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Long time asymptotic behavior of the focusing nonlinear Schrödinger equation

Michael Borghese [a], Robert Jenkins [a,*], Kenneth D.T.-R. McLaughlin [b]

[a] *Department of Mathematics, University of Arizona, Tucson, AZ 85721, USA*
[b] *Department of Mathematics, Colorado State University, Fort Collins, CO 80523, USA*

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Abstract

We study the Cauchy problem for the focusing nonlinear Schrödinger (fNLS) equation. Using the $\overline{\partial}$ generalization of the nonlinear steepest descent method we compute the long-time asymptotic expansion of the solution $\psi(x,t)$ in any fixed space-time cone $C(x_1,x_2,v_1,v_2) = \left\{(x,t) \in \mathbb{R}^2 : x = x_0 + vt \text{ with } x_0 \in [x_1,x_2],\ v \in [v_1,v_2]\right\}$ up to an (optimal) residual error of order $\mathcal{O}\left(t^{-3/4}\right)$. In each cone C the leading order term in this expansion is a multi-soliton whose parameters are modulated by soliton–soliton and soliton–radiation interactions as one moves through the cone. Our results require that the initial data possess one $L^2(\mathbb{R})$ moment and (weak) derivative and that it not generate any spectral singularities.

Keywords: Focusing; Nonlinear Schrödinger; Long time asymptotics; Integrable systems; Riemann–Hilbert; Soliton resolution

1. Introduction

In this paper we study the long time asymptotic behavior of the focusing nonlinear Schrödinger (fNLS) equation on $\mathbb{R} \times \mathbb{R}_+$:

$$i\psi_t + \frac{1}{2}\psi_{xx} + |\psi|^2\psi = 0, \qquad \psi(x,0) = \psi_0(x). \tag{1.1}$$

The long time behavior of the *defocusing* NLS equation—equation (1.1) with the sign of cubic nonlinearity reversed—has been thoroughly studied [34,9,13,11,12,14]. In the defocusing case, one finds that as $t \to \infty$,

$$\psi(x,t) = t^{-1/2}\alpha(z_0)e^{ix^2/(2t) - i\nu(z_0)\log(4t)} + \mathcal{E}(x,t) \tag{1.2}$$

* Corresponding author.
E-mail addresses: michael.borghese@gmail.com (M. Borghese), rjenkins@math.arizona.edu (R. Jenkins), kenmcl@rams.colostate.edu (K.D.T.-R. McLaughlin).

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where

$$\nu(z) = -\frac{1}{2\pi}\log(1-|r(z)|^2), \quad |\alpha(z)|^2 = \nu(z)^2, \tag{1.3}$$

and

$$\arg\alpha(z) = \frac{1}{\pi}\int_{-\infty}^{z}\log(z-s)d(\log(1-|r(s)|^2)) + \frac{\pi}{4} + \arg\Gamma(i\nu(z)) - \arg r(z). \tag{1.4}$$

Here $z_0 = -x/(2t)$, Γ is the gamma function, and r is the so-called reflection coefficient for the potential $\psi_0(x)$ described below. Estimates for the size of the error term $\mathcal{E}(x,t)$ depend on smoothness and decay assumptions on ψ_0. The leading term without estimates was first obtained in [34]. Using the nonlinear steepest descent method [10], it was shown in [11,13] that if ψ_0 had a high degree of smoothness and decay then $\mathcal{E}(x,t) = \mathcal{O}\left(t^{-1}\log t\right)$. This was later improved [12] to $\mathcal{E}(x,t) = \mathcal{O}\left(t^{-(1/2+\kappa)}\right)$ for any $0 < \kappa < 1/4$ under the much weaker assumption that ψ_0 belonged to the weighted Sobolev space

$$H^{1,1} = \left\{ f \in L^2(\mathbb{R}) : xf,\ f' \in L^2(R) \right\}. \tag{1.5}$$

Recently, McLaughlin and Miller [29,30], developed a method of asymptotic analysis of Riemann–Hilbert problems (RHPs) based on $\overline{\partial}$ problems, rather than the asymptotic analysis of singular integrals on contours. This was successfully adapted to study the defocusing NLS equation both for finite mass initial data [14] and finite density initial data [8]; the latter of which supports soliton solutions. The advantages of this method are two fold: 1) it avoids delicate estimates involving L^p estimates of Cauchy projection operators (central to the work in [12]), and 2) it improves error estimates without additional restrictions on the initial data. The result in [14], which can be shown to be sharp, is that for $\psi_0 \in H^{1,1}$, the error $\mathcal{E}(x,t) = \mathcal{O}\left(t^{-3/4}\right)$.

In this work we apply these $\overline{\partial}$-techniques to the inverse scattering transform (IST) for fNLS to obtain the long-time asymptotic behavior of solutions to (1.1). The long-time behavior of solutions of fNLS are necessarily more detailed than in the defocusing case due to the presence of solitons which correspond to discrete spectrum of the non self-adjoint ZS-AKNS (Dirac) scattering operator associated with fNLS (cf. (2.1a) below). Given initial data $\psi_0 \in H^{1,1}(\mathbb{R})$ the ZS-AKNS operator for (1.1) allows for (complex conjugate pairs of) discrete spectrum anywhere in $\mathbb{C}\backslash\mathbb{R}$. In the defocusing case the ZS-AKNS operator is self-adjoint and the discrete spectrum is empty for finite mass initial data; a non-empty discrete spectrum is possible for the finite density type data studied in [8], but it is restricted to lie in a fixed interval of the real axis set by the boundary conditions. The description of the minimal scattering data for the forward/inverse scattering transform is necessarily more complicated in the focusing case.

Let us briefly consider the minimal scattering data for (1.1). More details are given in Section 2 and the references therein. Associated with any point in the simple discrete spectrum, $z_k \in \mathbb{C}^+$, is a nonzero complex number c_k called a norming constant. The real axis is the continuous spectrum of the ZS-AKNS operator along which we define a *reflection coefficient* $r : \mathbb{R} \to \mathbb{C}$. In the focusing case, the reflection coefficient r may take any value in $\mathbb{C}$; it is also possible that r may posses singularities along the real line—such points are called *spectral singularities*. When spectral singularities exist it is possible for there to be a (countably) infinite discrete spectrum which must accumulate at a spectral singularity; if no spectral singularities exist, the discrete spectrum is finite. For initial data ψ_0 which produces only simple discrete spectrum and has no spectral singularities, the minimal scattering data for fNLS is the collection $\mathcal{D} = \{r(z), \{(z_k, c_k)\}_{k=1}^N\}$. This is the classical scattering map $\mathcal{S} : \psi_0 \mapsto \mathcal{D}$ for fNLS. As described in [3,4] such initial data is generic. In the general, non-generic case, where spectral singularities or higher order discrete spectrum may exist, the classical scattering map is replaced by $\mathcal{S} : \psi_0 \mapsto v$ where v is a certain matrix defined along a contour Γ consisting of the real axis and a closed circle around infinity as described in [36].

Both the focusing and defocusing NLS equations are linearized by the scattering map $\mathcal{S}$; for a potential ψ_0 evolving according to (1.1) the scattering data evolution is trivial: $\mathcal{D}(t) = \{r(z)e^{2iz^2t}, \{(z_k, c_k e^{2iz_k^2 t})\}_{k=1}^N\}$ (or $v(t) = e^{-iz^2t\sigma_3} v e^{iz^2t\sigma_3}$ in the general case). It is often remarked in the literature that the scattering map $\mathcal{S}$ is a kind of nonlinear Fourier transform, and indeed it preserves regularity and smoothness in the same way; as shown in [37] the scattering map is a bijective (in fact bi-Lipschitz) map from $H^{j,k}(\mathbb{R})$ to $H^{k,j}(\Gamma)$ for any $j > 0$ and $k \geq 1$ (in the classical setting without spectral singularities this reduces to the reflection coefficient $r \in H^{k,j}(\mathbb{R})$). However, it is a trivial calculation that in order for the time evolving scattering data to persist in the weighted Sobolev space $H^{k,j}$ one must have $j \geq k$. It follows that the largest space $H^{j,k}$ from which the IST for (1.1) is well defined is $H^{1,1}$, and this is precisely the space in which we will work.

Scattering data $\{r \equiv 0, \{(z_k, c_k)\}_{k=1}^N\}$ for which the reflection coefficient vanishes identically correspond to N-soliton solutions of (1.1). If the spectrum consists of a single point, $\sigma_d = \{(\xi + i\eta, c)\}$ the corresponding solution of (1.1) is the one-soliton

$$\begin{gathered}\psi_{\text{sol}}(x,t) = \psi_{\text{sol}}(x,t;\{(\xi+i\eta,c)\}) = 2\eta \operatorname{sech}(2\eta(x+2\xi t - x_0))e^{-2i(\xi x + (\xi^2-\eta^2)t)}e^{-i\phi_0},\\ \|\psi_{\text{sol}}(\cdot,t)\|^2_{L^2(\mathbb{R})} = 4\eta\end{gathered} \tag{1.6}$$

where the phase shift x_0 and constant ϕ_0 are

$$x_0 = \frac{1}{2\eta}\log\left|\frac{c}{2\eta}\right|, \qquad \phi_0 = \frac{\pi}{2} + \arg(c). \tag{1.7}$$

This solution is a localized pulse with speed $v = -2\xi$ and maximum amplitude 2η. When $N > 1$ the solution of (1.1) with scattering data $\{r \equiv 0,\ \sigma_d = \{(z_k, c_k)\}_{k=1}^N\}$, which we label $\psi_{\text{sol}}(x,t;\sigma_d)$, is called an N-soliton solution (corresponding to the discrete scattering data σ_d). The long-time behavior of the N-soliton is a straightforward exercise in linear algebra and goes back to [35]. Generically, the solution breaks apart into N independent one-solitons; each traveling at distinct speed $v_k = -2\operatorname{Re} z_k$. When the points of discrete spectrum do not have distinct real parts the long-time behavior is more complicated; we give a streamlined review of this in Appendix B. Likewise, in the absence of solitons the steepest descent analysis is significantly simpler; the procedure in [14] goes through with only superficial changes of certain signs. One can use [14] as a primer for working through the more involved analysis needed here to deal with solitons.

1.1. Main results and remarks

Our main result describes the asymptotic behavior of the solution (1.1) as $t \to \infty$, for generic initial data $\psi_0 \in H^{1,1}(\mathbb{R})$. In order to state our results we define the following quantities derived from given scattering data $\{r, \{(z_k, c_k)\}_{k=1}^N\}$. Let $\mathcal{Z}$ denote the projection of the discrete scattering data $\sigma_d = \{(z_k, c_k)\}_{k=1}^N$ onto its first coordinate $\mathcal{Z} = \{z_k\}_{k=1}^N \subset \mathbb{C}^+$; define

$$\kappa(s) = -\frac{1}{2\pi}\log(1 + |r(s)|^2), \tag{1.8}$$

and for any real number ξ let

$$\begin{aligned}\Delta_\xi^- &= \{k \in \{0,1,\dots,N\} : \operatorname{Re} z_k < \xi\}\\ \Delta_\xi^+ &= \{k \in \{0,1,\dots,N\} : \operatorname{Re} z_k > \xi\}.\end{aligned} \tag{1.9}$$

Given any real interval $\mathcal{I} = [a,b]$ let

$$\begin{aligned}&\mathcal{Z}(\mathcal{I}) = \{z_k \in \mathcal{Z} : \operatorname{Re} z_k \in \mathcal{I}\} && Z^-(\mathcal{I}) = \{z_k \in \mathcal{Z} : \operatorname{Re} z_k < a\}\\ &N(\mathcal{I}) = |\mathcal{Z}(\mathcal{I})|, && Z^+(\mathcal{I}) = \{z_k \in \mathcal{Z} : \operatorname{Re} z_k > b\}\end{aligned} \tag{1.10}$$

For $\xi \in \mathcal{I}$ let

$$\begin{gathered}\Delta_\xi^-(\mathcal{I}) = \{k \in \{0,1,\dots,N\} : a \leq \operatorname{Re} z_k < \xi\},\\ \Delta_\xi^+(\mathcal{I}) = \{k \in \{0,1,\dots,N\} : \xi < \operatorname{Re} z_k \leq b\},\\ \sigma_d^\pm(\mathcal{I}) = \{(z_k, c_k^\pm(\mathcal{I})) : z_k \in \mathcal{Z}(\mathcal{I})\}\\ c_k^\pm(\mathcal{I}) = c_k \prod_{z_j \in \mathcal{Z}^\mp(\mathcal{I})}\left(\frac{z_k - z_j}{z_k - z_j^*}\right)^2 \exp\left(\pm 2i\int_\xi^{\mp\infty}\frac{\kappa(s)}{s - z_k}ds\right).\end{gathered} \tag{1.11}$$

Finally, given pairs of velocities $v_1 \leq v_2$ and points $x_1 \leq x_2$ define the cone

$$C(x_1,x_2,v_1,v_2) := \left\{(x,t) \in \mathbb{R}^2 : x = x_0 + vt \text{ with } x_0 \in [x_1,x_2],\ v \in [v_1,v_2]\right\}. \tag{1.12}$$

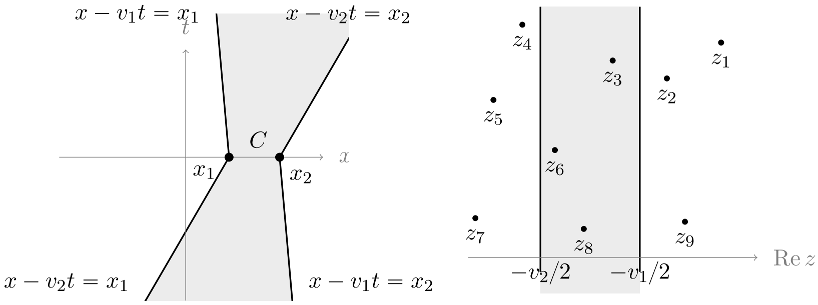

Fig. 1.1. Given initial data ψ_0 with scattering data $\{r, \{(z_k, c_k)\}_{k=1}^N\}$, the asymptotic behavior of $\psi(x,t)$, the solution of (1.1), as $|t| \to \infty$ with $(x,t) \in C(x_1, x_2, v_1, v_2)$ is described to leading order by the $N(\mathcal{I})$-soliton $\psi_{\mathrm{sol}}(x, t; \widehat{\sigma}_d^{\pm}(\mathcal{I}))$ corresponding to the reflectionless scattering data given by $\mathcal{Z}(\mathcal{I})$ and connection coefficients $\widehat{c}_k^{\pm}(\mathcal{I})$ modified by the self-interaction between solitons and with the reflection coefficient as described in Theorem 1.1. In the example here, the original data has nine points of discrete spectrum, but inside the cone C the solution is asymptotically described by a 3-soliton with discrete spectrum $\mathcal{Z}(\mathcal{I}) = \{z_3, z_6, z_8\}$.

Theorem 1.1. *Let $\psi(x,t)$ be the solution of (1.1) corresponding to initial data $\psi(x, t=0) = \psi_0(x) \in H^{1,1}(\mathbb{R})$ and suppose that ψ_0 is generic, i.e., it satisfies Assumption 2.1. Let $\{r, \{z_k, c_k\}_{k=1}^N\}$ denote the scattering data generated from ψ_0. Fix $x_1, x_2, v_1, v_2 \in \mathbb{R}$ with $x_1 \le x_2$ and $v_1 \le v_2$. Let $\mathcal{I} = [-v_2/2, -v_1/2]$, and let $\xi = -x/(2t)$. Then, as $t \to \pm\infty$ with $(x,t) \in C(x_1, x_2, v_1, v_2)$, with C as defined in (1.12), we have*

$$\psi(x,t) = \psi_{\mathrm{sol}}(x, t; \sigma_d^{\pm}(\mathcal{I})) + t^{-1/2} f^{\pm}(x,t) + \mathcal{O}\left(t^{-3/4}\right).$$

Here, $\psi_{\mathrm{sol}}(x, t; \sigma_d^{\pm}(\mathcal{I}))$ is the $N(\mathcal{I})$ soliton corresponding to the modified discrete scattering data (see Fig. 1.1) given by (1.11) and

$$f^{\pm}(x,t) = m_{11}(\xi; x, t)^2 \alpha(\xi, \pm) e^{ix^2/(2t) \mp i\kappa(\xi) \log|4t|} + m_{12}(\xi; x, t)^2 \alpha(\xi, \pm)^* e^{-ix^2/(2t) \pm i\kappa(\xi)\log|4t|}, \tag{1.13}$$

with

$$|\alpha(\xi, \pm)|^2 = |\kappa(\xi)|, \tag{1.14}$$

and

$$\arg\alpha(\xi, \pm) = \pm\frac{\pi}{4} \pm \arg\Gamma(i\kappa(\xi)) - \arg r(\xi) - 4\sum_{k \in \Delta_\xi^{\mp}} \arg(\xi - z_k) \mp 2\int_{\mp\infty}^{\xi} \log|\xi - s| \mathrm{d}_s\kappa(s). \tag{1.15}$$

The coefficients $m_{11}(\xi; x, t)$ and $m_{12}(\xi; x, t)$ are the entries in the first row of the solution of RHP B.2 with discrete scattering data $\sigma_d^{\pm}(\mathcal{I})$ and $\Delta = \Delta_\xi^{\mp}(\mathcal{I})$ evaluated at $z = \xi$.

Our result is essentially optimal. For initial data in the weakest possible weighted Sobolev space $H^{j,k}$ in which the IST can be formulated, we derive an asymptotic description up to a residual $\mathcal{O}\left(t^{-3/4}\right)$ error; this is the same order that arises in the long-time analysis of the free Schrödinger equation.[1] We avoid the consideration of spectral singularities only to limit the length of the paper. Even subject to spectral singularities, our results should still hold

[1] The solution of $i\phi_t + \frac{1}{2}\phi_{xx} = 0$ on the line is

$$\phi(x,t) = \frac{1}{\sqrt{2\pi i t}} \int_{\mathbb{R}} \phi_0(y) e^{i(x-y)^2/(2t)} dy = \frac{e^{ix^2/(2t)}}{\sqrt{2\pi i t}} \left[\widehat{\phi}_0\left(\frac{x}{t}\right) + \int_{\mathbb{R}} \phi_0(y) e^{-ixy/t} \left(e^{iy^2/(2t)} - 1\right) dy \right].$$

The Schwarz inequality bounds the final integral by a term proportional to $t^{-1/4} \| \langle \cdot \rangle \psi_0 \|_{L^2(\mathbb{R})}$.

for any $(x, t) \in C(x_1, x_2, v_1, v_2)$, such that the spectral interval $\mathcal{I}$ associated with the cone C does not contain any spectral singularities.

Remark 1.1. Theorem 1.1 requires that $\psi_0 \in H^{1,1}(\mathbb{R})$ so that the IST has nice mapping properties, but our asymptotic results depend only on the $H^1(\mathbb{R})$ norm of r. In particular, the long-time calculations presented below go through without change for any $\psi_0 \in L^2(\mathbb{R}, (1 + |x|)dx)$ satisfying Assumption 2.1.

Remark 1.2. Spectral singularities may exist for data in any weighted Sobolev space $H^{j,k}$; there are even examples [36, Example 3.3.16] of Schwartz class data for which spectral singularities occur. However, if the initial data decays exponentially, i.e., for some $c > 0$, $\int_{\mathbb{R}} e^{c|x|}|\psi_0(x)|dx < \infty$ then it is easily shown that the discrete spectrum cannot accumulate on the real axis. Isolated spectral singularities may still occur.

In Theorem 1.1 we give the asymptotic description in cones in order to accommodate many situations at once. In particular by considering small cones instead of fixed frames of reference we are able to account for uncertainties in the computation (or measurement) of the scattering data and thus speed of the resulting solitons. We believe that such a description should also be useful to study non-integrable perturbations of fNLS where the discrete spectrum would no longer be stationary.

The formulae above can be simplified greatly in special cones $C(x_1, x_2, v_1, v_2)$. If the reference cone C does not correspond to any of the soliton speeds, i.e., if we have $|\xi - \operatorname{Re} z_k| \geq c > 0$ for all $(x, t) \in C$ and $k = 1, \ldots, N$, then $\psi_{\text{sol}}(x, t)$, $m_{11}(\xi) - 1$, and $m_{12}(\xi)$ are each identically zero so the asymptotic description reduces to

$$\psi(x, t) = t^{-1/2}\alpha(\xi, \pm)e^{ix^2/(2t) \mp i\kappa(\xi)\log|4t|} + \mathcal{O}\left(t^{-3/4}\right), \qquad t \to \pm\infty. \tag{1.16}$$

This is the analog of the defocusing result (1.2).

Next, consider a cone $C = C(x_1, x_2, v_1, v_2)$ which contains a single soliton speed, that is, suppose that $z_k = \xi_k + i\eta_k \in \mathcal{Z}$ is the only point of the discrete spectrum whose real part lies in the interval $\mathcal{I} = [-v_2/2, -v_1/2]$ and let c_k be its associated norming constant. Then as $t \to \pm\infty$ with $(x, t) \in C_k$ the asymptotic solution reduces to

$$\begin{aligned} &\psi(x, t) = \psi_{\text{sol}}(x, t; (z_k, \widehat{c}_k^{\pm})) + \mathcal{O}\left(t^{-1/2}\right) \qquad t \to \pm\infty \\ &\psi_{\text{sol}}(x, t; (z_k, \widehat{c}_k^{\pm})) = 2\eta_k \operatorname{sech}\left(2\eta_k(x - x_k^{\pm} + 2\xi_k t)\right) e^{-2i(\xi_k x - (\eta_k^2 - \xi_k^2)t)} e^{-i\phi_k^{\pm}} \end{aligned} \tag{1.17a}$$

where the angle variables are given by

$$\begin{aligned} x_k^{\pm} &= \frac{1}{2\eta_k}\log\left|\frac{c_k}{2\eta_k}\right| + \frac{1}{\eta_k}\sum_{\substack{z_j \in \mathcal{Z} \\ \pm(\xi_k - \xi_j) > 0}} \log\left|\frac{z_k - z_j}{z_k - z_j^*}\right| \pm \int_{\mp\infty}^{-x/(2t)} \frac{\kappa(s)ds}{(s - \xi_k)^2 + \eta_k^2} \\ \phi_k^{\pm} &= \frac{\pi}{2} + \arg c_k + 2\sum_{\substack{z_j \in \mathcal{Z} \\ \pm(\xi_k - \xi_j) > 0}} \arg\left(\frac{z_k - z_j}{z_k - z_j^*}\right) \mp 2\int_{\mp\infty}^{-x/(2t)} \frac{(s - \xi_k)\kappa(s)}{(s - \xi_k)^2 + \eta_k^2}ds \end{aligned} \tag{1.17b}$$

The last two terms in each expression above describe the asymptotic effect of the soliton–soliton interaction and the interaction of the soliton with the radiative component of the solution respectively. The total shifts in position and phase angle acquired by any simple one-soliton as it interacts with the other solitons and radiation are given by

$$\begin{aligned} x_k^+ - x_k^- &= \frac{1}{\eta_k}\sum_{j \neq k} \operatorname{sgn}(\xi_k - \xi_j)\log\left|\frac{z_k - z_j}{z_k - z_j^*}\right| + \int_{-\infty}^{\infty} \frac{\operatorname{sgn}(\xi_k - s)\kappa(s)}{(s - \xi_k)^2 + \eta_k^2}ds \\ \phi_k^+ - \phi_k^- &= 2\sum_{j \neq k} \operatorname{sgn}(\xi_k - \xi_j)\arg\left(\frac{z_k - z_j}{z_k - z_j^*}\right) + 2\int_{-\infty}^{\infty} \frac{|s - \xi_k|\kappa(s)}{(s - \xi_k)^2 + \eta_k^2}ds. \end{aligned} \tag{1.18}$$

Formulas (1.17b)–(1.18) agree with the early work of Alonso [27,28], who formally calculated identical formulae for the asymptotic angle variables using the Gel'fand–Levitan–Marchenko method under stronger assumptions on the initial data.

The above formulae establish soliton resolution for data satisfying Assumption 2.1 in the following precise sense. Suppose that the discrete scattering data $\{z_k\}_{k=1}^N$ have $M \le N$ distinct real parts, and we reindex the discrete spectral data as $\{(z_{j,k}, c_{j,k})\}_{j,k=1}^{M,\mu_j}$ where

$$\sum_{j=1}^{M} \mu_j = N, \qquad \operatorname{Re} z_{j,k} = \xi_j,\ k = 1, \dots, \mu_j. \tag{1.19}$$

Then as $t \to \pm\infty$ the solution ψ separates at leading order into M spatially localized quasi-periodic waves—breathers—traveling at characteristic speeds $v_j = -2\xi_j$, $j = 1, \dots, M$, plus a radiating correction of order $t^{-1/2}$:

$$\psi(x,t) = \sum_{j=1}^{M} \psi_{\mathrm{sol}}\left(x, t; \{(z_{j,k}, \widehat{c}_{j,k}^{\pm})\}_{k=1}^{\mu_j}\right) + \mathcal{O}\left(t^{-1/2}\right), \quad t \to \pm\infty. \tag{1.20}$$

If all of the discrete spectral points have distinct real parts (so that $M = N$), then one recovers the typical resolution into a sum of N one-solitons, each given by (1.17). In the simplest non-typical situation, suppose that the scattering data $\{r, \{(z_k, c_k)\}_{k=1}^N\}$ has exactly one pair of spectral points having the same real part, which for simplicity we label $z_1 = \xi + i\eta_1$ and $z_2 = \xi + i\eta_2$. Then as $|t| \to \infty$ with $x + 2\xi t = \mathcal{O}(1)$ we have

$$\psi(x,t) = 4 \frac{\eta_1 \gamma_1^{\pm}\left(1 + \mu|\gamma_2^{\pm}|^2\right) + \eta_2 \gamma_2^{\pm}\left(1 + \mu|\gamma_1^{\pm}|^2\right)}{1 + |\gamma_1^{\pm}|^2 + |\gamma_2^{\pm}|^2 + 2(1-\mu)\operatorname{Re}\left(\gamma_1^{\pm}\gamma_2^{\pm *}\right) + \mu^2|\gamma_1^{\pm}\gamma_2^{\pm}|^2} + \mathcal{O}\left(|t|^{-1/2}\right), \tag{1.21}$$

as $t \to \pm\infty$ where

$$\gamma_k^{\pm} = e^{-2\eta_k(x - x_k^{\pm} + 2\xi t)} e^{-2i(\xi x - (\eta_k^2 - \xi^2)t)} e^{-i\phi_k^{\pm}}, \quad k = 1, 2, \tag{1.22}$$

and $\mu = \left|\frac{\eta_1 - \eta_2}{\eta_1 + \eta_2}\right|^2 \in (0,1)$; $x_k^{\pm}$ and $\phi_k^{\pm}$ are as given by (1.17b).

Remark 1.3. Though we say that initial data ψ_0 whose spectrum has distinct real parts are generic (in the sense that small perturbations of any non-generic initial datum will be generic) there are important classes of non-generic data. The so called Klaus–Shaw 'single lobe' potentials, $\psi_0(x) = A(x)e^{-2ikx + i\phi_0}$ with $k, \phi_0 \in \mathbb{R}$ and $A(x)$ a bounded piecewise smooth function which is nondecreasing to the left of some x_0 and nonincreasing to the right of x_0, are such that their discrete spectrum are confined to lie along the line $\operatorname{Re} z = k$, [24]. Such potentials have been extensively studied in the semi-classical limit [23,32,6] where the size of the discrete spectrum becomes asymptotically large.

1.2. Organization of the rest of the paper and notation

Throughout the paper we assume that the reader is familiar with the inverse scattering transform and Riemann–Hilbert problems. The reader who wishes to re-familiarize themselves with these topics is encouraged to see [3,5,36,4] as well as the more recent monograph [33]. In Section 2 we describe the forward scattering transform step of the IST in greater detail collecting the necessary results for our later work and provide references for their proofs. The section ends with the characterization of the inverse scattering transform in terms of a Riemann–Hilbert problem RHP 2.1. Sections 3–6 describe the steepest descent analysis of RHP 2.1 for $t \to \infty$. The analysis for $t \to -\infty$ is essentially the same and is summarized in Appendix C. Section 3 describes the initial conjugation of RHP 2.1 to better condition the problem for asymptotic analysis in a given frame of reference. Section 4 introduces the $\overline{\partial}$ analysis to define extensions of the jump matrix for the non-linear steepest descent method. In Section 5 we construct a global model solution which captures the leading order asymptotic behavior of the solution. Removing this component of the solution results in a small-norm $\overline{\partial}$ problem which is analyzed in Section 6. The proof of Theorem 1.1 is given in Section 7.

We close this introduction with some comments on our notational conventions. With regard to complex variables, given a variable z or a function (scalar, vector, or matrix-valued) $f(z)$, we denote by z^* and $f(z)^*$ their respective complex conjugates; for non-scalar functions f^* does not indicate the conjugate transpose. We use capital letters M, $M^{(p)}$, and E to denote the solutions of various RHPs. Here, the p in the superscript is used as a problem index, to avoid having an alphabet of solutions to various RHPs. The solutions of these problems are sectionally meromorphic away from given oriented contours $\Sigma^{(p)}$ (again p acts as a problem index) where they take continuous non-tangential boundary values. We use subscripts $\pm$ to refer to these boundary values: if M is the function of interest M_+ (resp. M_-) refers to the non-tangential boundary value along Σ from the left (resp. right) as one traverses the contour Σ respecting its orientation. Finally, the symbol $\overline{\partial}$ denotes the derivative with respect to z^*, if $z = x + iy$, then $\overline{\partial} f = \frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y})$.

2. Results of scattering theory for focusing NLS

The fNLS equation can be integrated [1,35] using the ZS-AKNS operator associated with Lax pair for fNLS:

$$(\partial_x - \mathcal{L})\Phi = 0, \qquad \mathcal{L} = -iz\sigma_3 + \Psi, \tag{2.1a}$$

$$(i\partial_t - \mathcal{B})\Phi = 0, \qquad \mathcal{B} = iz\mathcal{L} + \frac{1}{2}\sigma_3(\Psi^2 - \Psi_x), \tag{2.1b}$$

where

$$\Psi = \Psi(x,t) = \begin{pmatrix} 0 & \psi(x,t) \\ -\psi(x,t)^* & 0 \end{pmatrix},$$

and σ_3 is the third Pauli matrix $\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\sigma_2 = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$, $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. The existence of a simultaneous solution of this overdetermined system of equations requires that the potential $\Psi(x,t)$ satisfy the zero-curvature equation,

$$i\mathcal{L}_t - \mathcal{B}_x + [\mathcal{L}, \mathcal{B}] = i\Psi_t + \frac{1}{2}\sigma_3\Psi_{xx} - \sigma_3\Psi^2 = 0, \tag{2.2}$$

which is just a restatement of (1.1).

In the forward scattering step given initial data $\psi_0(x)$ one constructs solutions $\Phi(x,z)$ of (2.1a) with $z \in \mathbb{R}$; in particular one constructs the two *Jost solutions* $\Phi^{(\pm)}(x,z) = m^{(\pm)}(x,z)e^{-izx\sigma_3}$, which satisfy

$$\partial_x m = -iz[\sigma_3, m] + \Psi m, \tag{2.3}$$

and the normalization conditions

$$\lim_{x\to\pm\infty} m^{(\pm)}(x,z) = I, \quad z \in \mathbb{R}. \tag{2.4}$$

These solutions can be expressed as Volterra type integrals

$$m^{(\pm)}(x,z) = I + \int_{\pm\infty}^{x} e^{iz(x-y)\sigma_3}\Psi(y)m^{(\pm)}(y,z)e^{-iz(x-y)\sigma_3}dy \tag{2.5}$$

By iteration one shows that these equations have bounded continuous solutions in both x and z whenever $\psi_0 \in L^1(\mathbb{R})$.

As the matrix $\mathcal{L}$ in (2.1a) is traceless, the determinant of any solution Φ is independent of x and it follows that $\det\Phi^{(\pm)} = \det m^{(\pm)} \equiv 1$. Also, if $m(x,z)$ is any solution of (2.3) then $\widetilde{m}(x,z) = \sigma_2 m(x,z^*)^*\sigma_2$ (complex conjugate but no transpose) also solves (2.3). For $z \in \mathbb{R}$, $\sigma_2 m^{(\pm)}(x,z^*)^*\sigma_2$ also satisfies (2.4) and it follows that

$$m^{(\pm)}(x,z) = \sigma_2 m^{(\pm)}(x,z)^*\sigma_2, \quad z \in \mathbb{R}. \tag{2.6}$$

For $z \in \mathbb{R}$ both $m^{(+)}$ and $m^{(-)}$ define a fundamental solution matrix for (2.3) and so there exists a continuous matrix function $S(z)$, *the scattering matrix*, satisfying

$$\begin{gathered} \Phi^{(-)}(x;z) = \Phi^{(+)}(x;z)S(z), \qquad z \in \mathbb{R}, \\ S(z) = \begin{pmatrix} a(z) & -b(z)^* \\ b(z) & a(z)^* \end{pmatrix}, \qquad \det S(z) = |a(z)|^2 + |b(z)|^2 = 1 \end{gathered} \tag{2.7}$$

the coefficients $a(z)$ and $b(z)$ can be expressed as

$$
\begin{aligned}
a(z) &= \det\left[m_1^{(-)}, m_2^{(+)}\right] = 1 + \int_{\mathbb{R}} \psi(y)^* m_{12}^{(+)}(y)dy = 1 + \int_{\mathbb{R}} \psi(y) m_{21}^{(-)}(y)dy, \\
b(z) &= \det\left[m_1^{(+)}, m_1^{(-)}\right] = -\int_{\mathbb{R}} \psi(y)^* e^{-2izy} m_{11}^{(+)}(y)dy = -\int_{\mathbb{R}} \psi(y)^* e^{-2izy} m_{11}^{(-)}(y)dy
\end{aligned}
\tag{2.8}
$$

where

$$
m^{(\pm)} = \left(m_1^{(\pm)}, m_2^{(\pm)}\right) = \begin{pmatrix} m_{11}^{(\pm)} & m_{12}^{(\pm)} \\ m_{21}^{(\pm)} & m_{22}^{(\pm)} \end{pmatrix}. \tag{2.9}
$$

The following results are standard, proofs and details can be found in the literature, see for example [3,4,12]. Let $m_j^{(\pm)}$ denote the jth column of $m^{(\pm)}$ and e_j denote the jth column of the identity matrix:

- $m_1^{(-)}(x,z)$, $m_2^{(+)}(x,z)$ and $a(z)$ extend analytically to $z \in \mathbb{C}^+$ with continuous boundary values on $\mathbb{R}$. As $z \to \infty$ in $\mathbb{C}^+$, $m_1^{(-)}(x,z) \to e_1$, $m_2^{(+)}(x,z) \to e_2$ and $a(z) \to 1$. Analogous statements hold for the other pair of columns for $z \in \mathbb{C}^-$. Generally, $b(z)$ is defined only for $z \in \mathbb{R}$.
- At any $z_k \in \mathbb{C}^+$ for which $a(z_k) = 0$, the solutions $\Phi_1^{(-)}(x, z_k)$ and $\Phi_2^{(+)}(x, z_k)$ are linearly dependent. Specifically, a *norming constant* c_k exists such that:

  $$\Phi_1^{(-)}(x, z_k) = c_k \Phi_2^{(+)}(x, z_k). \tag{2.10}$$

  These solutions decay exponentially as $x \to \mp\infty$ respectively; this indicates that z_k is an L^2 eigenvalue of (2.1a) with eigenfunction $\Phi_1^{(-)}(x, z_k)$. The symmetry (2.6) implies that

  $$\Phi_2^{(-)}(x, z_k^*) = -c_k^* \Phi_1^{(+)}(x, z_k^*), \tag{2.11}$$

  which shows that eigenvalues come in complex conjugate pairs.
- The *reflection coefficient* r and *transmission coefficient* τ are defined by

  $$r(z) = \frac{b(z)}{a(z)} \qquad \tau(z) = \frac{1}{a(z)} \tag{2.12}$$

  and it follows from (2.7) that $1 + |r(z)|^2 = |\tau(z)|^2$ for each $z \in \mathbb{R}$.
- The properties of the scattering coefficients are similar to those of the Fourier transform. Given initial data Ψ_0 in the weighted Sobolev space

  $$H^{j,k}(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}) : \partial_x^j f,\ |x|^k f \in L^2(\mathbb{R}) \right\} \tag{2.13}$$

  the scattering coefficients $a(z) - 1 \in H^{k,1}$ and $b(z) \in H^{k,j}$. It follows that a,b, and their first $k-1$ derivatives are each bounded continuous functions. In the absence of spectral singularities (real zeros of $a(z)$), there also exist $c \in (0,1)$ such that $c < |a(z)| < 1/c$. From these facts, it follows that the map $\mathcal{R} : \psi_0 \mapsto r$ is a map from $H^{j,k}$ to $H^{k,j}$ (cf. [12]).

To avoid having to deal with the many pathologies possible when considering general initial data, we introduce the following working assumption which we will assume in all that follows.

Assumption 2.1. The initial data ψ_0 for the Cauchy problem (1.1) for fNLS generates generic scattering data in the sense that:

i) There are no spectral singularities, i.e., there exist a constant $c > 0$ such that $|a(z)| \geq c$ for any $z \in \mathbb{R}$.
ii) The discrete spectrum is simple, i.e., every zero of $a(z)$ in $\mathbb{C}^+$ is simple.

As discussed above, the absence of spectral singularities guarantees that the discrete spectrum is finite.

For initial data ψ_0 satisfying Assumption 2.1, the collection $\mathcal{D} = \{r(z), \{z_k, c_k\}_{k=1}^N\}$ is called the scattering data for $\psi_0(x)$ and the map $\mathcal{S} : \psi_0 \mapsto \mathcal{D}$ is called the (forward) scattering map. The essential fact of integrability is that if the potential $\psi_0(x)$ evolves according to (1.1) then the evolution of the scattering data $\mathcal{D}$ is trivial

$$\mathcal{D}(t) = \left\{r(z,t), \{z_k(t), c_k(t)\}_{k=1}^N\right\} = \left\{r(z)e^{2itz^2}, \{z_k, c_k e^{2itz_k^2}\}_{k=1}^N\right\}. \tag{2.14}$$

The inverse scattering map $\mathcal{S}^{-1} : \mathcal{D}(t) \mapsto \psi(x,t)$ seeks to recover the solution of (1.1) from its scattering data. This is done as follows: from the (now time evolving) Jost function $\Phi^{(\pm)}(x,t;z) = m^{(\pm)}(x,t;z)e^{-izx\sigma_3}$ one constructs the function

$$M(z) = M(z;x,t) := \begin{cases} \left[\dfrac{m_1^{(-)}(x,t;z)}{a(z)}, m_2^{(+)}(x,t;z)\right] & : \quad z \in \mathbb{C}^+ \\ \sigma_2 M(z^*;x,t)^*\sigma_2 & : \quad z \in \mathbb{C}^-. \end{cases} \tag{2.15}$$

For data ψ_0 satisfying Assumption 2.1, the matrix M defined above is the solution of the following Riemann–Hilbert problem.

Riemann–Hilbert Problem 2.1. *Find an analytic function $M : \mathbb{C}\backslash(\mathbb{R} \cup \mathcal{Z} \cup \mathcal{Z}^*) \to SL_2(\mathbb{C})$ with the following properties*

1. $M(z) = I + \mathcal{O}\left(z^{-1}\right)$ *as* $z \to \infty$.
2. *For each $z \in \mathbb{R}$ (with $\mathbb{R}$ oriented left-to-right), M takes continuous boundary values $M_\pm(z)$ which satisfy the jump relation $M_+(z) = M_-(z)V(z)$ where*

$$V(z) = \begin{pmatrix} 1 + |r(z)|^2 & r^*(z)e^{-2it\theta(z)} \\ r(z)e^{2it\theta(z)} & 1 \end{pmatrix}, \tag{2.16}$$

where

$$\theta = \theta(z;x,t) = z^2 - 2\xi z = (z-\xi)^2 - \xi^2, \quad \xi = -x/(2t). \tag{2.17}$$

3. *$M(z)$ has simple poles at each $z_k \in \mathcal{Z}$ and $z_k^* \in \mathcal{Z}^*$ at which*

$$\begin{aligned} \operatorname*{Res}_{z_k} M &= \lim_{z \to z_k} M \begin{pmatrix} 0 & 0 \\ c_k e^{2it\theta} & 0 \end{pmatrix}, \\ \operatorname*{Res}_{z_k^*} M &= \lim_{z \to z_k^*} M \begin{pmatrix} 0 & -c_k^* e^{-2it\theta} \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{2.18}$$

It's a simple consequence of Liouville's theorem that if a solution exists, it is unique. The existence of solutions of RHP 2.1 for all $(x,t) \in \mathbb{R}^2$ follows by means of Zhou's vanishing lemma argument [36] after replacing the poles by jumps along small circular contours in a standard way. Expanding this solution as $z \to \infty$, $M = I + z^{-1}M^{(1)}(x,t) + o\left(z^{-1}\right)$ and inserting this into (2.3) one finds that

$$M = I + \frac{1}{2iz}\begin{bmatrix} -\int_x^\infty |\psi(s,t)|^2 ds & \psi(x,t) \\ \psi(x,t)^* & \int_x^\infty |\psi(s,t)|^2 ds \end{bmatrix} + o\left(z^{-1}\right), \tag{2.19}$$

and it follows that the solution of (1.1) is given by

$$\psi(x,t) = \lim_{z \to \infty} 2izM_{12}(z;x,t). \tag{2.20}$$

For non-generic potentials various parts of the above characterization must be altered. There can exist points $z \in \mathbb{R}$ for which $a(z) = 0$, in which case $M_\pm(z)$ fail to exist; these are called *spectral singularities*. The number of points in the discrete spectrum of (2.1a) may be infinite, due to the first property of the solution m, the discrete spectrum must accumulate at a spectral singularity along the real axis. In the absence of spectral singularities the discrete spectrum is finite. Smoothness and decay of the initial data does not preclude the existence of spectral singularities; in [36] an explicit example is given of a Schwartz-class potential which generates an infinite discrete spectrum accumulating at $z = 0$. Finally, even in the case of a finite discrete spectrum, poles may coalesce resulting in higher order singularities at certain points of the discrete spectrum; in this case the pole conditions (2.18) must be altered. For simplicity we will consider here only the generic setting. Special cases of a single spectral singularity and of an infinite number of solitons have been partially described in [22,21].

3. Conjugation

The function $M(z; x, t)$ defined by (2.15) which solves RHP 2.1, is normalized such that it has identity asymptotics as $x \to +\infty$ with t fixed. It is not unreasonable to assume that the RHP should be well conditioned as $t \to \infty$ along a characteristic $x = vt$ where $v \gg 1$. However, along an arbitrary characteristic there is no reason to expect that M will remain near identity. In this section we describe a transformation $M \mapsto M^{(1)}$ which renormalizes the RHP such that it is well behaved as $t \to \infty$ along an arbitrary characteristic.

Let $\xi = -x/(2t)$. Recall the partition $\Delta_\xi^\pm$ of N defined by (1.9) This partition splits the residue coefficients c_k in (2.18) into two sets: As $t \to \infty$ with $x \geq -2\xi t$, it follows from (2.17) that $\operatorname{Im}(\theta(z_k)) < 0$ for each $k \in \Delta_\xi^-$, and thus the residue coefficient in (2.18) at each z_k for $k \in \Delta_\xi^-$ grows without bound as $t \to \infty$. Similarly, for z_k with $k \in \Delta_\xi^+$, the residue coefficients are bounded or near zero. See Fig. 3.1.

The first step in our analysis is to introduce a transformation which renormalizes the Riemann–Hilbert problem such that it is well conditioned for $t \to \infty$ with ξ fixed. In order to arrive at a problem which is well normalized, we introduce the function

$$
\begin{aligned}
T(z) = T(z,\xi) &= \prod_{k \in \Delta_\xi^-} \left(\frac{z - z_k^*}{z - z_k} \right) \exp\left(i \int_{-\infty}^{\xi} \frac{\kappa(s)}{s - z} ds \right), \\
\kappa(s) &= -\frac{1}{2\pi} \log(1 + |r(s)|^2).
\end{aligned}
\tag{3.1}
$$

A standard result of the forward scattering theory [15] is the following trace formula for the transmission coefficient

$$
\frac{1}{a(z)} = \prod_{k=1}^{N} \left(\frac{z - z_k^*}{z - z_k} \right) \exp\left(i \int_{-\infty}^{\infty} \frac{\kappa(s)}{s - z} ds \right) \tag{3.2}
$$

from which we see that our function $T(z, \xi)$ is a partial transmission coefficient which approaches the total transmission $1/a(z)$ as $\xi \to \infty$.

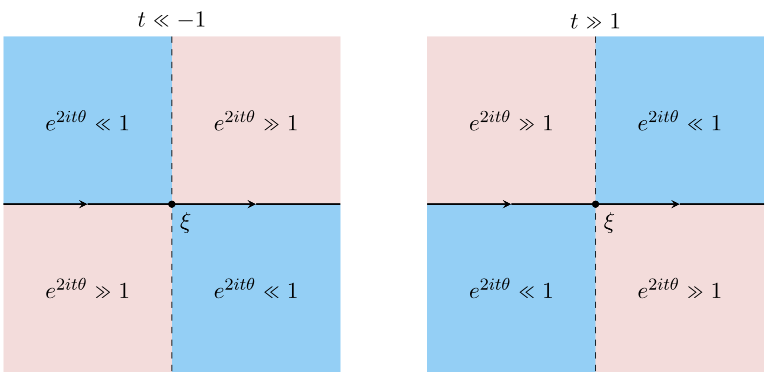

Fig. 3.1. The exponential factor $e^{2it\theta}$ governs the growth and decay of the jump matrix V and residue conditions (2.16)–(2.18). Depicted above are the regions of growth and decay of $e^{2it\theta}$ for large $|t|$. Notice that the regions are reversed when t changes sign.

Proposition 3.1. *The function $T(z)$ defined by (3.1) has the following properties:*

(a) *T is meromorphic in $\mathbb{C}\backslash(-\infty,\xi]$. For each $k\in\Delta_\xi^-$, $T(z)$ has a simple pole at z_k and a simple zero at z_k^*; elsewhere in $\mathbb{C}\backslash(-\infty,\xi]$, T is nonzero and analytic.*
(b) *For $z\in\mathbb{C}\backslash(-\infty,\xi]$, $T(z^*)^*=1/T(z)$.*
(c) *For $z\in(-\infty,\xi)$, the boundary values $T_\pm$ satisfy*

$$T_+(z)/T_-(z)=1+|r(z)|^2,\quad z\in(-\infty,\xi). \tag{3.3}$$

(d) *As $|z|\to\infty$ with $|\arg(z)|\le c<\pi$,*

$$T(z)=1+\frac{i}{z}\left[2\sum_{k\in\Delta_\xi^-}\operatorname{Im} z_k-\int_{-\infty}^{\xi}\kappa(s)ds\right]+\mathcal{O}\left(z^{-2}\right). \tag{3.4}$$

(e) *As $z\to\xi$ along any ray $\xi+e^{i\phi}\mathbb{R}_+$ with $|\phi|\le c<\pi$*

$$\left|T(z,\xi)-T_0(\xi)(z-\xi)^{i\kappa(\xi)}\right|\le C\|r\|_{H^1(\mathbb{R})}|z-\xi|^{1/2} \tag{3.5}$$

where $T_0(\xi)$ is the complex unit

$$T_0(\xi)=\prod_{k\in\Delta_\xi^-}\left(\frac{\xi-z_k^*}{\xi-z_k}\right)e^{i\beta(\xi,\xi)}=\exp\left[i\left(\beta(\xi,\xi)-2\sum_{k\in\Delta_\xi^-}\arg(\xi-z_k)\right)\right],$$

$$\beta(z,\xi)=-\kappa(\xi)\log(z-\xi+1)+\int_{-\infty}^{\xi}\frac{\kappa(s)-\chi(s)\kappa(\xi)}{s-z}ds,$$

and $\chi(s)$ is the characteristic function of the interval $(\xi-1,\xi)$ and the logarithm is principally branched along $(-\infty,\xi-1]$.

Proof. Parts (a)–(c) are elementary consequences of the definition (3.1) and the Sokhotski–Plemelj formula. For part (d) one geometrically expands the product term and the factor $(s-z)^{-1}$ for large z, and uses the fact that $\|\kappa\|_{L^1(\mathbb{R})}\le\frac{1}{2\pi}\|r\|_{L^2(\mathbb{R})}$ to bound the remainder in the integral term for z bounded away from the contour of integration. For part (e) we write

$$\begin{aligned}T(z,\xi)&=\prod_{k\in\Delta_\xi^-}\left(\frac{z-z_k^*}{z-z_k}\right)\exp\left(i\int_{\xi-1}^{\xi}\frac{\kappa(\xi)}{s-z}ds+i\int_{-\infty}^{\xi}\frac{\kappa(s)-\chi(s)\kappa(\xi)}{s-z}ds\right)\\&=\prod_{k\in\Delta_\xi^-}\left(\frac{z-z_k^*}{z-z_k}\right)(z-\xi)^{i\kappa(\xi)}\exp\left(i\beta(z,\xi)\right).\end{aligned} \tag{3.6}$$

The result then follows from the facts that

$$\left|(z-\xi)^{i\kappa(\xi)}\right|\le e^{-\pi\kappa(\xi)}=\sqrt{1+|r(\xi)|^2} \tag{3.7}$$

and using Lemma 23.3 of [4]:

$$|\beta(z,\xi)-\beta(\xi,\xi)|\le C\|r\|_{H^1(\mathbb{R})}|z-\xi|^{1/2}.\quad\square \tag{3.8}$$

We define a new unknown function $M^{(1)}$ using our partial transmission coefficient

$$M^{(1)}(z)=M(z)T(z)^{-\sigma_3} \tag{3.9}$$

Proposition 3.2. *The function $M^{(1)}$ defined by (3.9) satisfies the following Riemann–Hilbert problem.*

Riemann–Hilbert Problem 3.1. *Find an analytic function* $M^{(1)} : \mathbb{C}\backslash(\mathbb{R} \cup \mathcal{Z} \cup \mathcal{Z}^*) \to SL_2(\mathbb{C})$ *with the following properties:*

1. $M^{(1)}(z) = I + \mathcal{O}\left(z^{-1}\right)$ *as* $z \to \infty$.
2. *For each* $z \in \mathbb{R}$, *the boundary values* $M_\pm^{(1)}(z)$ *satisfy the jump relation* $M_+^{(1)}(z) = M_-^{(1)}(z)V^{(1)}(z)$ *where*

$$V^{(1)}(z) = \begin{cases} \begin{pmatrix} 1 & r^*(z)T(z)^2 e^{-2it\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r(z)T(z)^{-2}e^{2it\theta} & 1 \end{pmatrix} & z \in (\xi, \infty) \\ \begin{pmatrix} 1 & 0 \\ \frac{r(z)T_-(z)^{-2}}{1+|r(z)|^2} e^{2it\theta} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{r^*(z)T_+(z)^2}{1+|r(z)|^2} e^{-2it\theta} \\ 0 & 1 \end{pmatrix} & z \in (-\infty, \xi) \end{cases} \tag{3.10}$$

3. $M^{(1)}(z)$ *has simple poles at each* $z_k \in \mathcal{Z}$ *and* $z_k^* \in \mathcal{Z}^*$ *at which*

$$\begin{aligned} \operatorname*{Res}_{z_k} M^{(1)} &= \begin{cases} \lim\limits_{z\to z_k} M^{(1)} \begin{pmatrix} 0 & c_k^{-1}(1/T)'(z_k)^{-2}e^{-2it\theta} \\ 0 & 0 \end{pmatrix} & k \in \Delta_\xi^- \\ \lim\limits_{z\to z_k} M^{(1)} \begin{pmatrix} 0 & 0 \\ c_k T(z_k)^{-2}e^{2it\theta} & 0 \end{pmatrix} & k \in \Delta_\xi^+ \end{cases} \\ \operatorname*{Res}_{z_k^*} M^{(1)} &= \begin{cases} \lim_{z\to z_k^*} M^{(1)} \begin{pmatrix} 0 & 0 \\ -(c_k^*)^{-1}T'(z_k^*)^{-2}e^{2it\theta} & 0 \end{pmatrix} & k \in \Delta_\xi^- \\ \lim_{z\to z_k^*} M^{(1)} \begin{pmatrix} 0 & -c_k^* T(z_k^*)^2 e^{-2it\theta} \\ 0 & 0 \end{pmatrix} & k \in \Delta_\xi^+ \end{cases} \end{aligned} \tag{3.11}$$

Proof. That $M^{(1)}$ is unimodular, analytic in $\mathbb{C}\backslash(\mathbb{R}\cup\mathcal{Z}\cup\mathcal{Z}^*)$, and approaches identity as $z \to \infty$ follows directly from its definition, Proposition 3.1 and the properties of M. The jump (3.10) follows directly from using the factorizations of V, (2.16), given by

$$V^{(1)}(z) = \begin{cases} T(z)^{\sigma_3} \begin{pmatrix} 1 & r(z)^* e^{-2it\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r(z)e^{2it\theta} & 1 \end{pmatrix} T(z)^{-\sigma_3} & z > \xi \\ T_-(z)^{\sigma_3} \begin{pmatrix} 1 & 0 \\ \frac{r(z)e^{2it\theta}}{1+|r(z)|^2} & 1 \end{pmatrix} \left(\frac{T_+(z)}{T_-(z)}\right)^{\sigma_3} \begin{pmatrix} 1 & \frac{r(z)^* e^{-2it\theta}}{1+|r(z)|^2} \\ 0 & 1 \end{pmatrix} T_+(z)^{-\sigma_3} & z < \xi \end{cases} \tag{3.12}$$

to the right and left of $z = \xi$ on the real line respectively and making use of the jump relation (3.3) satisfied by $T(z)$ on $(-\infty, \xi)$. Concerning the residues, since $T(z)$ is analytic at each z_k, z_k^* with $k \in \Delta_\xi^+$, the residue conditions at these poles are an immediate consequence of (3.1). For $k \in \Delta_\xi^-$, $T(z)$ has a zero at z_k^* and a pole at z_k, so that $M_1^{(1)} = M_1(z)T(z)^{-1}$ has a removable singularity at z_k, but acquires a pole at z_k^*. For $M_2^{(1)} = M_2(z)T(z)$ the situation is reversed; it has a pole at z_k and a removable singularity at z_k^*. At z_k we have

$$\begin{aligned} M_1^{(1)}(z_k) &= \lim_{z\to z_k} M_1(z)T(z)^{-1} = \operatorname*{Res}_{z_k} M_1(z) \cdot (1/T)'(z_k) \\ &= c_k e^{2it\theta_k} M_2(z_k)(1/T)'(z_k), \\ \operatorname*{Res}_{z_k} M_2^{(1)}(z) &= \operatorname*{Res}_{z=z_k} M_2(z)T(z) = M_2(z_k)\left[(1/T)'(z_k)\right]^{-1} \\ &= c_k^{-1}\left[(1/T)'(z_k)\right]^{-2} e^{-2it\theta} M_1^{(1)}(z_k), \end{aligned} \tag{3.13}$$

from which the first formula in (3.11) clearly follows. The computation of the residue at z_k^* for $k \in \Delta_\xi^-$ is similar. $\square$

4. Introducing $\overline{\partial}$ extensions of jump factorization

The next step in our analysis is to introduce factorizations of the jump matrix whose factors admit continuous–but not necessarily analytic–extensions off the real axis following the ideas in [29,30,14,8]. In particular, the construction in [14] is essentially the reduction of what is presented here if there were no solitons present in the initial data. Using these extensions we define a new unknown that deforms the oscillatory jump along the real axis onto new contours along which the jumps are decaying. The price we pay for this non-analytic transformation is that the new unknown has nonzero $\overline{\partial}$ derivatives inside the regions in which the extensions are introduced and satisfies a hybrid $\overline{\partial}$/Riemann–Hilbert problem.

Define the contours

$$\Sigma_k = \xi + e^{i(2k-1)\pi/4}\mathbb{R}_+, \quad k = 1, 2, 3, 4, \tag{4.1}$$

oriented with increasing real part and denote the six open sectors in $\mathbb{C}$—separated by $\mathbb{R}$ and the collection of Σ_k, $k = 1, \dots, 4$—by Ω_k, $k = 1, \dots, 6$ starting with the sector Ω_1 between $[\xi, \infty)$ and Σ_1 and numbered consecutively continuing counterclockwise, see Fig. 4.1. Define

$$\Sigma_R = \mathbb{R} \cup \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Sigma_4. \tag{4.2}$$

Additionally, let

$$\rho = \frac{1}{2} \min_{\substack{\lambda,\mu \in \mathcal{Z} \cup \mathcal{Z}^* \\ \lambda \neq \mu}} |\lambda - \mu|. \tag{4.3}$$

Note that, as poles come in conjugate pairs and (by assumption) no pole lies on the real axis, we have $\rho \leq \operatorname{dist}(\mathcal{Z}, \mathbb{R})$. Let $\chi_{\mathcal{Z}} \in C_0^\infty(\mathbb{C}, [0, 1])$ be supported near the discrete spectrum such that

$$\chi_{\mathcal{Z}}(z) = \begin{cases} 1 & \operatorname{dist}(z, \mathcal{Z} \cup \mathcal{Z}^*) < \rho/3 \\ 0 & \operatorname{dist}(z, \mathcal{Z} \cup \mathcal{Z}^*) > 2\rho/3 \end{cases} \tag{4.4}$$

The steepest descent method of Deift and Zhou uses factorizations like those in (3.10) to deform the jump matrix onto new contours in the plane on which they decay, see (4.14)–(4.15) below. The deformation requires extensions of the off-diagonal entries on the right hand side of (3.10), which are a priori only defined on the jump contour, into the appropriate regions of the complex plane. We define extensions of the off-diagonal entries of (3.10) in the following lemma.

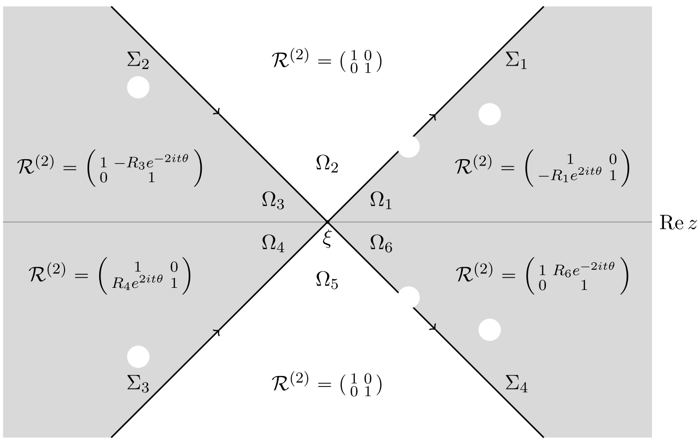

Fig. 4.1. The contours Σ_k and regions Ω_k $k = 1, \dots, 6$ defining the non-analytic change of variables $M^{(2)} = M^{(1)}\mathcal{R}^{(2)}$. The support of the $\bar{\partial}$-derivatives, $\overline{\partial} M^{(2)} = M^{(2)}\overline{\partial}\mathcal{R}^{(2)}$, is shaded in gray.

Lemma 4.1. *It is possible to define functions $R_j : \overline{\Omega}_j \to \mathbb{C}$, $j = 1, 3, 4, 6$, with boundary values satisfying*

$$R_1(z) = \begin{cases} r(z)T(z)^{-2} & z \in (\xi, \infty) \\ r(\xi)T_0(\xi)^{-2}(z-\xi)^{-2i\kappa(\xi)}(1-\chi_{\mathcal{Z}}(z)) & z \in \Sigma_1 \end{cases} \tag{4.5a}$$

$$R_3(z) = \begin{cases} \dfrac{r(z)^*}{1+|r(z)|^2} T_+(z)^2 & z \in (-\infty, \xi) \\ \dfrac{r(\xi)^*}{1+|r(\xi)|^2}(\xi)^2(z-\xi)^{2i\kappa(\xi)}(\chi_{\mathcal{Z}}(z)) & z \in \Sigma_2 \end{cases} \tag{4.5b}$$

$$R_4(z) = \begin{cases} \dfrac{r(z)}{1+|r(z)|^2} T_-(z)^{-2} & z \in (-\infty, \xi) \\ \dfrac{r(\xi)}{1+|r(\xi)|^2} T_0(\xi)^{-2}(z-\xi)^{-2i\kappa(\xi)}(1-\chi_{\mathcal{Z}}(z)) & z \in \Sigma_3 \end{cases} \tag{4.5c}$$

$$R_6(z) = \begin{cases} r(z)^* T(z)^2 & z \in (\xi, \infty) \\ r(\xi)^* T_0(\xi)^2 (z-\xi)^{2i\kappa(\xi)}(1-\chi_{\mathcal{Z}}(z)) & z \in \Sigma_4 \end{cases} \tag{4.5d}$$

such that for a fixed constant $c_1 = c_1(\psi_0)$, and a fixed cutoff function $\chi_{\mathcal{Z}} \in C_0^\infty(\mathbb{C}, [0,1])$ satisfying (4.4) we have[2]

$$\begin{aligned} \left|R_j(z)\right| &\le c_1 \sin^2(\arg(z-\xi)) + c_1 \langle \operatorname{Re} z\rangle^{-1/2}, \\ \left|\overline{\partial} R_j(z)\right| &\le c_1 \overline{\partial}\chi_{\mathcal{Z}}(z) + c_1 \left|r'(\operatorname{Re} z)\right| + c_1 |z-\xi|^{-1/2}, \\ \overline{\partial} R_j(z) &= 0 \quad \textit{if } \operatorname{dist}(z, \mathcal{Z} \cup \mathcal{Z}^*) \le \rho/3. \end{aligned} \tag{4.6}$$

Moreover, if we set $R : (\mathbb{C} \setminus \Sigma_R) \to \mathbb{C}$ by $R(z)\big|_{z \in \Omega_j} = R_j(z)$, (with $R_2(z) = R_5(z) = 0$), the extension can be made such that $R(z^*)^* = R(z)$.

Proof. Using the constant $T_0(\xi)$ defined in Proposition 3.1, define the functions

$$\begin{aligned} f_1(z) &= r(\xi)T(z)^2 T_0(\xi)^{-2}(z-\xi)^{-2i\kappa(\xi)} \qquad z \in \overline{\Omega}_1 \\ f_3(z) &= \frac{r(\xi)^*}{1+|r(\xi)|^2} T(z)^2 T_0(\xi)^{-2}(z-\xi)^{-2i\kappa(\xi)} z \in \overline{\Omega}_3. \end{aligned} \tag{4.7}$$

Let $z - \xi = se^{i\phi}$. Define, for $z \in \overline{\Omega}_j$, $j = 1, 3$, the extensions

$$\begin{aligned} R_1(z) &= \left[f_1(z) + (r(\operatorname{Re} z) - f_1(z))\cos(2\phi)\right] T(z)^{-2}(1-\chi_{\mathcal{Z}}(z)), \\ R_3(z) &= \left[f_3(z) + \left(\frac{r(\operatorname{Re} z)^*}{1+|r(\operatorname{Re} z)|^2} - f_3(z)\right)\cos(2\phi)\right] T(z)^2(1-\chi_{\mathcal{Z}}(z)). \end{aligned} \tag{4.8}$$

The extensions R_4 and R_6 are defined using part (b) of Proposition 3.1 and choosing $\chi_{\mathcal{Z}}(z)$ to respect Schwarz symmetry; we define $R_4 = R_3(z^*)^*$ and $R_6(z) = R_1(z^*)^*$ which preserves the Schwarz reflection symmetry.

We give the rest of the details for R_1 only. The other cases are easily inferred. Clearly, $R_1(z)$ satisfies the boundary conditions of the lemma as $\cos(2\phi)$ vanishes on Σ_1 and $\chi_{\mathcal{Z}}(z)$ is zero on the real axis. First consider $|R_1(z)|$. We have

$$\begin{aligned} |R_1(z)| &\le \left|2 f_1(z) T(z)^{-2}(1-\chi_{\mathcal{Z}}(z))\right| \sin^2(\phi) + \left|T(z)^{-2}(1-\chi_{\mathcal{Z}}(z))\cos(2\phi)\right| |r(\operatorname{Re} z)| \\ &\lesssim \sin^2(\phi) + \langle \operatorname{Re} z\rangle^{-1/2}. \end{aligned} \tag{4.9}$$

Where in the first line, we have bounded the first factor in each term of the sum on the left hand side using Proposition 3.1 and equations (3.7), (4.4), and (4.7) noting that the poles of $T(z)^{-2}$ are outside the support of $(1-\chi_{\mathcal{Z}}(z))$. In the second line we've used the fact that $r \in H^1(\mathbb{R})$ implies that r is Hölder-1/2 continuous and $|r(u)| \lesssim \langle u\rangle^{-1/2}$.

[2] Here and elsewhere $\langle \cdot \rangle := \sqrt{1+(\cdot)^2}$.

Since $\overline{\partial} = (\partial_x + i\partial_y)/2 = e^{i\phi}(\partial_s + i\rho^{-1}\partial_\phi)/2$, we have

$$\begin{aligned}\overline{\partial} R_1(z) &= -\left[f_1(z) + (r(\operatorname{Re} z) - f_1(z))\cos 2\phi\right] T(z)^{-2}\overline{\partial}\chi_{\mathcal{Z}}(z) \\ &\quad + \left[\frac{1}{2} r'(\operatorname{Re} z)\cos(2\phi) - i e^{i\phi}\frac{r(\operatorname{Re}(z)) - f_1(z)}{|z-\xi|}\sin(2\phi)\right] T(z)^{-2}(1 - \chi_{\mathcal{Z}}(z)).\end{aligned} \tag{4.10}$$

We arrive at (4.6) by using the continuity and decay of $r(\operatorname{Re} z)$ described above and the fact that as both $1 - \chi_{\mathcal{Z}}(z)$ and $\overline{\partial}\chi_{\mathcal{Z}}(z)$ are supported away from the discrete spectrum, the poles and zeros of $T(z)$ do not affect the bound. This gives the first two terms in the bound. For the last term we write

$$|r(\operatorname{Re} z) - f_1(z)| \le |r(\operatorname{Re} z) - r(\xi)| + |r(\xi) - f_1(z)| \tag{4.11}$$

and use Cauchy–Schwarz to bound each term as follows:

$$|r(\operatorname{Re} z) - r(\xi)| \le \left| \int_{\xi}^{\operatorname{Re} z} r'(s)ds \right| \le \|r\|_{H^1(\mathbb{R})}|z-\xi|^{1/2} \tag{4.12}$$

and

$$|r(\xi) - f_1(z)| \le |r(\xi)|(1+|r(\xi)|^2)\left|T(z,\xi)^2 - T_0(\xi)^2(z-\xi)^{2i\kappa(\xi)}\right| \le C_\xi \|r\|_{H^1(\mathbb{R})}|z-\xi|^{1/2}. \tag{4.13}$$

The last estimate uses (3.5) and (3.7) to bound $T(z,\xi)$ and $(z-\xi)^{i\kappa(\xi)}$ in a neighborhood of $z = \xi$. The bound (4.6) for $z \in \Omega_1$ follows immediately. $\square$

We use the extension in Lemma 4.1 and the factorized jump matrices in (3.10) to define a new unknown function

$$M^{(2)}(z) = M^{(1)}(z)\mathcal{R}^{(2)}(z) \tag{4.14}$$

$$\mathcal{R}^{(2)} = \begin{cases} \begin{pmatrix} 1 & 0 \\ -R_1(z)e^{2it\theta} & 1 \end{pmatrix} & z \in \Omega_1 \\ \begin{pmatrix} 1 & -R_3(z)e^{-2it\theta} \\ 0 & 1 \end{pmatrix} & z \in \Omega_3 \\ \begin{pmatrix} 1 & 0 \\ R_4(z)e^{2it\theta} & 1 \end{pmatrix} & z \in \Omega_4 \\ \begin{pmatrix} 1 & R_6(z)e^{-2it\theta} \\ 0 & 1 \end{pmatrix} & z \in \Omega_6 \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & z \in \Omega_2 \cup \Omega_5 \end{cases} \tag{4.15}$$

Let $\Sigma^{(2)} = \bigcup_{j=1}^{4} \Sigma_k$. It is an immediate consequence of Lemma 4.1 and RHP 3.1 that $M^{(2)}$ satisfies the following $\overline{\partial}$-Riemann–Hilbert problem.

$\overline{\partial}$-Riemann–Hilbert Problem 4.1. *Find a function $M^{(2)} : \mathbb{C}\backslash(\Sigma^{(2)} \cup \mathcal{Z} \cup \mathcal{Z}^*) \to SL_2(\mathbb{C})$ with the following properties.*

1. *$M^{(2)}$ is continuous with sectionally continuous first partial derivatives in $\mathbb{C} \setminus (\Sigma^{(2)} \cup \mathcal{Z} \cup \mathcal{Z}^*)$.*
2. *$M^{(2)}(z) = I + \mathcal{O}\left(z^{-1}\right)$ as $z \to \infty$.*
3. *For $z \in \Sigma^{(2)}$, the boundary values satisfy the jump relation $M_+^{(2)}(z) = M_-^{(2)}(z)V^{(2)}(z)$, where*

$$V^{(2)}(z) = I + (1 - \chi_{\mathcal{Z}}(z))\delta V^{(2)},$$

$$\delta V^{(2)}(z) = \begin{cases} \begin{pmatrix} 0 & 0 \\ r(\xi)T_0(\xi)^{-2}(z-\xi)^{-2i\kappa(\xi)}e^{2it\theta} & 0 \end{pmatrix} & z \in \Sigma_1 \\ \begin{pmatrix} 0 & \frac{r(\xi)^* T_0(\xi)^2}{1+|r(\xi)|^2}(z-\xi)^{2i\kappa(\xi)}e^{-2it\theta} \\ 0 & 0 \end{pmatrix} & z \in \Sigma_2 \\ \begin{pmatrix} 0 & 0 \\ \frac{r(\xi)T_0^{-2}(\xi)}{1+|r(\xi)|^2}(z-\xi)^{-2i\kappa(\xi)}e^{2it\theta} & 0 \end{pmatrix} & z \in \Sigma_3 \\ \begin{pmatrix} 0 & r(\xi)^* T_0(\xi)^2(z-\xi)^{2i\kappa(\xi)}e^{-2it\theta} \\ 0 & 0 \end{pmatrix} & z \in \Sigma_4 \end{cases} \tag{4.16}$$

4. *For* $\mathbb{C} \setminus (\Sigma^{(2)} \cup \mathcal{Z} \cup \mathcal{Z}^*)$ *we have* $\overline{\partial} M^{(2)}(z) = M^{(2)}(z)\overline{\partial}\mathcal{R}^{(2)}(z)$ *where*

$$\overline{\partial}\mathcal{R}^{(2)}(z) = \begin{cases} \begin{pmatrix} 0 & 0 \\ -\overline{\partial}R_1(z)e^{2it\theta} & 0 \end{pmatrix} & z \in \Omega_1 \\ \begin{pmatrix} 0 & -\overline{\partial}R_3(z)e^{-2it\theta} \\ 0 & 0 \end{pmatrix} & z \in \Omega_3 \\ \begin{pmatrix} 0 & 0 \\ \overline{\partial}R_4(z)e^{2it\theta} & 0 \end{pmatrix} & z \in \Omega_4 \\ \begin{pmatrix} 0 & \overline{\partial}R_6(z)e^{-2it\theta} \\ 0 & 0 \end{pmatrix} & z \in \Omega_6 \\ \mathbf{0} & \textit{elsewhere} \end{cases} \tag{4.17}$$

5. $M^{(2)}(z)$ *has simple poles at each* $z_k \in \mathcal{Z}$ *and* $z_k^* \in \mathcal{Z}^*$ *at which*

$$\begin{aligned} \operatorname*{Res}_{z_k} M^{(2)} &= \begin{cases} \lim_{z\to z_k} M^{(2)} \begin{pmatrix} 0 & c_k^{-1}(1/T)'(z_k)^{-2}e^{-2it\theta} \\ 0 & 0 \end{pmatrix} & k \in \Delta_\xi^- \\ \lim_{z\to z_k} M^{(2)} \begin{pmatrix} 0 & 0 \\ c_k T(z_k)^{-2}e^{2it\theta} & 0 \end{pmatrix} & k \in \Delta_\xi^+ \end{cases} \\ \operatorname*{Res}_{z_k^*} M^{(2)} &= \begin{cases} \lim_{z\to z_k^*} M^{(2)} \begin{pmatrix} 0 & 0 \\ -(c_k^*)^{-1}T'(z_k^*)^{-2}e^{2it\theta} & 0 \end{pmatrix} & k \in \Delta_\xi^- \\ \lim_{z\to z_k^*} M^{(2)} \begin{pmatrix} 0 & -c_k^* T(z_k^*)^2 e^{-2it\theta} \\ 0 & 0 \end{pmatrix} & k \in \Delta_\xi^+ \end{cases} \end{aligned} \tag{4.18}$$

Remark 4.1. In the $\overline{\partial}$-RHP for $M^{(2)}$ above, it is useful to recall how the extensions $R_j(z)$ are defined in Lemma 4.1, particularly the second condition in (4.6). Though (4.17) may seem to suggest that $M^{(2)}$ is non-analytic near points of the discrete spectrum, the $\overline{\partial}$-derivative vanishes in small neighborhoods of each point of the discrete spectrum so that $M^{(2)}$ is analytic in each neighborhood.

The $\overline{\partial}$-Riemann–Hilbert Problem 4.1 is now ideally conditioned for large t asymptotic analysis. It has jump matrices which approach identity point-wise, each residue coefficient corresponding to a soliton whose speed differs from ξ is exponentially small, and Lemma 4.1 controls the $\overline{\partial}$ derivatives in a manageable way. The final two sections construct the solution $M^{(2)}$ as follows:

1. The $\overline{\partial}$ component of $\overline{\partial}$-RHP 4.1 is ignored, and we prove the existence of a solution of the resulting pure Riemann–Hilbert problem and compute its asymptotic expansion.
2. Conjugating off the solution of the first step, we arrive at a pure $\overline{\partial}$ problem which we show has a solution and bound its size.

Unwinding the series of transformations that led from the M to $M^{(2)}$ we recover the solution RHP 2.1 and then from (2.20) we recover a long time asymptotic expansion of the solution $q(x,t)$ of fNLS for our class of initial data.

5. Removing the Riemann–Hilbert component of the solution

In this section we build a solution $M^{(2)}_{\mathrm{RHP}}$ to the Riemann–Hilbert problem that results from the $\overline{\partial}$-RHP for $M^{(2)}$ by dropping the $\overline{\partial}$ component. Specifically:

$$\text{Let } M^{(2)}_{\mathrm{RHP}} \text{ be the solution of the Riemann–Hilbert problem resulting from setting } \overline{\partial}\mathcal{R}^{(2)} \equiv 0 \text{ in } \overline{\partial}\text{-RHP 4.1.} \tag{5.1}$$

In this section we will prove that the solution $M^{(2)}_{\mathrm{RHP}}$ exists and construct its asymptotic expansion for large t. Before we embark upon this adventure, we first show that if $M^{(2)}_{\mathrm{RHP}}$ exists, it reduces $\overline{\partial}$-RHP 4.1 to a pure $\overline{\partial}$ problem.

Proposition 5.1. *Suppose that $M^{(2)}_{\mathrm{RHP}}$ is a solution of the Riemann–Hilbert problem described in (5.1), then the ratio*

$$M^{(3)}(z) := M^{(2)}(z) M^{(2)}_{\mathrm{RHP}}(z)^{-1} \tag{5.2}$$

is a continuously differentiable function satisfying the following $\overline{\partial}$-problem.

$\overline{\partial}$ Problem 5.1. *Find a function $M^{(3)} : \mathbb{C} \to SL_2(\mathbb{C})$ with the following properties.*

1. *$M^{(3)}$ is continuous with sectionally continuous first partial derivatives in $\mathbb{C} \setminus (\mathbb{R} \cup \Sigma^{(2)})$.*
2. *$M^{(3)}(z) = I + \mathcal{O}\left(z^{-1}\right)$ as $z \to \infty$.*
3. *For $z \in \mathbb{C}$, we have*

$$\overline{\partial} M^{(3)}(z) = M^{(3)}(z) W^{(3)} \tag{5.3}$$

where $W^{(3)} := M^{(2)}_{\mathrm{RHP}}(z) \overline{\partial}\mathcal{R}^{(2)}(z) M^{(2)}_{\mathrm{RHP}}(z)^{-1}$ and $\overline{\partial}\mathcal{R}^{(2)}$ is defined by (4.17).

Proof. Both $M^{(2)}$ and $M^{(2)}_{\mathrm{RHP}}$ are unimodular and approach identity as z tends to infinity. It follows from (5.2) that $M^{(3)}$ inherits these properties as well as continuous differentiability in $\mathbb{C}\setminus\Sigma^{(2)}$. Since both $M^{(2)}$ and $M^{(2)}_{\mathrm{RHP}}$ satisfy the same jump relation (4.16), we have

$$\begin{aligned} {M^{(3)}_-}^{-1} M^{(3)}_+ &= M^{(2)}_{\mathrm{RHP}-}(z) M^{(2)}_-(z)^{-1} M^{(2)}_+(z) M^{(2)}_{\mathrm{RHP}+}(z)^{-1} \\ &= M^{(2)}_{\mathrm{RHP}-}(z) V^{(2)}(z) \left(M^{(2)}_{\mathrm{RHP}-}(z) V^{(2)}(z) \right)^{-1} = I, \end{aligned} \tag{5.4}$$

from which it follows that $M^{(3)}$ and its first partials extend continuously to $\Sigma^{(2)}$.

Both $M^{(2)}$ and $M^{(2)}_{\mathrm{RHP}}$ are analytic in some deleted neighborhood of each point of discrete spectrum z_k and satisfy the residue relation (4.18). Let N_k denote the constant (in z) nilpotent matrix which appears in the left side of (4.18), then we have the Laurent expansions

$$\begin{aligned} M^{(2)}(z) &= C_0 \left[\frac{N_k}{z - z_k} + I \right] + \mathcal{O}(z - z_k), \\ M^{(2)}_{\mathrm{RHP}}(z)^{-1} &= \left[\frac{-N_k}{z - z_k} + I \right] \widehat{C}_0 + \mathcal{O}(z - z_k), \end{aligned} \tag{5.5}$$

where C_0 and $\widehat{C}_0$ are the constant terms in the Laurent expansions of $M^{(2)}(z)$ and $M^{(2)}_{\mathrm{RHP}}(z)^{-1}$ respectively. This implies that

$$M^{(2)}(z)M^{(2)}_{\mathrm{RHP}}(z)^{-1}=\mathcal{O}(1)\,, \tag{5.6}$$

i.e., $M^{(3)}$ has only removable singularities at each z_k. The last property follows immediately from the definition of $M^{(3)}$, exploiting the fact that $M^{(2)}_{\mathrm{RHP}}$ has no $\bar\partial$ component:

$$\bar\partial M^{(3)}(z)=\bar\partial M^{(2)}(z)M^{(2)}_{\mathrm{RHP}}(z)^{-1}=M^{(2)}\bar\partial\mathcal{R}^{(2)}(z)M^{(2)}_{\mathrm{RHP}}(z)^{-1}=M^{(3)}W^{(3)}(z). \quad \square \tag{5.7}$$

5.1. Constructing the model problems

We will construct the solution $M^{(2)}_{\mathrm{RHP}}$ by seeking a solution of the form

$$M^{(2)}_{\mathrm{RHP}}(z)=\begin{cases} E(z)M^{(\mathrm{out})}(z) & |z-\xi|>\rho/2 \\ E(z)M^{(\xi)}(z) & |z-\xi|<\rho/2 \end{cases} \tag{5.8}$$

where $M^{(\mathrm{out})}$ and $M^{(\xi)}$ are models which we construct below, and the error $E(z)$, the solution of a small-norm Riemann–Hilbert problem, we will prove exists and bound it asymptotically.

5.1.1. The outer model: an N-soliton potential

The matrix $M^{(2)}_{\mathrm{RHP}}$ is meromorphic away from the contour $\Sigma^{(2)}$ on which its boundary values satisfy the jump relation (4.16). However, at any distance from the saddle point $z=\xi$, the jump is uniformly near identity. Specifically, let $\mathcal{U}_\xi$ denote the open neighborhood

$$\mathcal{U}_\xi=\{z\,:\,|z-\xi|<\rho/2\}, \tag{5.9}$$

on which $M^{(2)}_{\mathrm{RHP}}$ is pole free. Using (4.16) with the spectral bound (4.3), and also recalling the definition (2.17) of θ we have

$$\|V^{(2)}-I\|_{L^\infty(\Sigma^{(2)})}=\mathcal{O}\left(\rho^{-2}e^{-\sqrt{2}t|z-\xi|^2}\right), \tag{5.10}$$

which is exponentially small in $\Sigma^{(2)}\backslash\mathcal{U}_\xi$, since $|z-\xi|\geq\rho/2$ outside $\mathcal{U}_\xi$. This estimate justifies constructing a model solution outside $\mathcal{U}_\xi$ which ignores the jumps completely.

Riemann–Hilbert Problem 5.1. *Find an analytic function $M^{(\mathrm{out})}:(\mathbb{C}\setminus\mathcal{Z}\cup\mathcal{Z}^*)\to SL_2(\mathbb{C})$ such that*

1. $M^{(\mathrm{out})}(z)=I+\mathcal{O}\left(z^{-1}\right)$ *as $z\to\infty$.*
2. $M^{(\mathrm{out})}$ *has simple poles at each $z_k\in\mathcal{Z}$ and $z_k^*\in\mathcal{Z}^*$ satisfying the residue relations in (4.18) with $M^{(\mathrm{out})}$ replacing $M^{(2)}$.*

Proposition 5.2. *There exist a unique solution $M^{(\mathrm{out})}$ of RHP 5.1. Specifically,*

$$M^{(\mathrm{out})}(z)=m^{\Delta_\xi^-}(z\,|\,\sigma_d^{out}) \tag{5.11}$$

where $m^{\Delta_\xi^-}$ is the solution of RHP B.2 with $\Delta=\Delta_\xi^-$ and $\sigma_d^{(out)}:=\{z_k,\widetilde{c}_k(\xi)\}_{k=1}^N$ with

$$\widetilde{c}_k(\xi)=c_k\exp\left(\frac{i}{\pi}\int_{-\infty}^{\xi}\log(1+|r(s)|^2)\frac{ds}{s-z_k}\right). \tag{5.12}$$

Moreover,

$$\lim_{z\to\infty}2izM^{(\mathrm{out})}_{12}(z;x,t)=\psi_{\mathrm{sol}}(x,t;\sigma_d^{out})$$

where $\psi_{\mathrm{sol}}(x,t;\sigma_d^{(out)})$ is the N-soliton solution of (1.1) corresponding to the discrete scattering data $\sigma_d^{(out)}$.

Proof. Observe that the conditions defining $M^{(\mathrm{out})}$ are identical to those defining $m^{\Delta_\xi^-}(z\,|\,\sigma_d^{\mathrm{out}})$ in RHP B.2 with $\Delta=\Delta_\xi^-$ and $\sigma_d=\sigma_d^{(\mathrm{out})}$. Proposition B.1 establishes the existence and uniqueness of solutions to RHP B.2 and the large z behavior follows from (B.16). $\square$

5.1.2. Local model near the saddle point $z = \xi$

For $z \in \mathcal{U}_\xi$ the bound (5.10) gives a point-wise, but not uniform estimate on the decay of the jump $V^{(2)}$ to identity. In order to arrive at a uniformly small jump Riemann–Hilbert problem for the function E, implicitly defined by (5.8) we introduce a different local model $M^{(\xi)}$ which exactly matches the jumps of $M^{(2)}_{\mathrm{RHP}}$ on $\Sigma^{(2)} \cap \mathcal{U}_\xi$. In order to motivate the model, recall the definition (2.17) of θ, and let $\zeta = \zeta(z)$ denote the rescaled local variable

$$\zeta = \zeta(z) = 2\sqrt{t}(z-\xi) \quad \Rightarrow \quad 2t\theta = \zeta^2/2 - 2t\xi^2 \tag{5.13}$$

which maps $\mathcal{U}_\xi$ to an expanding neighborhood of $\zeta = 0$. Additionally, let

$$r_\xi := r(\xi) T_0(\xi)^{-2} e^{2i(\kappa(\xi)\log(2\sqrt{t}) - t\xi^2)}. \tag{5.14}$$

Then, since $1 - \chi_{\mathcal{Z}}(z) \equiv 1$ for $z \in \mathcal{U}_\xi$, the jumps of $M^{(2)}_{\mathrm{RHP}}$ in $\mathcal{U}_\xi$ can be expressed as

$$V^{(2)}(z)\Big|_{z\in\mathcal{U}_\xi} = \begin{cases} \begin{pmatrix} 1 & 0 \\ r_\xi\, \zeta(z)^{-2i\kappa(\xi)} e^{i\zeta(z)^2/2} & 1 \end{pmatrix} & z \in \Sigma_1 \\ \begin{pmatrix} 1 & \frac{r_\xi^*}{1+|r_\xi|^2} \zeta(z)^{2i\kappa(\xi)} e^{-i\zeta(z)^2/2} \\ 0 & 1 \end{pmatrix} & z \in \Sigma_2 \\ \begin{pmatrix} 1 & 0 \\ \frac{r_\xi}{1+|r_\xi|^2} \zeta(z)^{-2i\kappa(\xi)} e^{i\zeta(z)^2/2} & 1 \end{pmatrix} & z \in \Sigma_3 \\ \begin{pmatrix} 1 & r_\xi^*\, \zeta(z)^{2i\kappa(\xi)} e^{-i\zeta(z)^2/2} \\ 0 & 1 \end{pmatrix} & z \in \Sigma_4, \end{cases} \tag{5.15}$$

which are *exactly* the jumps of the parabolic cylinder model problem (A.3) which was first introduced and solved in [17], and later applied to the MKdV in [10]. The solution is described in Appendix A. Using (A.4) we define the local model $M^{(\xi)}$ in (5.8) by

$$M^{(\xi)}(z) = M^{(\mathrm{out})}(z) M^{(\mathrm{PC})}(\zeta(z), r_\xi), \qquad z \in \mathcal{U}_\xi, \tag{5.16}$$

which satisfies the jump $V^{(2)}$ of $M^{(2)}_{\mathrm{RHP}}$ as $M^{(\mathrm{out})}$ is an analytic and bounded function in $\mathcal{U}_\xi$.

5.2. The small-norm Riemann–Hilbert problem for $E(z)$

Using the functions $M^{(\mathrm{out})}$ and $M^{(\xi)}$ defined by Proposition 5.2 and (5.16) respectively, (5.8) implicitly defines an unknown $E(z)$ which is analytic in $\mathbb{C}\backslash\Sigma^{(E)}$,

$$\Sigma^{(E)} = \partial\mathcal{U}_\xi \cup (\Sigma^{(2)} \backslash \mathcal{U}_\xi), \tag{5.17}$$

where we orient $\partial\mathcal{U}_\xi$ clockwise. It is straightforward to show that $E(z)$ must satisfy the following Riemann–Hilbert problem.

Riemann–Hilbert Problem 5.2. *Find a holomorphic function $E : \mathbb{C}\backslash\Sigma^{(E)} \to SL_2(\mathbb{C})$ with the following properties*

1. $E(z) = I + \mathcal{O}\left(z^{-1}\right)$ *as* $z \to \infty$.
2. *For each* $z \in \Sigma^{(E)}$ *the boundary values* $E_\pm(z)$ *satisfy* $E_+(z) = E_-(z) V^{(E)}(z)$ *where*

$$V^{(E)}(z) = \begin{cases} M^{(\mathrm{out})}(z) V^{(2)}(z) M^{(\mathrm{out})}(z)^{-1} & z \in \Sigma^{(2)} \backslash \mathcal{U}_\xi \\ M^{(\mathrm{out})}(z) M^{(\mathrm{PC})}(\zeta(z), r_\xi) M^{(\mathrm{out})}(z)^{-1} & z \in \partial\mathcal{U}_\xi \end{cases} \tag{5.18}$$

Starting from (5.18) and using (5.10) for $z \in \mathbb{C}\backslash\mathcal{U}_\xi$ and, using (5.13), (A.7) and the boundedness of $M^{(\mathrm{out})}$ for $z \in \mathcal{U}_\xi$, one finds that

$$|V_E(z) - I| = \begin{cases} \mathcal{O}\left(\rho^{-2} e^{-\sqrt{2}t|z-\xi|^2}\right) & z \in \Sigma^{(E)} \backslash \mathcal{U}_\xi \\ \mathcal{O}\left(t^{-1/2}\right) & z \in \partial\mathcal{U}_\xi, \end{cases} \tag{5.19}$$

and it follows that

$$\| \langle \cdot \rangle^k (V_E - I) \|_{L^p(\Sigma^{(E)})} = \mathcal{O}\left(t^{-1/2}\right) \qquad p \in [1, \infty],\ k \geq 0. \tag{5.20}$$

This uniformly vanishing bound on $V_E - I$ establishes RHP 5.2 as a small-norm Riemann–Hilbert problem, for which there is a well known existence and uniqueness theorem [13,12,36]. In fact, we may write

$$E(z) = I + \frac{1}{2\pi i} \int\limits_{\Sigma^{(E)}} \frac{(I + \eta(s))(V_E(s) - I)}{s - z} ds \tag{5.21}$$

where $\eta \in L^2(\Sigma^{(E)})$ is the unique solution of

$$\left(\mathbf{1} - C_{V^{(E)}}\right) \eta = C_{V^{(E)}} I. \tag{5.22}$$

The singular integral operator $C_{V^{(E)}} : L^2(\Sigma^{(E)}) \to L^2(\Sigma^{(E)})$ is defined by

$$C_{V^{(E)}} f = C_-(f(V_E - I)), \tag{5.23}$$

$$C_- f(z) \lim_{z \to \Sigma_-^{(E)}} \frac{1}{2\pi i} \int\limits_{\Sigma^{(E)}} f(s) \frac{ds}{s - z}, \tag{5.24}$$

where C_- is the well known Cauchy projection operator. It's well known that $\|C_-\|_{L^2_{\text{op}}(\Gamma)}$ is bounded for a very large class of contours Γ including the class of finite unions of analytic curves with finite intersection which includes $\Sigma^{(E)}$. It then follows from (5.20) and (5.23) that

$$\|C_{V^{(E)}}\|_{L^2_{\text{op}}(\Sigma^{(E)})} \lesssim \|C_-\|_{L^2_{\text{op}}(\Sigma^{(E)})} \|V^{(E)} - I\|_{L^\infty(\Sigma^{(E)})} \lesssim \mathcal{O}\left(t^{-1/2}\right), \tag{5.25}$$

which guarantees the existence of the resolvent operator $(\mathbf{1} - C_{V^{(E)}})^{-1}$ and thus of both η and E.

The existence of the solution $E(z)$ completes the definition of $M^{(2)}_{\text{RHP}}(z)$ given by (5.8) which in turn solves (5.1) and thus also justifies the transformation (5.2) of Proposition 5.1 to an unknown $M^{(3)}$ which satisfies the pure $\overline{\partial}$-Problem 5.1.

In order to reconstruct the solution $\psi(x, t)$ of (1.1) we need the large z behavior of the solution of RHP 2.1. This will include the large z expansion of E which we give here. Geometrically expanding $(s - z)^{-1}$ for z large in (5.21), which is justified by the finiteness of moments in (5.20), we have

$$E(z) = I + z^{-1} E_1 + \mathcal{O}\left(z^{-2}\right) \tag{5.26}$$

where

$$E_1 = -\frac{1}{2\pi i} \int\limits_{\Sigma^{(E)}} (I + \eta(s))(V^{(E)}(s) - I)\, ds. \tag{5.27}$$

Then using (5.22)–(5.25) and the bounds on $V_E - I$ in (5.19)–(5.20) we have

$$E_1 = -\frac{1}{2\pi i} \oint\limits_{\partial\mathcal{U}_\xi} \left(V^E(s) - I\right) ds + \mathcal{O}\left(t^{-1}\right). \tag{5.28}$$

This last integral, using (5.18), (A.7), and (5.13), can be asymptotically computed by residues yielding (recall that $\partial\mathcal{U}_\xi$ is clockwise oriented) to leading order

$$E_1(x, t) = \frac{1}{2i\sqrt{t}} M^{(\text{out})}(\xi; x, t) \begin{pmatrix} 0 & \beta_{12}(r_\xi) \\ -\beta_{21}(r_\xi) & 0 \end{pmatrix} M^{(\text{out})}(\xi; x, t)^{-1} + \mathcal{O}\left(t^{-1}\right), \tag{5.29}$$

where, using (A.6) and (5.14), we have

$$\beta_{12}(r_\xi) = \beta_{21}(r_\xi)^* = \alpha(\xi, +) e^{ix^2/(2t) - i\kappa(\xi) \log|4t|}. \tag{5.30}$$

Here

$$|\alpha(\xi,+)|^2 = |\kappa(\xi)|, \tag{5.31}$$

$$\arg\alpha(\xi,+) = \frac{\pi}{4} + \arg\Gamma(i\kappa(\xi)) - \arg r(\xi) - 4\sum_{k\in\Delta_\xi^-}\arg(\xi - z_k) - 2\int_{-\infty}^{\xi}\log|\xi - s|\mathrm{d}_s\kappa(s). \tag{5.32}$$

6. Analysis of the remaining $\bar{\partial}$-problem

$\bar{\partial}$-Problem 5.1 is equivalent to the integral equation

$$M^{(3)}(z) = I - \frac{1}{\pi}\iint_{\mathbb{C}}\frac{\bar{\partial}M^{(3)}(s)}{s-z}dA(s) = I - \frac{1}{\pi}\iint_{\mathbb{C}}\frac{M^{(3)}(s)W^{(3)}(s)}{s-z}dA(s), \tag{6.1}$$

where $dA(s)$ is Lebesgue measure on the plane.

Equation (6.1) can be written using operator notation as

$$(\mathbf{1} - S)[M^{(3)}(z)] = I, \tag{6.2}$$

where S is the solid Cauchy operator

$$S[f](z) = -\frac{1}{\pi}\iint_{\mathbb{C}}\frac{f(s)W^{(3)}(s)}{s-z}dA(s). \tag{6.3}$$

The following lemma shows that for sufficiently large t the operator S is small-norm, so that the resolvent operator $(I-S)^{-1}$ exists and can be expressed as a Neumann series.

Proposition 6.1. *There exists a constant C such that for all $t>0$, the operator (6.3) satisfies the inequality*

$$\|S\|_{L^\infty\to L^\infty} \le Ct^{-1/4}. \tag{6.4}$$

Proof. We detail the case for matrix functions having support in the region Ω_1, the case for the other regions follows similarly. Let $A\in L^\infty(\Omega_1)$ and $s=u+iv$, then from (4.17) and (5.3) it follows that

$$\begin{aligned}|S[A](z)| &\le \iint_{\Omega_1}\frac{|A(s)M^{(2)}_{\mathrm{RHP}}(s)W^{(2)}(s)M^{(2)}_{\mathrm{RHP}}(s)^{-1}|}{|s-z|}dA(s)\\ &\le \|A\|_{L^\infty(\Omega_1)}\|M^{(2)}_{\mathrm{RHP}}\|_{L^\infty(\Omega_1^\sharp)}\|M^{(2)\,-1}_{\mathrm{RHP}}\|_{L^\infty(\Omega_1^\sharp)}\iint_{\Omega_1}\frac{|\bar{\partial}R_1(s)|e^{-4tv(u-\xi)}|}{|s-z|}dA(s),\end{aligned} \tag{6.5}$$

where $\Omega_1^\sharp := \Omega_1\cap\operatorname{supp}(1-\chi_{\mathcal{Z}})$ is bounded away from the poles z_k of $M^{(2)}_{\mathrm{RHP}}$, so that $\|(M^{(2)}_{\mathrm{RHP}})^{\pm1}\|_{L^\infty(\Omega_1^\sharp)}$ are finite.

Using (4.6) the result follows using the estimates in Appendix D to bound the final integral term in (6.5):

$$\|S\|_{L^\infty\to L^\infty} \le C\,(I_1+I_2+I_3) \le Ct^{-1/4}, \tag{6.6}$$

where

$$I_1 = \iint_{\Omega_1}\frac{|\chi_{\mathcal{Z}}(s)|e^{-4tv(u-\xi)}}{|s-z|}dA(s), \qquad I_2 = \iint_{\Omega_1}\frac{|r'(u)|e^{-4tv(u-\xi)}}{|s-z|}dA(s), \tag{6.7}$$

and

$$I_3 = \iint_{\Omega_1}\frac{|s-\xi|^{-1/2}e^{-4tv(u-\xi)}}{|s-z|}dA(s). \qquad \square \tag{6.8}$$

To recover the long-time asymptotic behavior of $\psi(x,t)$ using (2.20) it is necessary to determine the asymptotic behavior of the coefficient of the z^{-1} term in the Laurent expansion of $M^{(3)}$ at infinity. An integral representation of this term is given by the expansion

$$M^{(3)}(z) = I - \frac{1}{\pi} \iint_{\mathbb{C}} \frac{M^{(3)}(s)W^{(3)}(s)}{s-z} dA(s) = I + \frac{M_1}{z} + \frac{1}{\pi} \iint_{\mathbb{C}} \frac{sM^{(3)}(s)W^{(3)}(s)}{z(s-z)} dA(s), \tag{6.9}$$

where

$$M_1^{(3)} = \frac{1}{\pi} \iint_{\mathbb{C}} M^{(3)}(s)W^{(3)} dA(s). \tag{6.10}$$

Proposition 6.2. *For all $t > 0$ there exists a constant c such that*

$$|M_1^{(3)}| \le ct^{-3/4}. \tag{6.11}$$

A proof of Proposition 6.2 is detailed in Appendix D.

7. Long time asymptotics for focusing NLS

We are now ready to prove Theorem 1.1. We give the details of the proof for $t \to +\infty$, for $t \to -\infty$ one simply replaces the formulae for the various components with their counterparts for negative times found in Appendix C.

Proof of Theorem 1.1. Inverting the sequence of transformations (3.9), (4.14), (5.2), and (5.8) the solution of RHP 2.1 is given by

$$M(z) = M^{(3)}(z)E(z)M^{(\mathrm{out})}(z)\mathcal{R}^{(2)}(z)T(z)^{\sigma_3}, \quad z \in \mathbb{C} \setminus \mathcal{U}_\xi. \tag{7.1}$$

The solution of (1.1) can now be recovered using (2.20).

Taking $z \to \infty$ vertically, eventually $z \in \Omega_2$ so that $\mathcal{R}^{(2)} = I$; in the vertical direction (3.4) gives

$$T(z)^{\sigma_3} = I + \frac{T_1\sigma_3}{z} + \mathcal{O}\left(z^{-2}\right), \qquad T_1 = 2i \sum_{\Delta^-} \operatorname{Im} z_k - \int_{-\infty}^{\xi} \kappa(s)ds.$$

Now

$$M = \left(I + \frac{M_1^{(3)}}{z} + \cdots\right)\left(I + \frac{E_1}{z} + \cdots\right)\left(I + \frac{M_1^{(\mathrm{out})}}{z} + \cdots\right)\left(I + \frac{T_1\sigma_3}{z} + \cdots\right), \tag{7.2}$$

and consequently the coefficient of the z^{-1} in the Laurent expansion of M is given by

$$M_1 = M_1^{(3)} + E_1 + M_1^{(\mathrm{out})} + T_1\sigma_3. \tag{7.3}$$

Using the reconstruction formula (2.20) and Proposition 6.2 we have

$$\psi(x,t) = 2i(M_1^{(\mathrm{out})})_{12} + 2i(E_1)_{12} + \mathcal{O}\left(t^{-3/4}\right). \tag{7.4}$$

Applying Proposition 5.2 to the first term and using (5.29)–(5.32) to evaluate the second term, we have

$$\psi(x,t) = \psi_{\mathrm{sol}}(x,t;\sigma_d^{\mathrm{out}}) + t^{-1/2}f^+(x,t) + \mathcal{O}\left(t^{-3/4}\right), \tag{7.5}$$

where f^+ is as given in (1.13) and $\psi_{\mathrm{sol}}(x,t;\sigma_d^{\mathrm{out}})$ is the N-soliton generated from scattering data σ_d^{out} defined in Proposition 5.2. To complete the proof, given a cone $C(x_1,x_2,v_1,v_2)$ as defined in Theorem 1.1, we apply Corollary B.3 to replace $\psi_{\mathrm{sol}}(x,t;\sigma_d^{\mathrm{out}})$ with $\psi_{\mathrm{sol}}(x,t;\sigma_d^+(\mathcal{I}))$ up to exponential errors which are absorbed into the $\mathcal{O}\left(t^{-3/4}\right)$ term. □

Conflict of interest statement

Conflicts of interest: none.

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Appendix A. The parabolic cylinder model problem

Here we describe the solution of the parabolic cylinder model problem introduced by [17] and later by [10]. It appears frequently in the literature of long-time asymptotic calculations for integrable nonlinear waves [14,16,18,25,26,20]. Let $\Sigma_{PC} = \bigcup_{j=1}^{4} \Sigma_j$, where Σ_j denotes the complex contour

$$\Sigma_j = \left\{ \zeta \in \mathbb{C} \mid \arg \zeta = \frac{2j-1}{4}\pi \right\}, \quad j = 1, \dots, 4, \tag{A.1}$$

oriented with increasing real part. Let Ω_j, $j = 1, \dots 6$ denote the six maximally connected open sectors in $\mathbb{C} \backslash (\Sigma_{PC} \cup \mathbb{R})$, where Ω_1 denotes the sector abutting the positive real axis from above, the rest labelled sequentially as one encircles the origin in a counterclockwise fashion. (See Fig. A.1.) Finally, fix $r \in \mathbb{C}$ and let

$$\kappa = \kappa(r) := -\frac{1}{2\pi} \log(1 + |r|^2). \tag{A.2}$$

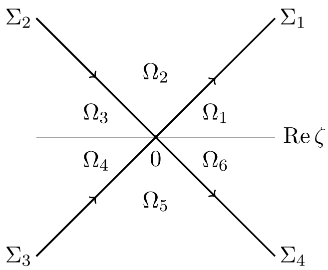

Fig. A.1. The contours Σ_j and sectors Ω_j in the ζ-plane defining RHP A.1.

Then consider the following Riemann–Hilbert problem

Parabolic Cylinder Model Riemann–Hilbert Problem A.1. *Fix $r \in \mathbb{C}$, find an analytic function $M^{(PC)}(\cdot, r) : \mathbb{C} \backslash \Sigma^{(PC)} \to SL_2(\mathbb{C})$ such that*

1. $M^{(PC)}(\zeta, r) = I + \frac{M^{(PC)(1)}(r)}{\zeta} + \mathcal{O}\left(\zeta^{-2}\right)$ *uniformly as* $\zeta \to \infty$.
2. *For* $\zeta \in \Sigma^{(PC)}$, *the continuous boundary values* $M_{\pm}^{(PC)}(\zeta, r)$ *satisfy the jump relation* $M_{+}^{(PC)}(\zeta, r) = M_{-}^{(PC)}(\zeta, r) V^{(PC)}(\zeta, r)$ *where*

$$V^{(PC)}(\zeta,r)=\begin{cases}\begin{pmatrix}1 & 0\\ r\zeta^{-2i\kappa}e^{i\zeta^2/2} & 1\end{pmatrix} & \arg\zeta=\pi/4\\ \begin{pmatrix}1 & r^*\zeta^{2i\kappa}e^{-i\zeta^2/2}\\ 0 & 1\end{pmatrix} & \arg\zeta=-\pi/4\\ \begin{pmatrix}1 & \frac{r^*}{1+|r|^2}\zeta^{2i\kappa}e^{-i\zeta^2/2}\\ 0 & 1\end{pmatrix} & \arg\zeta=3\pi/4\\ \begin{pmatrix}1 & 0\\ \frac{r}{1+|r|^2}\zeta^{-2i\kappa}e^{i\zeta^2/2} & 1\end{pmatrix} & \arg\zeta=-3\pi/4\end{cases} \tag{A.3}$$

RHP A.1 has an explicit solution $M^{(PC)}(\zeta,r)$ which is expressed in terms of $D_a(\pm z)$, solutions of the parabolic cylinder equation, $\left(\frac{\partial^2}{\partial z^2}+\left(\frac{1}{2}-\frac{z^2}{2}+a\right)\right)D_a(z)=0$ [31, Chapter 12], as follows:

$$M^{(PC)}(\zeta,r)=\Phi(\zeta,r)\mathcal{P}(\zeta,r)e^{\frac{i}{4}\zeta^2\sigma_3}\zeta^{-i\kappa\sigma_3} \tag{A.4}$$

where

$$\begin{aligned}\mathcal{P}(\zeta,r)&=\begin{cases}\begin{pmatrix}1 & 0\\ -r & 1\end{pmatrix} & \zeta\in\Omega_1\\ \begin{pmatrix}1 & \frac{-r^*}{1+|r|^2}\\ 0 & 1\end{pmatrix} & \zeta\in\Omega_3\\ \begin{pmatrix}1 & 0\\ \frac{r}{1+|r|^2} & 1\end{pmatrix} & \zeta\in\Omega_4\\ \begin{pmatrix}1 & r^*\\ 0 & 1\end{pmatrix} & \zeta\in\Omega_6\\ I & \zeta\in\Omega_2\cup\Omega_5\end{cases}\\ \Phi(\zeta,r)&=\begin{cases}\begin{pmatrix}e^{-\frac{3\pi\kappa}{4}}D_{i\kappa}\left(e^{\frac{-3i\pi}{4}}\zeta\right) & -i\beta_{12}e^{\frac{\pi}{4}(\kappa-i)}D_{-i\kappa-1}\left(e^{-\frac{i\pi}{4}}\zeta\right)\\ i\beta_{21}e^{\frac{-3\pi}{4}(\kappa+i)}D_{i\kappa-1}\left(e^{\frac{-3i\pi}{4}}\zeta\right) & e^{\frac{\pi\kappa}{4}}D_{-i\kappa}\left(e^{\frac{-i\pi}{4}}\zeta\right)\end{pmatrix} & \zeta\in\mathbb{C}^+\\ \begin{pmatrix}e^{\frac{\pi\kappa}{4}}D_{i\kappa}\left(e^{\frac{i\pi}{4}}\zeta\right) & -i\beta_{12}e^{\frac{-3\pi}{4}(\kappa-i)}D_{-i\kappa-1}\left(e^{\frac{3i\pi}{4}}\zeta\right)\\ i\beta_{21}e^{\frac{\pi}{4}(\kappa+i)}D_{i\kappa-1}\left(e^{\frac{i\pi}{4}}\zeta\right) & e^{\frac{-3\pi\kappa}{4}}D_{-i\kappa}\left(e^{\frac{3i\pi}{4}}\zeta\right)\end{pmatrix} & \zeta\in\mathbb{C}^-\end{cases}\end{aligned} \tag{A.5}$$

and β_{12} and β_{21} are the complex constants

$$\beta_{12}=\beta_{12}(r)=\frac{\sqrt{2\pi}e^{i\pi/4}e^{-\pi\kappa/2}}{r\Gamma(-i\kappa)},\qquad \beta_{21}=\beta_{21}(r)=\frac{-\sqrt{2\pi}e^{-i\pi/4}e^{-\pi\kappa/2}}{r^*\Gamma(i\kappa)}=\frac{\kappa}{\beta_{12}}. \tag{A.6}$$

A derivation of this result is given in [10], a direct verification of the solution in given in the appendix of [19]. The essential fact for our needs is the asymptotic behavior of the solution given in the above references, as is easily verified using the well known asymptotic behavior of $D_a(z)$,

$$M^{(PC)}(\zeta,r)=I+\frac{1}{\zeta}\begin{pmatrix}0 & -i\beta_{12}(r)\\ i\beta_{21}(r) & 0\end{pmatrix}+\mathcal{O}\left(\zeta^{-2}\right). \tag{A.7}$$

Appendix B. Meromorphic solutions of the focusing NLS Riemann–Hilbert problem

Here we consider the solutions of the Riemann–Hilbert problem associated with the fNLS equation, RHP 2.1, for which the reflection coefficient $r(z)\equiv 0$. In this case the unknown function is analytic across the real axis and has isolated poles in the plane, i.e., the solution is meromorphic. The resulting, *reflectionless*, solutions of fNLS, $\psi(x,t)$, derived from the solution of the Riemann–Hilbert problem, are multi-solitons. Here we give a simple proof of the existence and uniqueness of solutions of this problem and briefly discuss some well known results concerning the asymptotic behavior of these solutions as $t \to \infty$.

Given a finite discrete spectrum and their associated normalization constants, $\sigma_d = \{(z_k, c_k)\}_{k=1}^N$, denote the solution of the reflectionless Riemann–Hilbert problem, RHP B.1, associated to (1.1) by $m(z; x, t \,|\, \sigma_d)$.

Riemann–Hilbert Problem B.1. *Given discrete data $\sigma_d = \{(z_k, c_k)\}_{k=1}^N \subset \mathbb{C}^+ \times \mathbb{C}_*$ let $\mathcal{Z} = \{z_k\}_{k=1}^N$. Find an analytic function $m : \mathbb{C} \setminus (\mathcal{Z} \cup \mathcal{Z}^*) \to SL_2(\mathbb{C})$ with the following properties.*

1. $m(z; x, t \,|\, \sigma_d) = I + \mathcal{O}\left(z^{-1}\right)$ *as* $z \to \infty$.
2. *Each point of $\mathcal{Z} \cup \mathcal{Z}^*$ is a simple pole of $m(z; x, t \,|\, \sigma_d)$. They satisfy the residue conditions*

$$\begin{aligned} \operatorname*{Res}_{z=z_k} m(z; x, t \,|\, \sigma_d) &= \lim_{z \to z_k} m(z; x, t \,|\, \sigma_d) n_k \\ \operatorname*{Res}_{z=z_k^*} m(z; x, t \,|\, \sigma_d) &= \lim_{z \to z_k^*} m(z; x, t \,|\, \sigma_d) \sigma_2 n_k^* \sigma_2 \end{aligned} \tag{B.1}$$

where n_k is the nilpotent matrix,

$$n_k = \begin{pmatrix} 0 & 0 \\ \gamma_k(x, t) & 0 \end{pmatrix} \qquad \gamma_k(x, t) := c_k \exp(2i(t z_k^2 + x z_k)). \tag{B.2}$$

In what follows we will omit the dependence on x, t, and/or σ_d and write $m(z \,|\, \sigma_d)$ or just $m(z)$ for $m(z; x, t \,|\, \sigma_d)$ when the context is clear. It's a direct consequence of the uniqueness of the solution (which follows from Liouville's theorem) and the symmetries in RHP B.1 (or more generally in RHP 2.1) that the solution of RHP B.1 must possess the symmetry $m(z \,|\, \sigma_d) = \sigma_2 m(z^* \,|\, \sigma_d)^* \sigma_2$. It follows that any solution of RHP B.1 must admit a partial fraction expansion of the form

$$m(z; x, t \,|\, \sigma_d) = I + \sum_{k=1}^N \frac{1}{z - z_k} \begin{pmatrix} \alpha_k(x, t) & 0 \\ \beta_k(x, t) & 0 \end{pmatrix} + \frac{1}{z - z_k^*} \begin{pmatrix} 0 & -\beta_k(x, t)^* \\ 0 & \alpha_k(x, t)^* \end{pmatrix} \tag{B.3}$$

for coefficients $\alpha_k(x, t)$, $\beta_k(x, t)$ to be determined.

Proposition B.1. *Given data $\sigma_d = \{(z_k, c_k)\}_{k=1}^N \subset \mathbb{C}^+ \times \mathbb{C}_*$ such that $z_j \neq z_k$ for $j \neq k$, there exists a unique solution of RHP B.1 for each $(x, t) \in \mathbb{R}^2$.*

Proof. Inserting the partial fraction expansion (B.3) into the residue conditions (B.1) leads to, after some renormalization, the following linear system of equations for $j = 1, \dots, N$,

$$\widehat{\alpha}_j + \sum_{k=1}^N \frac{{\gamma_j}^{1/2} {\gamma_k^*}^{1/2}}{z_j - z_k^*} \widehat{\beta}_k^* = 0, \qquad \widehat{\beta}_j^* - \sum_{k=1}^N \frac{{\gamma_j^*}^{1/2} \gamma_k^{1/2}}{z_j^* - z_k} \widehat{\alpha}_k, = {\gamma_j^*}^{1/2} \tag{B.4}$$

where we've defined the renormalized parameters

$$\widehat{\alpha}_j = \alpha_j / \gamma_j^{1/2}, \quad \text{and} \quad \widehat{\beta}_j^* = \beta_j^* / {\gamma_j^*}^{1/2}, \tag{B.5}$$

and for brevity we've suppressed the (x, t) dependence of α_j, β_j, and γ_j. Letting $\widehat{\alpha} = (\widehat{\alpha}_1, \dots, \widehat{\alpha}_N)^\intercal$, $\widehat{\beta} = (\widehat{\beta}_1, \dots, \widehat{\beta}_N)^\intercal$, $\gamma^{1/2} = (\gamma_1^{1/2}, \dots, \gamma_N^{1/2})^\intercal$, and A be the $N \times N$ matrix with entries

$$A_{jk} = \frac{-i {\gamma_j^*}^{1/2} \gamma_k^{1/2}}{(z_j^* - z_k)}, \qquad j, k = 1, \dots, N \tag{B.6}$$

the system (B.4) is equivalent to the block matrix equation

$$\begin{bmatrix} I_N & -i A^* \\ -i A & I_N \end{bmatrix} \begin{bmatrix} \widehat{\alpha} \\ \widehat{\beta}^* \end{bmatrix} = \begin{bmatrix} 0 \\ {\gamma^*}^{1/2} \end{bmatrix}. \tag{B.7}$$

Note that A^* denotes only the complex, not hermitian, conjugate of A. Equation (B.7) will have a unique solution if and only if

$$\det\begin{bmatrix} I_N & -i\,A^* \\ -i\,A & I_N \end{bmatrix} = \det\left(I_N + AA^*\right) \neq 0. \tag{B.8}$$

Clearly, A is hermitian. Observing also that A has the inner product structure

$$A_{jk} = \int_0^\infty \gamma_j^{*\,1/2}\gamma_k^{1/2} e^{i(z_k - z_j^*)s}\, ds = \left\langle \gamma_j{}^{1/2} e^{iz_j s},\ \gamma_k{}^{1/2} e^{iz_k s}\right\rangle \tag{B.9}$$

where the functions $f_j(s) = \gamma_j{}^{1/2} e^{iz_j s}$ are linearly independent in $L^2(\mathbb{R}_+)$ since $z_j \neq z_k$ by assumption. It follows that A is positive definite. Let $A^{1/2}$ denote the unique positive definite square root of A. Now the eigenvalues of $AA^* = A^{1/2}\left(A^{1/2}A^*\right)$ are the same as those of $A^{1/2}(A^*)A^{1/2}$ which is itself positive definite. If we denote these eigenvalues as $\{\lambda_k\}_{k=1}^N \subset \mathbb{R}_+$ then it follows that

$$\det\left(I_n + AA^*\right) = \prod_{k=1}^N (1 + \lambda_k) > 0. \tag{B.10}$$

This proves the proposition. □

B.1. Renormalizations of the reflectionless Riemann–Hilbert problem

The Riemann–Hilbert problem RHP B.1 which encodes the N-soliton solutions of (1.1) arises from a particular choice of normalization in the forward scattering step of the IST. Specifically, recalling that $\Phi_1^{(-)}(x,t;z)$ and $\Phi_2^{(+)}(x,t;z)$ denote the first and second columns respectively of the left and right normalized Jost functions $\Phi^{(\pm)}(x,t;z)$ of the ZS-AKNS scattering problem, (2.1a), the matrix $m(z\,|\,\sigma_d)$ in RHP B.1 is defined for $z \in \mathbb{C}^+$ as

$$m(z;x,t\,|\,\sigma_d) = \left[\left.\frac{\Phi_1^{(-)}(x,t;z)}{a(z)}\,\right|\,\Phi_2^{(+)}(x,t;z)\right] e^{i(tz^2 + xz)\sigma_3}, \quad a(z) = \prod_{k=1}^N\left(\frac{z - z_k}{z - z_k^*}\right), \tag{B.11}$$

where $1/a(z)$ is the transmission coefficient of the reflectionless initial data. This choice of normalization ensures that for any fixed t, $\lim_{x\to+\infty} m(z;x,t) = I$, but is not the only choice available to us.

Motivated by examples in the literature (e.g. [7], [2]) we introduce the following transformation which prepares the residue coefficients for asymptotic analysis. Let $\Delta \subseteq \{1, 2, \ldots, N\}$ and $\nabla = \Delta^c = \{1, \ldots, N\}\backslash\Delta$. Define

$$a_\Delta(z) = \prod_{k\in\Delta}\left(\frac{z - z_k}{z - z_k^*}\right) \qquad \text{and} \qquad a_\nabla(z) = \frac{a(z)}{a_\Delta(z)} = \prod_{k\in\nabla}\left(\frac{z - z_k}{z - z_k^*}\right). \tag{B.12}$$

The renormalization

$$m^\Delta(z\,|\,\sigma_d) = m(z\,|\,\sigma_d)a_\Delta(z)^{\sigma_3} = \left[\left.\frac{\Phi_1^{(-)}(x,t;z)}{a_\nabla(z)}\,\right|\,\frac{\Phi_2^{(+)}(x,t;z)}{a_\Delta(z)}\right] e^{i(tz^2 + xz)\sigma_3}, \tag{B.13}$$

then splits the poles between the columns of $m^\Delta(z\,|\,\sigma_d)$ according to the choice of Δ. It's a simple calculation to show that the renormalization m^Δ satisfies a modified discrete Riemann–Hilbert problem.

Riemann–Hilbert Problem B.2. *Given discrete data $\sigma_d = \{(z_k, c_k)\}_{k=1}^N \subset \mathbb{C}^+ \times \mathbb{C}_*$ and $\Delta \subseteq \{1, \ldots, N\}$ find an analytic function $m^\Delta : \mathbb{C} \setminus (\mathcal{Z} \cup \mathcal{Z}^*) \to SL_2(\mathbb{C})$ with the following properties.*

1. $m^\Delta(z;x,t\,|\,\sigma_d) = I + \mathcal{O}\left(z^{-1}\right)$ *as* $z \to \infty$.

2. *Each point of $\mathcal{Z} \cup \mathcal{Z}^*$ is a simple pole of $m^{\Delta}(z; x, t \,|\, \sigma_d)$, they satisfy the residue conditions*

$$\begin{aligned} \operatorname*{Res}_{z=z_k} m^{\Delta}(z; x, t \,|\, \sigma_d) &= \lim_{z \to z_k} m^{\Delta}(z; x, t \,|\, \sigma_d) n_k^{\Delta} \\ \operatorname*{Res}_{z=z_k^*} m^{\Delta}(z; x, t \,|\, \sigma_d) &= \lim_{z \to z_k^*} m^{\Delta}(z; x, t \,|\, \sigma_d) \sigma_2 (n_k^{\Delta})^* \sigma_2 \end{aligned} \tag{B.14}$$

where n_k is the nilpotent matrix,

$$n_k^{\Delta} = \begin{cases} \begin{pmatrix} 0 & 0 \\ \gamma_k(x,t) a_{\Delta}(z_k)^2 & 0 \end{pmatrix} & k \in \nabla \\ \begin{pmatrix} 0 & \gamma_k(x,t)^{-1} a_{\Delta}{}'(z_k)^{-2} \\ 0 & 0 \end{pmatrix} & k \in \Delta, \end{cases} \qquad \gamma_k(x,t) := c_k \exp(2i(tz_k^2 + xz_k)), \tag{B.15}$$

and a_{Δ} is as defined in (B.12).

As $m^{\Delta}(z; x, t \,|\, \sigma_d)$ is an explicit transformation of $m(z; x, t \,|\, \sigma_d)$, it follows directly from Proposition B.1 that RHP B.2 has a unique solution whenever the poles $z_k \in \mathcal{Z}$ are distinct. Moreover, if $\psi_{\mathrm{sol}}(x,t) = \psi_{\mathrm{sol}}(x, t; \sigma_d)$ denotes the N-soliton solution of (1.1) encoded by RHP B.1, then using (2.19) and (B.13) we have

$$m^{\Delta}(z \,|\, \sigma_d) = I + \frac{1}{2iz} \begin{bmatrix} -\int_x^{\infty} |\psi_{\mathrm{sol}}(s,t)|^2 ds + \sum_{k \in \Delta} 4 \operatorname{Im} z_k & \psi_{\mathrm{sol}}(x,t) \\ \psi_{\mathrm{sol}}^*(x,t) & \int_x^{\infty} |\psi_{\mathrm{sol}}(s,t)|^2 ds - \sum_{k \in \Delta} 4 \operatorname{Im} z_k \end{bmatrix} + \mathcal{O}\left(z^{-2}\right). \tag{B.16}$$

This shows that each normalization encodes ψ_{sol} in the same way. The advantage of the nonstandard normalizations is, as we will see below, that by choosing Δ correctly, other asymptotic limits in which $t \to \infty$ with $-x/2t = \xi$ bounded are under better asymptotic control. The new sums appearing on the diagonal entries above, when compared to (2.20), represent the squared L^2 mass of the solitons corresponding to each z_k, $k \in \Delta$.

B.2. *Long time behavior of soliton solutions*

If $N = 1$, then the scattering data consists of only a single point $\sigma_d = \{(\xi + i\eta, c_1)\}$. In this case, the algebraic system for $\alpha_1(x,t)$ and $\beta_1(x,t)$ implied by (B.1)–(B.3) is trivial. Using (2.20), the solution of (1.1), $\psi(x, t; \sigma_d) = -2i\beta_1(x,t)^*$, is given by

$$\begin{aligned} &\psi(x, t; \sigma_d) = 2\eta \operatorname{sech}(2\eta(x + 2\xi t - x_0))\, e^{-2i(\xi x + (\xi^2 - \eta^2)t)} e^{-i\phi_0}, \\ &\qquad x_0 = \frac{1}{2\eta} \log \left| \frac{c_1}{2\eta} \right|, \qquad \phi_0 = \frac{\pi}{2} + \arg(c_1), \end{aligned} \tag{B.17}$$

which is a localized traveling wave of maximum amplitude $2 \operatorname{Im} z_0$ traveling at speed $-2 \operatorname{Re} z_0$; the normalization constant c determines the initial location and constant phase shift of the solution.

For $N > 1$ exact formulas for the solution become ungainly, and we will not present them here. However, as is well known, the N-soliton solutions undergo elastic collisions and asymptotically separate as $t \to \infty$ into, generically, N single-soliton solutions traveling at speeds $-2 \operatorname{Re} z_k$, one for each point in the discrete spectrum $\{z_k\}_{k=1}^N$ which defines RHP B.1. The exception, of course, is the non-generic case in which two (or more) points of discrete spectrum lie on a vertical line $\xi + i\mathbb{R}$.

The following proposition and its corollary make this precise. Recall the notation (1.9)–(1.12) used in Theorem 1.1 and let

$$\mu = \mu(\mathcal{I}) = \min_{z_k \in \mathcal{Z} \setminus \mathcal{Z}(\mathcal{I})} \{ \operatorname{Im}(z_k) \operatorname{dist}(\operatorname{Re} z_k, \mathcal{I}) \}. \tag{B.18}$$

Proposition B.2. *Given discrete scattering data $\sigma_d = \{(z_k, c_k)\}_{k=1}^N \subset \mathbb{C}^+ \times (\mathbb{C} \setminus \{0\})$, fix $x_1, x_2, v_1, v_2 \in \mathbb{R}$ with $x_1 \leq x_2$ and $v_1 \leq v_2$. Let $\mathcal{I} = [-v_2/2, -v_1/2]$. Then, as $t \to \pm\infty$ with $(x,t) \in C(x_1, x_2, v_1, v_2)$ we have*

$$m^{\Delta_\xi^{\mp}}(z; x, t \,|\, \sigma_d) = \left(I + \mathcal{O}\left(e^{-4\mu|t|} \right) \right) m^{\Delta_\xi^{\mp}(\mathcal{I})}(z; x, t \,|\, \sigma_d^{\pm}(\mathcal{I})) \tag{B.19}$$

for all z bounded away from $\mathcal{Z} \cup \mathcal{Z}^*$.

Here $\sigma_d^{\pm}(\mathcal{I})$ is the scattering data for the $N(\mathcal{I}) \leq N$ soliton given by

$$\sigma_d^{\pm}(\mathcal{I}) = \{(z_k, c_k^{\pm}(\mathcal{I})) : z_k \in \mathcal{Z}(\mathcal{I})\}, \qquad c_k^{\pm}(\mathcal{I}) = c_k \prod_{z_j \in \mathcal{Z}^{\mp}(\mathcal{I})} \left(\frac{z_k - z_j}{z_k - z_j^*} \right)^2 . \tag{B.20}$$

Corollary B.3. *Let $\psi_{\text{sol}}(x, t; \sigma_d)$ denote the N-soliton solution of the fNLS equation (1.1) corresponding to discrete scattering data $\sigma_d = \{(z_k, c_k)\}_{k=1}^N \subset \mathbb{C}^+ \times (\mathbb{C} \setminus \{0\})$. Let $\mathcal{I}$, $C(x_1, x_2, v_1, v_2)$ and $\sigma_d^{\pm}(\mathcal{I})$ be as given in Proposition B.2.*

Then as $t \to \pm\infty$, with $(x, t) \in C(x_1, x_2, v_1, v_2)$

$$\psi_{\text{sol}}(x, t; \sigma_d) = \psi_{\text{sol}}(x, t; \sigma_d^{\pm}(\mathcal{I})) + \mathcal{O}\left(e^{-4\mu t}\right) \tag{B.21}$$

where $\psi_{\text{sol}}(x, t; \sigma_d^{\pm}(\mathcal{I}))$ is the reduced $N(\mathcal{I})$-soliton solution of fNLS with scattering data $\sigma_d^{\pm}(\mathcal{I})$.

Proof of Proposition B.2. Observe that

$$|\gamma_k(x_0 + vt, t)| = |c_k| \exp[-2x_0 \operatorname{Im}(z_k)] \exp[-4t \operatorname{Im}(z_k) \operatorname{Re}(z_k + v/2)]. \tag{B.22}$$

The choice of normalization $\Delta = \Delta_\xi^{\mp}$ in RHP B.2 ensures that as $|t| \to \infty$ with $(x, t) \in C(x_1, x_2, v_1, v_2)$ that

$$\|n_k^{\Delta_\xi^{\mp}}\| = \begin{cases} \mathcal{O}(1) & z_k \in \mathcal{Z}(\mathcal{I}) \\ \mathcal{O}(\exp(-4\mu|t|)) & z_k \in \mathcal{Z} \setminus \mathcal{Z}(\mathcal{I}), \end{cases} \qquad t \to \pm\infty. \tag{B.23}$$

This suggests that the residues with $z_k \in \mathcal{Z} \setminus \mathcal{Z}(\mathcal{I})$ do not meaningfully contribute to the solution $m^{\Delta_\xi^{\mp}}$.

For each $z_k \in \mathcal{Z} \setminus \mathcal{Z}(\mathcal{I})$ we trade the residue for a near identity jump by introducing small disks D_k around each $z_k \in \mathcal{Z} \setminus \mathcal{Z}(\mathcal{I})$ whose radii are chosen sufficiently small that they are non-overlapping. We make the change of variables

$$m^{\Delta_\xi^{\mp}}(z \,|\, \sigma_d) = \begin{cases} \widehat{m}^{\Delta_\xi^{\mp}}(z) \left(I + \frac{n_k}{z - z_k} \right) & z \in D_k \\ \widehat{m}^{\Delta_\xi^{\mp}}(z) \left(I + \frac{\sigma_2 n_k^* \sigma_2}{z - z_k^*} \right) & z \in D_k^* \\ \widehat{m}^{\Delta_\xi^{\mp}}(z) & \text{elsewhere.} \end{cases} \tag{B.24}$$

The new unknown $\widehat{m}^{\Delta_\xi^{\mp}}(z)$ has jumps across each disk boundary which, by virtue of (B.23), satisfy

$$\widehat{m}_+^{\Delta_\xi^{\mp}}(z) = \widehat{m}_-^{\Delta_\xi^{\mp}}(z) \widehat{v}(z) \qquad z \in \partial D_k \cup \partial D_k^* \tag{B.25}$$

with

$$\|\widehat{v} - I\| = \mathcal{O}(\exp(-4\mu|t|)) \qquad z \in \partial D_k \cup \partial D_k^*. \tag{B.26}$$

Next, we observe that $m^{\Delta_\xi^{\mp}(\mathcal{I})}(z \,|\, \sigma_d^{\pm}(\mathcal{I}))$ has the same poles as $\widehat{m}^{\Delta_\xi^{\mp}}(z \,|\, \sigma_d)$ with exactly the same residue conditions. A simple calculation then shows that the quantity

$$e(z) = \widehat{m}^{\Delta_\xi^{\mp}}(z \,|\, \sigma_d) \left[m^{\Delta_\xi^{\mp}(\mathcal{I})}(z \,|\, \sigma_d^{\pm}(\mathcal{I})) \right]^{-1} \tag{B.27}$$

has no poles, and its jumps satisfy estimates identical to (B.26). Using the theory of small-norm Riemann–Hilbert problems, one shows that $e(z)$ exists and that $e(z) = I + \mathcal{O}\left(e^{-4\mu|t|}\right)$ for all sufficiently large $|t|$. It then follows from (B.24) and (B.27) that $m^{\Delta_\xi^{\mp}}(z; x, t \,|\, \sigma_d) = e(z) m^{\Delta_\xi^{\mp}(\mathcal{I})}(z; x, t \,|\, \sigma_d^{\pm}(\mathcal{I}))$ for z outside each disk D_k and D_k^*. The result follows immediately. □

Appendix C. Steepest descent analysis for large negative times

The steps in the steepest descent analysis of RHP 2.1 for $t \to -\infty$ mirror those presented in Sections 3–6 for $t \to \infty$. The differences that appear can be traced back to the fact that the regions of growth and decay of the exponential factors $e^{2it\theta}$ are reversed when one considers $t \to -\infty$, see Fig. 3.1. Here we briefly sketch those changes, leaving the detailed calculations to the interested reader.

The first step in the analysis, as in Section 3, is a conjugation to well-condition the problem for large-time analysis. Similar to (3.9) define

$$M^{(1)}(z) = M(z)T(z)^{-\sigma_3} \tag{C.1}$$

where now

$$T(z) = T(z,\xi) = \prod_{k\in\Delta_\xi^+}\left(\frac{z-z_k^*}{z-z_k}\right)\exp\left(i\int_\xi^\infty \frac{\kappa(s)}{s-z}ds\right) \tag{C.2}$$

satisfies properties similar to those in Proposition 3.1 subject to the obvious changes introduced by its redefinition for negative t. In particular, in property (d)

$$T(z) = 1 + \frac{i}{z}\left[2\sum_{k\in\Delta_\xi^+}\operatorname{Im} z_k - \int_\xi^\infty \kappa(s)ds\right] + \mathcal{O}\left(z^{-2}\right), \tag{C.3}$$

and in property e.

$$\begin{aligned} &\left|T(z,\xi) - T_0(\xi)(\xi-z)^{-i\kappa(\xi)}\right| \leq C\|r\|_{H^1(\mathbb{R})}|z-\xi|^{1/2} \\ &T_0(\xi) = \prod_{k\in\Delta_\xi^+}\left(\frac{\xi-z_k^*}{\xi-z_k}\right)\exp\left(-i\int_\xi^\infty \log(s-\xi)d_s\kappa(s)\right) \end{aligned} \tag{C.4}$$

where χ is the characteristic function of $[\xi, \xi+1]$. With these changes to T, $M^{(1)}$ satisfies RHP 3.1 with the intervals $(-\infty,\xi)$ and (ξ,∞) in (3.10) and index sets Δ_ξ^-, Δ_ξ^+ in (3.11) interchanged.

Next, non-analytic extensions of the jump matrices (3.10) are introduced to deform jump matrices onto contours along which they decay to identity as was done in Section 4. Define contours Σ_j' $j = 1,\dots,4$ and regions Ω_j', $j = 1,\dots,6$ as in Fig. C.1. One then proves analogously to Lemma 4.1, that there exist functions $R_j : \Omega_j' \to \mathbb{C}$, $j = 1,3,4,6$, satisfying[3]

$$R_1(z) = \begin{cases} r(z)T(z)^{-2} & z\in(-\infty,\xi) \\ r(\xi)T_0(\xi)^{-2}(\xi-z)^{2i\kappa(\xi)}(1-\chi_{\mathcal{Z}}(z)) & z\in\Sigma_1' \end{cases} \tag{C.5a}$$

$$R_3(z) = \begin{cases} \dfrac{r(z)^*}{1+|r(z)|^2}T_+(z)^2 & z\in(\xi,\infty) \\ \dfrac{r(\xi)^*}{1+|r(\xi)|^2}T_0(\xi)^2(\xi-z)^{-2i\kappa(\xi)}(1-\chi_{\mathcal{Z}}(z)) & z\in\Sigma_2' \end{cases} \tag{C.5b}$$

$$R_4(z) = \begin{cases} \dfrac{r(z)}{1+|r(z)|^2}T_-(z)^{-2} & z\in(\xi,\infty) \\ \dfrac{r(\xi)}{1+|r(\xi)|^2}T_0(\xi)^{-2}(\xi-z)^{2i\kappa(\xi)}(1-\chi_{\mathcal{Z}}(z)) & z\in\Sigma_3' \end{cases} \tag{C.5c}$$

$$R_6(z) = \begin{cases} r(z)^*T(z)^2 & z\in(-\infty,\xi) \\ r(\xi)^*T_0(\xi)^2(\xi-z)^{-2i\kappa(\xi)}(1-\chi_{\mathcal{Z}}(z)) & z\in\Sigma_4' \end{cases} \tag{C.5d}$$

[3] Note that the differences in sign between (C.4) and (3.5) have been incorporated into (C.5) compared to (4.5).

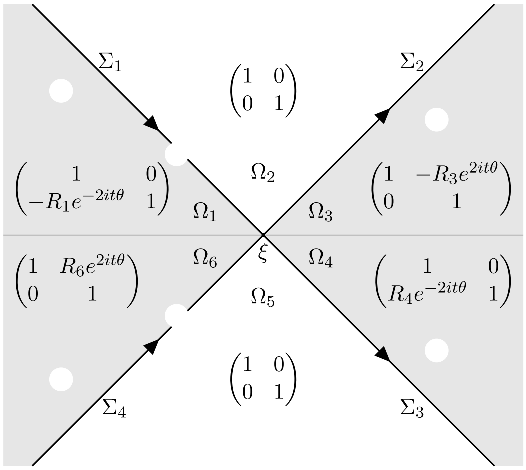

Fig. C.1. The contours $\Sigma^{(2)} = \bigcup_{k=1}^{4} \Sigma_k$ and regions Ω_k, $k = 1, \dots, 6$, used to define the transformation $M^{(2)} = M^{(1)}\mathcal{R}$ for $t < 0$. The non-analytic matrix $\mathcal{R}^{(2)}$ is shown for each region Ω_k. The support of the $\overline{\partial}$-derivative $W^{(2)} = \overline{\partial}\mathcal{R}^{(2)}$ is shaded in gray.

satisfying the bounds in (4.6). Once the functions in (C.5) are constructed, the transformation

$$M^{(2)}(z) = M^{(1)}(z)\mathcal{R}(z), \tag{C.6}$$

where $\mathcal{R}$ is defined in each sector Ω_j in Fig. C.1, defines a new unknown $M^{(2)}$ which satisfies

$\overline{\partial}$-Riemann–Hilbert Problem C.1. *Find a function $M^{(2)} : \mathbb{C}\backslash(\Sigma^{(2)} \cup \mathcal{Z} \cup \mathcal{Z}^*) \to SL_2(\mathbb{C})$ with the following properties.*

1. *$M^{(2)}$ is continuous with sectionally continuous first partial derivatives in $\mathbb{C} \setminus (\Sigma^{(2)} \cup \mathcal{Z} \cup \mathcal{Z}^*)$.*
2. *$M^{(2)}(z) = I + \mathcal{O}\left(z^{-1}\right)$ as $z \to \infty$.*
3. *For $z \in \Sigma^{(2)}$, the boundary values satisfy the jump relation $M_+^{(2)}(z) = M_-^{(2)}(z)V^{(2)}(z)$, where*

$$V^{(2)}(z) = I + (1 - \chi_{\mathcal{Z}}(z))\delta V^{(2)},$$
$$\delta V^{(2)}(z) = \begin{cases} \begin{pmatrix} 0 & 0 \\ r(\xi)T_0(\xi)^{-2}(\xi - z)^{2i\kappa(\xi)}e^{2it\theta} & 0 \end{pmatrix} & z \in \Sigma_1 \\ \begin{pmatrix} 0 & \frac{r(\xi)^* T_0(\xi)^2}{1+|r(\xi)|^2}(\xi - z)^{-2i\kappa(\xi)}e^{-2it\theta} \\ 0 & 0 \end{pmatrix} & z \in \Sigma_2 \\ \begin{pmatrix} 0 & 0 \\ \frac{r(\xi)T_0^{-2}(\xi)}{1+|r(\xi)|^2}(\xi - z)^{2i\kappa(\xi)}e^{2it\theta} & 0 \end{pmatrix} & z \in \Sigma_3 \\ \begin{pmatrix} 0 & r(\xi)^* T_0(\xi)^2(\xi - z)^{-2i\kappa(\xi)}e^{-2it\theta} \\ 0 & 0 \end{pmatrix} & z \in \Sigma_4 \end{cases} \tag{C.7}$$

4. *For $\mathbb{C} \setminus (\Sigma^{(2)} \cup \mathcal{Z} \cup \mathcal{Z}^*)$ we have $\overline{\partial}M^{(2)}(z) = M^{(2)}(z)\overline{\partial}\mathcal{R}^{(2)}(z)$ where $\mathcal{R}^{(2)}$ is defined in each Ω_k as shown in Fig. C.1.*
5. *$M^{(2)}(z)$ has simple poles at each $z_k \in \mathcal{Z}$ and $z_k^* \in \mathcal{Z}^*$ at which*

$$\operatorname*{Res}_{z_k} M^{(2)} = \begin{cases} \lim\limits_{z\to z_k} M^{(2)} \begin{pmatrix} 0 & 0 \\ c_k T(z_k)^{-2} e^{2it\theta} & 0 \end{pmatrix} & k \in \Delta_\xi^- \\ \lim\limits_{z\to z_k} M^{(2)} \begin{pmatrix} 0 & c_k^{-1}(1/T)'(z_k)^{-2} e^{-2it\theta} \\ 0 & 0 \end{pmatrix} & k \in \Delta_\xi^+ \end{cases}$$
$$\operatorname*{Res}_{z_k^*} M^{(2)} = \begin{cases} \lim\limits_{z\to z_k^*} M^{(2)} \begin{pmatrix} 0 & -c_k^* T(z_k^*)^{2} e^{-2it\theta} \\ 0 & 0 \end{pmatrix} & k \in \Delta_\xi^- \\ \lim\limits_{z\to z_k^*} M^{(2)} \begin{pmatrix} 0 & 0 \\ -(c_k^*)^{-1} T'(z_k^*)^{-2} e^{2it\theta} & 0 \end{pmatrix} & k \in \Delta_\xi^+ \end{cases} \tag{C.8}$$

The final steps of the analysis, mimicking Sections 5–6 are to first construct a solution $M^{(2)}_{\text{RHP}}$ of the Riemann–Hilbert components of RHP C.1, and then to use the solid Cauchy integral operator to prove that the remainder $M^{(3)} = M^{(2)}(M^{(2)}_{\text{RHP}})^{-1}$ is uniformly near identity with estimates identical to Proposition 6.2. The model $M^{(2)}_{\text{RHP}}$ again takes the form (5.8). Using the results of Appendix B, as $t \to -\infty$ with[4] $(x,t) \in C(x_1, x_2, v_1, v_2)$ the outer model takes the form

$$M^{(\text{out})}(z; x, t) = \left[I + \mathcal{O}\left(e^{-4\rho|t|}\right)\right] m^{\Delta_\xi^+(\mathcal{I})}(z; x, t \mid \sigma_d^-(\mathcal{I})) \tag{C.9}$$

with $\sigma_d^-(\mathcal{I})$ defined by (1.11).

The local model $M^{(\xi)}$ is constructed as in Section 5.1.2. Define

$$\zeta = \zeta(z) = 2\sqrt{-t}(z-\xi) \quad \Rightarrow \quad 2t\theta = -\zeta^2/2 - 2t\xi^2 \tag{C.10}$$
$$\widehat{r}_\xi := r(\xi)^* T_0(\xi)^2 e^{2i(t\xi^2 + \kappa(\xi)\log(2\sqrt{-t}))} \tag{C.11}$$

Then the local model $M^{(\xi)}$ is given by

$$M^{(\xi)}(z) = M^{(\text{out})}(z)\sigma_2 M^{(\text{PC})}(-\zeta(z), \widehat{r}_\xi)\sigma_2, \tag{C.12}$$

where $M^{(\text{PC})}(\zeta, r)$ is the solution of RHP A.1.

The residual error $E(z)$ now satisfies RHP 5.2 but with (5.18) now given by

$$V^{(E)}(z) = \begin{cases} M^{(\text{out})}(z) V^{(2)}(z) M^{(\text{out})}(z)^{-1} & z \in \Sigma^{(2)} \backslash \mathcal{U}_\xi \\ M^{(\text{out})}(z)\sigma_2 M^{(\text{PC})}(-\zeta(z), \widehat{r}_\xi)\sigma_2 M^{(\text{out})}(z)^{-1} & z \in \partial\mathcal{U}_\xi . \end{cases} \tag{C.13}$$

small-norm theory again can be used to show that E exists and satisfies $E(z) = I + z^{-1}E_1 + \mathcal{O}\left(z^{-2}\right)$ where

$$E_1 = \frac{1}{2i\sqrt{-t}} M^{(\text{out})}(\xi; x, t) \begin{pmatrix} 0 & -\beta_{21}(\widehat{r}_\xi) \\ \beta_{12}(\widehat{r}_\xi) & 0 \end{pmatrix} M^{(\text{out})}(\xi; x, t)^{-1} + \mathcal{O}\left(t^{-1}\right), \tag{C.14}$$

and using (A.6) and (C.11) we have

$$-\beta_{21}(\widehat{r}_\xi) = -\beta_{12}(\widehat{r}_\xi)^* = \alpha(\xi, -) e^{ix^2/(2t) + i\kappa(\xi)\log|4t|} \tag{C.15}$$

where

$$|\alpha(\xi, -)|^2 = |\kappa(\xi)| \tag{C.16}$$

$$\arg\alpha(\xi, -) = -\frac{\pi}{4} - \arg\Gamma(i\kappa(\xi)) - \arg r(\xi) - 4\sum_{k\in\Delta_\xi^+} \arg(\xi - z_k) - 2\int_\xi^\infty \log|\xi - s| \mathrm{d}_s\kappa(s) \tag{C.17}$$

[4] Here, we are reusing the notation of Theorem 1.1.

Appendix D. Details of calculations for the $\overline{\partial}$ problem

Proposition D.1. *There exist constants c_1, c_2, and c_3 such that for all $t > 0$, the integrals I_j, $j = 1, 2, 3$, defined by (6.7)–(6.8) satisfy the bounds*

$$|I_j| \le \frac{c_j}{t^{1/4}}, \qquad j = 1, 2, 3. \tag{D.1}$$

Proof. Our proof follows that found in [14]. Let $s = u + iv$ and $z = \alpha + i\beta$. Throughout we use the elementary fact $\left\|\frac{1}{s-z}\right\|^2_{L^2_u(v+\xi,\infty)} = \left(\int_{v+\xi}^{\infty} \frac{1}{(u-\alpha)^2+(v-\beta)^2}du\right)^{1/2} \le \int_{\mathbb{R}} \frac{1}{u^2+(v-\beta)^2}du = \frac{\pi}{|v-\beta|}$, to show that

$$\begin{aligned} |I_1| &\le \int_0^\infty \int_{v+\xi}^\infty \frac{|\chi_{\mathcal{Z}}(s)|}{|s-z|} e^{-4tv(u-\xi)} du\, dv \le \int_0^\infty e^{-4tv^2} \int_{v+\xi}^\infty \frac{|\chi_{\mathcal{Z}}(s)|}{|s-z|} du\, dv \\ &\le \int_0^\infty e^{-4tv^2} \|\chi_{\mathcal{Z}}(s)\|_{L^2_u(v+\xi,\infty)} \cdot \left\|\frac{1}{s-z}\right\|_{L^2_u(v+\xi,\infty)} dv \\ &\le c_1 \int_0^\infty \frac{e^{-4tv^2}}{|v-\beta|^{1/2}} dv \le c_1 t^{-1/4} \int_{\mathbb{R}} \frac{e^{-4(w+\sqrt{t}\beta)^2}}{|w|^{1/2}} \le c_1 t^{-1/4} \int_{\mathbb{R}} \frac{e^{-4w^2}}{|w|^{1/2}} \le c_1 t^{-1/4}. \end{aligned} \tag{D.2}$$

The bound for I_2 is similar to I_1. Recalling that $r \in H^{1,1}(\mathbb{R})$,

$$|I_2| \le \int_0^\infty e^{-4tv^2} \int_{v+\xi}^\infty \frac{|r'(u)|}{|s-z|} du\, dv \le \|r'(u)\|_{L^2(\mathbb{R})} \int_0^\infty e^{-4tv^2} \left\|\frac{1}{s-z}\right\|_{L^2_u(v+\xi,\infty)} dv \le \frac{c_2}{t^{1/4}}. \tag{D.3}$$

For I_3 choose $p > 2$ and q Hölder conjugate to p, then

$$\begin{aligned} |I_3| &\le \int_0^\infty e^{-4tv^2} \left\|(s-\xi)^{-1/2}\right\|_{L^p_u(v+\xi,\infty)} \left\|(s-z)^{-1}\right\|_{L^q_u(v+\xi,\infty)} dv \\ &\le c_p \int_0^\infty e^{-4tv^2} v^{1/p-1/2} |v-\beta|^{1/q-1} dv \end{aligned} \tag{D.4}$$

To bound this last integral observe that

$$\int_0^\beta e^{-tv^2} v^{1/p-1/2} (\beta - v)^{1/q-1} dv = \int_0^1 \beta^{1/2} e^{-t\beta^2 w^2} w^{1/p-1/2} (1-w)^{1/q-1} dw \tag{D.5}$$

$$\le ct^{-1/4} \int_0^1 w^{1/p-1} (1-w)^{1/q-1} dw \le Ct^{-1/4}, \tag{D.6}$$

where we've used the bound $e^{-m} \le m^{-1/4}$ for $m \ge 0$ to replace the exponential factor in the second integral. Finally

$$\int_\beta^\infty e^{-tv^2} v^{1/p-1/2} (v-\beta)^{1/q-1} dv \le \int_0^\infty e^{-tw^2} w^{-1/2} dw \le Ct^{-1/4}. \tag{D.7}$$

The result is confirmed. $\square$

Proposition D.2. *For all $t > 0$ there exists a constant c such that*

$$|M_1^{(3)}| \le ct^{-3/4}. \tag{D.8}$$

Proof. The proof given here follows calculations that can be found in [14]. Recalling that the set $\Omega_1^\sharp = \Omega_1 \cap \operatorname{supp}(1-\chi_{\mathcal{Z}})$ is bounded away from the poles of $M^{(2)}_{\mathrm{RHP}}$, we have

$$\begin{aligned}
|M_1^{(3)}| &\leq \frac{1}{\pi}\iint_{\Omega_1} |M^{(3)}(s)M^{(2)}_{\mathrm{RHP}}(s)W^{(2)}(s)M^{(2)}_{\mathrm{RHP}}(s)^{-1}|dA \\
&\leq \frac{1}{\pi}\|M^{(3)}\|_{L^\infty(\Omega)}\|M^{(2)}_{\mathrm{RHP}}\|_{L^\infty(\Omega^\sharp)}\|(M^{(2)}_{\mathrm{RHP}})^{-1}\|_{L^\infty(\Omega^\sharp)}\iint_{\Omega_1}|\bar\partial R_1 e^{2it\theta}|dA \\
&\leq C\left(\iint_{\Omega_1}|\chi_{\mathcal{Z}}(s)|e^{-4tv(u-\xi)}dA + \iint_{\Omega_1}|r'(u)|e^{-4tv(u-\xi)}dA + \iint_{\Omega_1}\frac{1}{|s-\xi|^{1/2}}e^{-4tv(u-\xi)}dA\right) \\
&\leq C(I_4+I_5+I_6).
\end{aligned} \tag{D.9}$$

We bound I_4 by applying the Cauchy–Schwarz inequality:

$$\begin{aligned}
|I_4| &\leq \int_0^\infty \|\chi_{\mathcal{Z}}\|_{L^2_u(v+\xi,\infty)}\left(\int_v^\infty e^{-8tuv}du\right)^{1/2}dv \\
&\leq ct^{-1/2}\int_0^\infty \frac{e^{-4tv^2}}{\sqrt{v}}dv \leq ct^{-3/4}\int_0^\infty \frac{e^{-4w^2}}{\sqrt{w}}dw \leq ct^{-3/4}.
\end{aligned} \tag{D.10}$$

The bound for I_5 follows in the same manner as for I_4. For I_6 we proceed as with I_3 applying Hölder's inequality with $2<p<4$

$$\begin{aligned}
|I_6| &\leq c\int_0^\infty v^{1/p-1/2}\left(\int_v^\infty e^{-4qtuv}du\right)^{1/q}dv \\
&\leq ct^{-1/q}\int_0^\infty v^{2/p-3/2}e^{-4tv^2}dv \leq ct^{-3/4}\int_0^\infty w^{2/p-3/2}e^{-4w^2}dw \leq ct^{-3/4},
\end{aligned} \tag{D.11}$$

where we have used the substitution $w=t^{1/2}v$ and the fact that $-1<\frac{2}{p}-\frac{3}{2}<-\frac{1}{2}$. □

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