrt{t}})} \right\}. \tag{B.5}$$

Proof. Let $t \in M(u)$. Let $k \in \mathbb{Z}$. Then by means of the transformation formula we get

$$\begin{aligned}\int\limits_{B_{\lambda^k}} |u(x,t)|^2 dx &= \lambda^{3k} \int\limits_{B_1} |u(\lambda^k x, t)|^2 dx = \lambda^k \int\limits_{B_1} |\lambda^k u(\lambda^k x, \lambda^{2k}\lambda^{-2k} t)|^2 dx \\ &= \lambda^k \int\limits_{B_1} |u(x, \lambda^{-2k} t)|^2 dx.\end{aligned}$$

In case $\lambda^{2k} \ge t$ we get

$$\|u(t)\|^2_{L^2(B_{\lambda^k})} \le \lambda^k \|u\|^2_{L^\infty(0,1;L^2(B_1))}.$$

On the contrary, if $\lambda^{2k} < t$ we find

$$\|u(t)\|_{L^2(B_{\lambda^k})} \le \|u(t)\|_{L^2(B_{\sqrt{t}})}.$$

Accordingly,

$$\|u(t)\|_{L^2(B_{\lambda^k})} \le c \max\left\{ \lambda^{k/2} \|u\|_{L^\infty(0,1;L^2(B_1))}, \|u(t)\|_{L^2(B_{\sqrt{t}})} \right\}.$$

This yields (B.5). □

Lemma B.5. *Let $u \in \mathcal{V}_{\lambda-DSS}$. Furthermore, let $F_{ij}, g_i : Q \to \mathbb{R}$ such that $F_{ij}, g_i \in L^1(Q_R)$ and for all $0 < R < +\infty$, $i, j = 1, 2, 3$. We suppose for all $t \in [0, +\infty)$ the function $u(\cdot, t) \in L^2_{loc}(\mathbb{R}^3)$ with $\nabla \cdot u(\cdot, t) = 0$ in the sense of distributions, and that for all $\varphi \in C^\infty_c(Q)$ with $\nabla \cdot \varphi = 0$ the following identity holds true*

$$\int\limits_Q u \cdot \frac{\partial \varphi}{\partial t} dx dt = \int\limits_Q F : \nabla\varphi + g \cdot \varphi dx dt. \tag{B.6}$$

Then, eventually redefining $u(t)$ for t in a set of measure zero, we have

$$u \in C_w([0, +\infty); L^2(B_R)) \quad \forall 0 < R < +\infty, \tag{B.7}$$

$$M(u) = [0, +\infty). \tag{B.8}$$

Proof. By $L(u) \subset [0, +\infty)$ we denote the set of all Lebesgue points of u, more precisely, we say $t \in L(u)$, if for every $0 < R < +\infty$

$$\frac{1}{\varepsilon}\int_{t}^{t+\varepsilon} u(\cdot,\tau)d\tau \to u(\cdot,t) \quad in \quad L^2(B_R) \quad as \quad \varepsilon\to+\infty.$$

By Lebesgue's differentiation theorem we have meas$([0,+\infty)\setminus L(u))=0$. Let $t\in L(u)$. By a standard approximation argument we deduce from (B.6) that for every $\varphi\in C^\infty_{\rm c}(Q)$ with $\nabla\cdot u=0$

$$-\int_{\mathbb{R}^3} u(t)\cdot\varphi(t)dx+\int_0^t\int_{\mathbb{R}^3} u\cdot\frac{\partial\varphi}{\partial t}dxds=\int_0^t\int_{\mathbb{R}^3} F:\nabla\varphi+g\cdot\varphi dxds. \tag{B.9}$$

Next, let $\{t_j\}$ be a sequence in $M(u)\cap L(u)$ such that $t_j\to t\in L(u)$ as $j\in+\infty$. Thanks to Lemma B.4 we are in a position to apply Lemma B.3. Thus, there exists a subsequence $\{t_{j_m}\}$ and $v\in L^2_{loc}(\mathbb{R}^3)$ with $\nabla\cdot v$ in the sense of distributions such that

$$u(t_{j_m})\to v \quad weakly\ in \quad L^2_{loc}(\mathbb{R}^3) \quad as \quad m\to+\infty.$$

Then, in (B.9) with $t=t_{j_m}$ letting $m\to\infty$, we see that for all $\varphi\in C^\infty_{\rm c}(Q)$ with $\nabla\cdot\varphi=0$ it holds

$$-\int_{\mathbb{R}^3} v\cdot\varphi(t)dx+\int_0^t\int_{\mathbb{R}^3} u\cdot\frac{\partial\varphi}{\partial t}dxds=\int_0^t\int_{\mathbb{R}^3} F:\nabla\varphi+g\cdot\varphi dxds. \tag{B.10}$$

On the other hand, recalling that $t\in L(u)$, the identity (B.9) holds true. Combining both (B.9) and (B.10) we deduce that for all $\psi\in C^\infty_{{\rm c},\sigma}(\mathbb{R}^3)$

$$\int_{\mathbb{R}^3}(v-u(t))\cdot\psi dx=0.$$

Consequently, $v-u(t)$ is a harmonic function. On the other hand, by the lower semi continuity of the L^2 norm we obtain from (B.5) that

$$\|u(t)-v\|_{L^2(B_R)}\le C\max\left\{R^{1/2}\|u\|_{L^\infty(0,1;L^2(B_1))},\|u(t)\|_{L^2(B_{\sqrt{t}})}\right\}. \tag{B.11}$$

Whence, $v=u(t)$. In particular, $u(s)\to u(t)$ weakly in $L^2_{loc}(\mathbb{R}^3)$ as $s\in M(u)\cap L(u)\to t$.

Let $t\in[0,+\infty)$. There exists a sequence $\{t_j\}$ in $M(u)\cap L(u)$ such that $t_j\to t$ as $j\to+\infty$. Thanks to Lemma B.4 and Lemma B.3 there exists a subsequence $\{t_{j_m}\}$ and $v\in L^2_{loc}(\mathbb{R}^3)$ with $\nabla\cdot v=0$ in the sense of distributions such that

$$u(t_{j_m})\to v \quad weakly\ in \quad L^2_{loc}(\mathbb{R}^3) \quad as \quad m\to+\infty.$$

Observing (B.9) with t_{j_m} in place of t and letting $m\to+\infty$, we obtain for all $\varphi\in C^\infty_{\rm c}(Q)$ with $\nabla\cdot\varphi=0$

$$-\int_{\mathbb{R}^3} v\cdot\varphi(t)dx+\int_0^t\int_{\mathbb{R}^3} u\cdot\frac{\partial\varphi}{\partial t}dxds=\int_0^t\int_{\mathbb{R}^3} F:\nabla\varphi+g\cdot\varphi dxds. \tag{B.12}$$

On the other hand, by the lower semi continuity of the L^2 norm from (B.5) it follows that

$$\|v\|_{L^2(B_R)}\le C\max\left\{R^{1/2}\|u\|_{L^\infty(0,1;L^2(B_1))},\|u(t)\|_{L^2(B_{\sqrt{t}})}\right\}. \tag{B.13}$$

For a second subsequence $\{t'_{j_m}\}$ with limit $w\in L^2_{loc}(\mathbb{R}^3)$ we derive the same property as v which leads to the fact that for all $\psi\in C^\infty_{{\rm c},\sigma}(\mathbb{R}^3)$

$$\int_{\mathbb{R}^3}(v-w)\cdot\psi dx=0.$$

Consequently, $v - w$ is a harmonic function. Now taking into account the estimate (B.13), which is satisfied for w too, we infer $v = w$. Thus, the limit is uniquely determined. In case $t \notin M(u) \cap L(u)$ we set $u(t) = v$. In particular, (B.13) yields for all $t \in [0, +\infty)$ the estimate

$$\|u(t)\|_{L^2(B_R)} \le C \max\left\{R^{1/2}\|u\|_{L^\infty(0,1;L^2(B_1))}, \|u(t)\|_{L^2(B_{\sqrt{t}})}\right\}. \tag{B.14}$$

Furthermore, observing (B.10) for $t \in L(u)$) and (B.12) otherwise, it follows that for all $t \in [0, +\infty)$ and for all $\varphi \in C_c^\infty(Q)$ with $\nabla\varphi = 0$

$$-\int_{\mathbb{R}^3} u(t) \cdot \varphi(t) dx + \int_0^t \int_{\mathbb{R}^3} u \cdot \frac{\partial \varphi}{\partial t} dx ds = \int_0^t \int_{\mathbb{R}^3} F : \nabla\varphi + g \cdot \varphi dx ds. \tag{B.15}$$

Next, let $t \in [0, +\infty)$, and let $\{t_j\}$ be any sequence in $[0, +\infty)$ with $t_j \to t$ as $j \to +\infty$. In view of (B.14) once more we may apply Lemma B.3, which yields a subsequence $\{t_{j_m}\}$ and $w \in L^2_{loc}(\mathbb{R}^3)$ such that

$$u(t_{j_m}) \to w \quad \textit{weakly in} \quad L^2_{loc}(\mathbb{R}^3) \quad \textit{as} \quad m \to +\infty.$$

Observing (B.15) with t_{j_m} in place of t and letting $m \to +\infty$, it follows that

$$-\int_{\mathbb{R}^3} w \cdot \varphi(t) dx + \int_0^t \int_{\mathbb{R}^3} u \cdot \frac{\partial \varphi}{\partial t} dx ds = \int_0^t \int_{\mathbb{R}^3} F : \nabla\varphi + g \cdot \varphi dx ds. \tag{B.16}$$

Combining (B.16) and (B.15) and verifying (B.13) for w by a similar reasoning as above, we conclude $w = u(t)$. This shows that $u \in C_w([0, +\infty); L^2_{loc}(\mathbb{R}^3))$.

It only remains to prove that $M(u) = [0, +\infty)$. To see this let $\{t_j\}$ be a sequence in $M(u)$ such that $t_j \to t$. By using the transformation formula of the Lebesgue integral together with Lemma B.2 (cf. also (B.3)), we calculate for all $\psi \in C_c^\infty(\mathbb{R}^3)$

$$\begin{aligned}\int_{\mathbb{R}^3} u(x,t)\psi(x) dx &= \lim_{j\to\infty} \int_{\mathbb{R}^3} u(x, t_j)\psi(x) dx \\ &= \lambda^{-3k} \lim_{j\to\infty} \int_{\mathbb{R}^3} u(\lambda^{-k}x, t_j)\psi(\lambda x) dx \\ &= \lambda^{-2k} \lim_{j\to\infty} \int_{\mathbb{R}^3} u(x, \lambda^{2k} t_j)\psi(\lambda x) dx \\ &= \lambda^{-2k} \int_{\mathbb{R}^3} u(x, \lambda^{2k} t)\psi(\lambda x) dx = \int_{\mathbb{R}^3} \lambda^k u(\lambda^k x, \lambda^{2k} t)\psi(x) dx.\end{aligned}$$

This yields $u(x,t) = \lambda^k u(\lambda^k x, \lambda^{2k} t)$ for almost every $(x,t) \in Q$, and thus $t \in M(u)$. □

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