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Hessian surfaces and local Lagrangian embeddings

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Abstract

In this paper we prove that any smooth surfaces can be locally isometrically embedded into $\mathbb{C}^2$ as Lagrangian surfaces. As a byproduct we obtain that any smooth surfaces are Hessian surfaces.

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1. Introduction

A *Hessian metric* g is a Riemannian metric g with the property that locally g can be written as the Hessian of some convex potential function φ. A Riemannian manifold (M, g) is called Hessian if g is Hessian.

Hessian metrics play an important role in a variety of applications. They arise in the study of affine differential geometry [11,17] and special Kähler manifolds [13,15], and also in many applied sciences, for example, in optimization [19], statistical manifolds [2] and information geometry [3,2].

A duality plays an important role in the study of the Hessian metric. Given any affine connection D on a Riemannian manifold (M, g), we can define a *g-dual connection* D^* by

$$g(D_X Y, Z) = g(Y, D^*_X Z).$$

It is clear that the Levi-Civita connection ∇ of g is self-dual. A *g-dually flat structure* is a pair of g-dual connections which are both flat. It is not difficult to check that a metric locally admits a g-dually flat structure if and only if it is Hessian.

A natural and fundamental question for the Hessian structure is to ask when a Riemannian manifold is a Hessian manifold. This question was raised in [12,2] in the notion of g-dually flat connections. For higher dimension $n \geq 3$,

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this question has a negative answer in general. See the work of Amari and Armstrong [2] and Bryant [4]. In fact, when $n \geq 3$, the system of partial differential equations that would need to be solved is overdetermined. When $n = 2$, all analytic surfaces are Hessian, which was also proved by Amari and Armstrong [2] and Bryant [4] independently by using the Cartan–Kähler theory. In this paper, we will prove the following result.

Theorem 1. *All smooth surfaces are Hessian.*

By Theorem 1 and the result due to Amari and Armstrong, and Bryant, the Hessian manifolds can also be seen as a natural higher dimensional generalization of surfaces, such as locally conformally flat manifolds and Kähler manifolds.

Theorem 1 follows from our study of local embedding theorem, which has its own interest.

Theorem 2. *All smooth surfaces can be locally embedded in $\mathbb{C}^2$ isometrically as Lagrangian surfaces.*

An immersed submanifold $i : M^n \subset \mathbb{C}^n$ is *Lagrangian* if the complex structure J maps the tangent space T_p at an arbitrary point $p \in M$ isometrically onto the corresponding normal space N_M. Equivalently, it satisfies

$$i^*(\omega) = 0,$$

where ω is the canonical symplectic form of $\mathbb{C}^n$.

Lagrangian submanifolds are important geometric objects and have been intensively studied in symplectic geometry. See for example [22], [5] and [7]. Moore and Morvan [18] studied whether a Riemannian manifold (M^n, g) can be embedded isometrically as a Lagrangian submanifold. See also an earlier work in [8]. With a general negative answer to higher dimensional case, Moore and Morvan used the Cartan–Kähler theory to prove that any analytic surfaces can be locally embedded in $\mathbb{C}^2$ isometrically as Lagrangian surfaces. This is a Lagrangian version of the Cartan–Janet Theorem for local embedding of analytic surfaces into $\mathbb{R}^3$. The latter states that any analytic surfaces can be locally embedded in $\mathbb{R}^3$ isometrically [16]. However, this is not known for smooth surfaces, namely whether all smooth surfaces can be locally embedded in $\mathbb{R}^3$ isometrically. There have been many interesting results concerning local embedding of surfaces in $\mathbb{R}^3$. See [14] and references therein. Nevertheless, Poznyak [20] proved that any smooth surfaces can be locally isometrically embedded in $\mathbb{R}^4$. Theorem 2 shows that any smooth surfaces can be locally embedded in $\mathbb{C}^2 = \mathbb{R}^4$ isometrically as a Lagrangian surface. Hence it is a refinement of the classical result of Poznyak [20].

The paper is arranged as follows. In Section 2, we recall the basic properties of the Hessian metric and the Lagrangian submanifold. In Section 3, we derive a partial differential system for Lagrangian surfaces and reduce the proof of Theorems 1 and 2 to Theorem 3. In Section 4, we prove Theorem 3 by solving a strictly hyperbolic system.

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2. Hessian structures and Lagrangian submanifolds

In this section, we will review the definition and properties of Hessian metrics and the Lagrangian submanifolds in $\mathbb{C}^n$, as well as their basic properties.

A Riemannian metric g on M is called a *Hessian metric* (or *affine Kähler*) if around every point $p \in M$ there is a neighborhood U and a flat connection D over U such that in the corresponding affine coordinates with respects to D the metric g can be written as the Hessian of some convex potential function φ, i.e.,

$$g = Dd\varphi.$$

In this paper we follow [1] for the definition of Hessian metric. This is slightly different from the definition in [21], where a flat connection D is required globally. If one uses the definition of Hessian metrics as in [21], then Theorem 1 imply that any 2-dimensional smooth surfaces is locally Hessian. The Hessian metric is also called affine Kähler metric by Cheng and Yau [11], since it is similar to the Kähler metric.

Lemma 1. *Let (M, g) be a Riemannian manifold. Then, g is a Hessian metric if and only if for every point $p \in M$ there is a neighborhood U and a 3-tensor $T \in \Gamma(TU \otimes TU \otimes TU)$ satisfying the following conditions:*

(1) *T is totally symmetric,*
(2) $\nabla_k T_{ijl} - \nabla_l T_{ijk} = 0$*, where* ∇ *is the Levi-Civita connection with respect to* g*,*
(3) $R_{ijkl} = -(T_{iks}T^s{}_{jl} - T_{ils}T^s{}_{jk})$*, where* R *is the Riemann curvature of* g *and* $T^s{}_{ij} = g^{sk}T_{kij}$*.*

Proof. This is a known result, which can be found in [21] and [1]. For the convenience of the reader and the completeness of the paper, we provide a proof.

"$\Leftarrow$" Using T we first define a local linear connection over U by

$$D_X Y = \nabla_X Y + \widetilde{T}(X, Y), \tag{2.1}$$

where $\widetilde{T} : TU \times TU \to TU$ is determined uniquely by

$$g(\widetilde{T}(X, Y), Z) = T(X, Y, Z),$$

or locally, $\widetilde{T}^k_{ij} = g^{kl}T_{lij} = T^k_{ij}$. With the assumptions given in Lemma 1, we can show that D is a flat connection. In fact, the curvature R^D can be computed locally by

$$R^D_{ijkl} = R_{ijkl} + (\nabla_i T_{jkl} - \nabla_j T_{ikl}) + \sum_s (T_{iks}T^s_{jl} - T_{ils}T^s_{jk}), \tag{2.2}$$

which vanishes due to (2) and (3). From the flatness of D we can find an affine coordinate system $\{x^1, x^2, \cdots, x^n\}$ on a (possible smaller) neighborhood $\widetilde{U}$ of $p \in M$ satisfying

$$D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = 0 \quad \text{for any } i, j = 1, \cdots, n. \tag{2.3}$$

From (2.1) and (2.3), we have

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = T^k_{ij} \frac{\partial}{\partial x^k}, \tag{2.4}$$

which means that T^k_{ij} equals to the Christoffel symbols of the Levi-Civita connection ∇. Assumptions (1) and (2) imply

$$\frac{\partial}{\partial x^k} T_{ijl} - \frac{\partial}{\partial x^l} T_{ijk} = 0. \tag{2.5}$$

We may assume that $\widetilde{U}$ is simply connected. From (2.5) we can use Poincaré's Lemma to get a function u_{ij} on $\widetilde{U}$ satisfying

$$T_{ijk} = \frac{\partial}{\partial x^k} u_{ij}.$$

Using the symmetry of T_{ijk} in i, j, we can choose u_{ij} such that $u_{ij} = u_{ji}$. The symmetry of T implies that $\frac{\partial}{\partial x^k} u_{ij} = \frac{\partial}{\partial x^j} u_{ik}$, which, in turn, implies that there is a function u_i on $\widetilde{U}$ such that $u_{ij} = \frac{\partial}{\partial x^j} u_i$. Since u_{ij}, and then $\frac{\partial}{\partial x^j} u_i$, are symmetric, there is function u on $\widetilde{U}$ such that $u_i = \frac{\partial}{\partial x^i} u$. With T being the Christoffel symbols of ∇, we can show

$$\frac{\partial g_{ij}}{\partial x^k} = 2T_{ijk}.$$

Now it is easy to check

$$g_{ij} = 2\frac{\partial^2 u}{\partial x^i \partial x^j}.$$

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Hence g is a Hessian metric.

"⇒". If g is Hessian, for any point p there is a neighborhood U and a flat connection D over U such that in the corresponding affine coordinates with respect to D the metric g can be written as the Hessian of some convex potential function φ, i.e.,

$$g = Dd\varphi.$$

Let $u = \frac{1}{2}\varphi$. Define $T \in \Gamma(TU \otimes TU \otimes TU)$ by the difference operator $\widetilde{T} : \Gamma(TU \otimes TU) \to \Gamma(TU)$ between the flat connection D and the Levi-Civita connection ∇ of g

$$\widetilde{T}(X, Y) = D_X Y - \nabla_X Y.$$

It is easy to check that

$$T_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial x^k} = \frac{\partial^3 u}{\partial x^i \partial x^j \partial x^k},$$

which is total symmetry. From (2.2) and the flatness of D we have

$$R_{ijkl} + (\nabla_i T_{jkl} - \nabla_j T_{ikl}) + \sum_s (T_{iks} T^s_{jl} - T_{ils} T^s_{jk}) = 0. \tag{2.6}$$

Since the first and the third term in (2.6) are antisymmetric in k and l, the second term should be also antisymmetric in k and l. This forces the second term vanishes, for it is symmetric in k and l. Therefore, (2), and then (3) hold. □

A Riemannian manifold (M, g) can be embedded into $\mathbb{C}^n$ as a Lagrangian submanifold, if there is an isometric embedding $i : M \to \mathbb{C}^n$ such that the pull-back of the canonical symplectic form ω vanishes, i.e., $i^*(\omega) = 0$. The canonical symplectic form ω is

$$\omega = \sum_{i=1}^n dx^i \wedge dy^i.$$

There are several equivalent definitions of Lagrangian submanifolds. See for instance [9]. There are many works concerning Lagrangian submanifolds in symplectic topology. Here we are interested in its differential-geometric aspect. Using the embedding i we can identifying the tangential space T_pM as a subspace of $\mathbb{C}^n$. Let N_pM be the normal space and J be the canonical complex structure of $\mathbb{C}^n$. The property of being Lagrangian is equivalent to $JT_pM = N_pM$ for all $p \in M$. Let $H : TM \otimes TM \to NM$ be the second fundamental form of the embedding; namely,

$$H(X, Y) = (\nabla_X Y)^\perp,$$

where $^\perp : \mathbb{C}^n \to N_pM$ is the orthonormal projection. From H, we define a 3-tensor $h \in \Gamma(TM \otimes TM \otimes TM)$ by

$$h(X, Y, Z) = \langle H(X, Y), JZ \rangle. \tag{2.7}$$

Lagrangian submanifolds have the following crucial properties.

Lemma 2. *Let (M, g) be a Riemannian manifold, which is embedded in $\mathbb{C}^n$ isometrically as a Lagrangian submanifold, and h be the 3-tensor defined by (2.7). Then,*

(1) *h is total symmetric,*
(2) *$\nabla_k h_{ijl} - \nabla_l h_{ijk} = 0$, where ∇ is the Levi-Civita connection with respect to g,*
(3) *$R_{ijkl} = h_{iks} h^s{}_{jl} - h_{ils} h^s{}_{jk}$, where R is the Riemann curvature of g.*

This result is also well-known. We will provide a proof in Section 3 below, for the convenience of the reader.

It is clear that the conditions (1) and (2) in Lemma 2 are the same as those in Lemma 1, while (3) has a difference sign. Hence we can study these two problems, the existence of Hessian metrics and Lagrangian local embedding problem, with the same approach. In fact, we will prove the following result in the next two sections.

Theorem 3. *Let (Σ, g) be a given 2-dimensional surface with a Riemannian metric g. For $\varepsilon \in \{-1, 1\}$ and any point $p \in \Sigma$ there exists a neighborhood U and a 3-tensor $T \in \Gamma(TU \otimes TU \otimes TU)$ satisfying the following conditions:*

(1) *T is total symmetric,*
(2) $\nabla_k T_{ijl} - \nabla_l T_{ijk} = 0$,
(3) $R_{ijkl} = \varepsilon(T_{iks}T^s{}_{jl} - T_{ils}T^s{}_{jk})$.

With Theorem 3 we are ready to prove the main results in this paper.

Proof of Theorem 1 and Theorem 2. From Theorem 3, we know that locally we have a 3-tensor T satisfying conditions (1)–(3) in Lemma 1. Then by Lemma 1 we know that g is Hessian.

From Theorem 3, we know that locally we have a 3-tensor h satisfying conditions (1)–(3) in Lemma 2. With a Frenet type theorem for Lagrangian surfaces, Theorem A in [10], we can find an Lagrangian embedding with the second fundamental form $h = T$. □

To end this Section, we remark that for any Hessian manifold (M, g) its tangential bundle TM with a canonical metric g^T is Kähler and (M, g) can be seen as a Lagrangian submanifold in TM. See Proposition 2.6 in [21]. By Weinstein's tubular neighborhood theorem, any Lagrangian submanifold in a symplectic manifold can be locally seen as a Lagrangian submanifold in $\mathbb{C}^n$ *symplectically*.

3. A partial differential system for Lagrangian surfaces

In this Section we first prove Lemma 2 in the case $n = 2$. Then we rewrite conditions (1)–(3) as a Lagrangian Gauss–Codazzi system, which will be solved in the next Section.

Let M^2 be a surface. Let $p \in M^2$ a point and U a local coordinate system of M around p. The metric of the surface M^2 is

$$ds^2 = Edu_1^2 + 2Fdu_1du_2 + Gdu_2^2, \quad u = (u_1, u_2) \in U$$

locally. Then, M^2 is locally isometrically embedded in $\mathbb{C}^2 = \mathbb{R}^4$ if around any point $p \in M$ there is an isometric embedding $i = (x, y, z, w)$ into $\mathbb{C}^2 = \mathbb{R}^4$ as a Lagrangian satisfying the following Lagrangian property

$$i^*(\omega) = 0, \tag{3.1}$$

where ω is the standard sympletic form on $\mathbb{R}^4 = \mathbb{C}^2$, i.e.,

$$\omega = dx \wedge dy + dz \wedge dw.$$

The equation for i being an isometric embedding is

$$Edu_1^2 + 2Fdu_1du_2 + Gdu_2^2 = dx^2 + dy^2 + dz^2 + dw^2. \tag{3.2}$$

Equation (3.1) can be rewritten as

$$\det\begin{pmatrix} x_{u_1} & x_{u_2} \\ z_{u_1} & z_{u_2} \end{pmatrix} + \det\begin{pmatrix} y_{u_1} & y_{u_2} \\ w_{u_1} & w_{u_2} \end{pmatrix} = 0. \tag{3.3}$$

Now let $\mathbf{r} : U \to \mathbb{R}^4$ denote the embedding i and $\mathbf{r}_1 = \frac{\partial \mathbf{r}}{\partial u_1}$ and $\mathbf{r}_2 = \frac{\partial \mathbf{r}}{\partial u_2}$. Set

$$\mathbf{n}_1 = J\mathbf{r}_1, \quad \mathbf{n}_2 = J\mathbf{r}_2.$$

If $\mathbf{r}$ is a Lagrangian embedding, then $\mathbf{n}_1$ and $\mathbf{n}_2$ are normal vector fields. The first fundamental form I of $\mathbf{r}$ is

$$I = d\mathbf{r} \cdot d\mathbf{r} = g_{ij}du_idu_j,$$

and the second fundamental form is

$$II = h_{ijk}du_idu_jdu_k,$$

where $h_{ijk} = \mathbf{n}_i \cdot \mathbf{r}_{jk}$. Let (g^{ij}) be the inverse of (g_{ij}) and set $g = g_{11}g_{22} - g_{12}^2$. It is clear that

$$\begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \frac{1}{g}\begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{pmatrix}.$$

The Christoffel symbol is

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl}(g_{il,j} + g_{lj,i} - g_{ij,l}),$$

and the curvature tensor is

$$R_{ijkl} = g_{lm}(\Gamma^m_{ij,k} - \Gamma^m_{ik,j} + \Gamma^n_{ij}\Gamma^m_{nk} - \Gamma^n_{ik}\Gamma^m_{nj}).$$

Since $\{\mathbf{r}_1, \mathbf{r}_2\}$ is a basis of the tangential bundle and $\{\mathbf{n}_1, \mathbf{n}_2\}$ a basis of the normal bundle, we can express $\mathbf{r}_{jk}$ as

$$\mathbf{r}_{jk} = \Gamma^l_{jk}\mathbf{r}_l + h^l_{jk}\mathbf{n}_l. \tag{3.4}$$

It is trivial to see that Γ^k_{ij} and $h^k_{ij} := g^{kl} h_{lij}$ are symmetric in i, j. It is also easy to check

$$\mathbf{n}_{ij} := \frac{\partial \mathbf{n}_i}{\partial u_j} = -h^l_{ij}\mathbf{r}_k + \Gamma^k_{ij}\mathbf{n}_k,$$

the Weingarten formula. For Lagrangian $\mathbf{r}$, it is well-known that $h_{ijk} = h_{ikj}$. This follows from

$$\begin{aligned} \mathbf{n}_{ij} \cdot \mathbf{r}_k &= -\mathbf{n}_i \cdot \mathbf{r}_{kj} = -\mathbf{n}_i \cdot \mathbf{r}_{jk} \\ &= \mathbf{r}_i \cdot J\mathbf{r}_{jk} = -\mathbf{r}_{ik} \cdot Jr_j = \mathbf{r}_k \cdot \mathbf{n}_{ji}. \end{aligned}$$

It follows, with the symmetry of h^k_{ij} in i, j, that

$$h_{ijk} = h_{jik} = h_{ikj}, \tag{3.5}$$

i.e., h_{ijk} is totally symmetric.

Define a 1-form $\alpha = H_i dx^i$, where $H_i = g^{jk} h_{ijk}$. We can show that α is a close form, i.e.,

$$\nabla_i H_j = \nabla_j H_i,$$

which is obtained from the Codazzi equation

$$\nabla_i h_{jkl} = \nabla_j h_{ikl}. \tag{3.6}$$

Here $\nabla_i h_{jkl}$ is the covariant derivative defined by

$$\nabla_i h_{jkl} = \frac{\partial}{\partial x^i} h_{jkl} - \Gamma^m_{ij} h_{mkl} - \Gamma^m_{ik} h_{jml} - \Gamma^m_{il} h_{jkm}.$$

Another important relation is the Gauss equation

$$R_{ijkl} = g^{nm} h_{ikn} h_{mjl} - g^{nm} h_{iln} h_{mjk}. \tag{3.7}$$

These two equations can be proved by a computation as follows. Differentiating (3.4), we get

$$\begin{aligned} \mathbf{r}_{ijk} &= \Gamma^l_{ij,k}\mathbf{r}_l + \Gamma^l_{ij}\mathbf{r}_{lk} + h^l_{ij,k}\mathbf{n}_l + h^l_{ij}\mathbf{n}_{lk} \\ &= (\Gamma^l_{ij,k} + \Gamma^m_{ij}\Gamma^l_{mk} - g^{ml} h^n_{ij} h_{nkm})\mathbf{r}_l \\ &\quad + (\Gamma^m_{ij} h^l_{mk} + h^l_{ij,k} + h^n_{ij}\Gamma^l_{nk})\mathbf{n}_l. \end{aligned}$$

Since $\mathbf{r}_{ijk} = \mathbf{r}_{ikj}$, we have (3.7) and (3.6). This provides a proof of Lemma 2, in the case $n = 2$. For the general dimension, it is completely the same.

Now we rewrite the Gauss equation and the Codazzi equation. Remember that locally we have a metric

$$g = E du_1^2 + 2F du_1 du_2 + G du_2^2.$$

Let K be its Gaussian curvature, which is determined by g, due to Gauss' Theorema Egregium. Set

$$L = h_{111}, M = h_{112}, N = h_{122}, O = h_{222}.$$

By the symmetry of h_{ijk}, other $h_{ijk}'s$ are determined by L, M, N, O. The second fundamental form is given by

$$II = Ldu_1^3 + 3Mdu_1^2du_2 + 3Ndu_1du_2^2 + Odu_2^3.$$

Now we write the Codazzi equation (3.6) and the Gauss equation (3.7) in terms of L, M, N and O as follows. A direct computation yields

$$\begin{aligned}
\nabla_2 h_{111} &= L_y - 3\Gamma_{21}^l h_{l11} \\
&= L_y - 3\Gamma_{21}^1 L - 3\Gamma_{21}^2 M, \\
\nabla_1 h_{211} &= M_x - \Gamma_{12}^l h_{l11} - 2\Gamma_{11}^l h_{2l1} \\
&= M_x - \Gamma_{12}^1 L - (\Gamma_{12}^2 + 2\Gamma_{11}^1)M - 2\Gamma_{11}^2 N, \\
\nabla_2 h_{211} &= M_y - \Gamma_{22}^l h_{l11} - 2\Gamma_{12}^l h_{2l1} \\
&= M_y - \Gamma_{22}^1 L - (\Gamma_{22}^2 + 2\Gamma_{12}^1)M - 2\Gamma_{12}^2 N, \\
\nabla_1 h_{122} &= N_x - \Gamma_{11}^l h_{l22} - 2\Gamma_{12}^l h_{1l2} \\
&= N_x - 2\Gamma_{12}^2 M - (\Gamma_{11}^1 + 2\Gamma_{12}^2)N - \Gamma_{11}^2 O, \\
\nabla_2 h_{122} &= N_y - \Gamma_{21}^l h_{l22} - 2\Gamma_{22}^l h_{1l2} \\
&= N_y - 2\Gamma_{22}^1 M - (\Gamma_{21}^1 + 2\Gamma_{22}^2)N - \Gamma_{21}^2 O, \\
\nabla_1 h_{222} &= O_x - 3\Gamma_{12}^l h_{l22} \\
&= O_x - 3\Gamma_{12}^1 N - 3\Gamma_{12}^2 O.
\end{aligned}$$

From (3.6) we have

$$\begin{aligned}
L_y - M_x &= 2\Gamma_{12}^1 L + 2(\Gamma_{21}^2 - \Gamma_{11}^1)M - 2\Gamma_{11}^2 N, \\
M_y - N_x &= \Gamma_{22}^1 L + (\Gamma_{22}^2 + 2\Gamma_{12}^1 - 2\Gamma_{12}^2)M \\
&\quad + (2\Gamma_{12}^2 - \Gamma_{11}^1 - 2\Gamma_{12}^2)N - \Gamma_{11}^2 O, \\
N_y - O_x &= 2\Gamma_{22}^1 M + 2(\Gamma_{22}^2 - \Gamma_{12}^1)N - 2\Gamma_{12}^2 O.
\end{aligned} \tag{3.8}$$

The Gauss equation is expressed by

$$(EG - F^2)K = G(LN - M^2) - F(LO - MN) + E(MO - N^2). \tag{3.9}$$

We call equations (3.9) and (3.8) *Lagrangian Gauss–Codazzi system*. Our aim is to solve this system locally, for a given metric g, i.e., E, F and G and the corresponding Gaussian curvature K.

4. A local solution of the Lagrangian Gauss–Codazzi system

In this section, we prove the existence of a local solution of the Lagrangian Gauss–Codazzi system by solving (3.8) and (3.9).

Theorem 4. *Let g be a smooth metric in a neighborhood of the origin in $\mathbb{R}^2$. Then, the Lagrangian Gauss–Codazzi system (3.8)–(3.9) admits a smooth solution in a neighborhood of the origin.*

Proof. We divide the proof into several steps.

Step 1. We note that (3.8) and (3.9) involve three differential equations and one algebraic equation for four unknown functions. We will reduce (3.8) and (3.9) to a system of three differential equations for three unknown functions. To this end, we eliminate O from (3.8). By (3.9), we have

$$O = \frac{1}{EM - FL}\big[(EG - F^2)K + G(M^2 - LN) + EN^2 - FMN\big]. \tag{4.1}$$

A straightforward calculation yields

$$\begin{aligned} O_x = &\frac{1}{EM - FL}\big[(FO - GN)L_x + (2GM - FN - EO)M_x + (2EN - FM - LG)N_x\big] \\ &+ \frac{1}{EM - FL}\big[((EG - F^2)K)_x + (N^2 - OM)E_x + (LO - MN)F_x + (M^2 - LN)G_x\big]. \end{aligned}$$

We now substitute O_x in the third equation in (3.8) and rewrite the resulting system for L, M and N. Set

$$U = (L, M, N)^T.$$

Then, U satisfies

$$U_y + A(U, x, y)U_x + B(U, x, y) = 0, \tag{4.2}$$

where

$$A = -\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_1 & a_2 & a_3 \end{pmatrix},$$

$$B = (b_1, b_2, b_3)^T,$$

with

$$a_1 = \frac{FO - GN}{EM - FL}, \quad a_2 = \frac{2GM - FN - EO}{EM - FL}, \quad a_3 = \frac{2EN - FM - LG}{EM - FL}, \tag{4.3}$$

and

$$\begin{aligned} b_1 &= -2\Gamma^1_{12}L - 2(\Gamma^2_{21} - \Gamma^1_{11})M + 2\Gamma^2_{11}N, \\ b_2 &= -\Gamma^1_{22}L - (\Gamma^2_{22} + 2\Gamma^1_{12} - 2\Gamma^2_{12})M - (2\Gamma^2_{12} - \Gamma^1_{11} - 2\Gamma^2_{12})N + \Gamma^2_{11}O, \\ b_3 &= \frac{1}{EM - FL}\big[((EG - F^2)K)_x + (N^2 - OM)E_x + (LO - MN)F_x + (M^2 - LN)G_x\big] \\ &\quad - 2\Gamma^1_{22}M - 2(\Gamma^2_{22} - \Gamma^1_{12})N + 2\Gamma^2_{12}O. \end{aligned}$$

We should note that the explicit expression of $B(U)$ is not important. We also point out that O in A and B is expressed by (4.1).

In the following, we will solve (4.2) in a neighborhood of the origin in $\mathbb{R}^2$. Set

$$\mathcal{F}(U) = U_y + A(U, x, y)U_x + B(U, x, y). \tag{4.4}$$

Step 2. Near the origin, we choose a geodesic coordinate system such that

$$g = Edx^2 + dy^2, \tag{4.5}$$

where $E(\cdot, 0) = 1$. In other words, $F \equiv 0$ and $G \equiv 1$. We claim that there exists a smooth function $U_0 = U_0(x, y)$ in a neighborhood of the origin such that $\mathcal{F}(U_0) = O(y)$ and the matrix $A(U_0(x, y), x, y)$ has three distinct real eigenvalues for any (x, y) close to the origin.

By (4.1) and (4.3), we have

$$O = \frac{1}{EM}\big[EK + M^2 - LN + EN^2\big],$$

and

$$a_1 = -\frac{N}{EM}, \quad a_2 = \frac{2M - EO}{EM}, \quad a_3 = \frac{2EN - L}{EM}.$$

By substituting the expression of O in a_2, we get

$$a_2 = \frac{M^2 - EK}{EM^2} + \frac{ELN - E^2N^2}{E^2M^2}.$$

Set

$$\alpha = \frac{N}{EM}, \quad \beta = \frac{L}{EM}, \quad k = \frac{M^2 - EK}{EM^2}. \tag{4.6}$$

Then,

$$a_1 = -\alpha, \quad a_2 = k + E\alpha\beta - E^2\alpha^2, \quad a_3 = 2E\alpha - \beta. \tag{4.7}$$

To calculate the eigenvalues of A, we note

$$\det(\lambda I - A) = \lambda^3 - a_3\lambda^2 + a_2\lambda - a_1.$$

This polynomial has three distinct real roots if and only of

$$\Delta(U) \equiv \frac{1}{27}\big(a_2 - \frac{a_3^2}{3}\big)^3 + \frac{1}{4}\big(a_1 - \frac{1}{3}a_2a_3 + \frac{2}{27}a_3^3\big)^2 < 0. \tag{4.8}$$

By substituting (4.7) in (4.8), we get

$$27^2\Delta = (3k - 7E^2\alpha^2 + 7E\alpha\beta - \beta^2)^3 \\ + \big[-(\frac{27}{2} + 9kE)\alpha + \frac{9}{2}k\beta + 17E^3\alpha^3 - \frac{51}{2}E^2\alpha^2\beta + \frac{21}{2}E\alpha\beta - \beta^3\big]^2.$$

We now expand $27^2\Delta$ as a polynomial of α. The leading term for α is given by

$$(-7^3 + 17^2)E^6\alpha^6,$$

with a negative coefficient. Therefore, $\Delta < 0$ if α is sufficiently large relative to β and k.

Take a large constant α_0. Set

$$U_{00} = (0, 1, \alpha_0)^T,$$

and

$$U_0(x, y) = U_{00} + yU_{01}(x), \tag{4.9}$$

where $U_{01}(x)$ is a vector-valued function of x to be determined. Recall $E = 1$ on $y = 0$. Then,

$$L = 0, \quad M = 1, \quad N = \alpha_0 \quad \text{on } y = 0,$$

and hence, by (4.6),

$$\alpha = \alpha_0, \quad \beta = 0, \quad k = 1 - K(\cdot, 0) \quad \text{on } y = 0.$$

Therefore, $\Delta(U_0) < 0$ on $y = 0$ and hence for small y. Then, $A(U_0)$ has three distinct real eigenvalues for small y.

Last, we substitute (4.9) in (4.4) and expand $\mathcal{F}(U_0)$ in terms of y. Then, we can find U_{01} so that the expression corresponding to y^0 in $\mathcal{F}(U_0)$ vanishes. In fact,

$$U_{01} = -B(U_{00}, \cdot, 0).$$

With this choice of U_{01}, we have that $\mathcal{F}(U_0) = O(y)$ and the matrix $A(U_0(x, y), x, y)$ has three distinct real eigenvalues for any (x, y) close to the origin, as claimed.

Step 3. We are ready to solve $\mathcal{F}(U) = 0$ near the origin. Without loss of generality, we assume that g is a smooth metric defined in $\mathbb{R} \times (-\varepsilon_0, \varepsilon_0)$, for some $\varepsilon_0 > 0$, given by (4.5) with $E = 1$ for $|x| \geq 1$. By Step 2, there exists a smooth function $U_0 = U_0(x, y)$ in $\mathbb{R} \times (-\varepsilon_0, \varepsilon_0)$ such that

$$A(U_0(x, y), x, y) \text{ has three distinct real eigenvalues for any } (x, y) \in \mathbb{R} \times (-\varepsilon_0, \varepsilon_0),$$

and

$$\mathcal{F}(U_0) = yF_0(x, y),$$

for some smooth function F_0 in $\mathbb{R} \times (-\varepsilon_0, \varepsilon_0)$.

Take $\varepsilon > 0$ sufficiently small and set

$$x = \varepsilon s, \quad y = \varepsilon t,$$

and

$$U(x, y) = U_0(x, y) + \varepsilon V(s, t).$$

Then,

$$U_x = U_{0,x} + V_s, \quad U_y = U_{0,y} + V_t,$$

and

$$\begin{aligned}\mathcal{F}(U) = {} & V_t + A(U_0 + \varepsilon V)V_s \\ & + \mathcal{F}(U_0) + \big[A(U_0 + \varepsilon V) - A(U_0)\big]U_{0,x} + B(U_0 + \varepsilon V) - B(U_0).\end{aligned}$$

We now write this as

$$\mathcal{G}(V, \varepsilon) = V_t + \big[C_0(s, t) + C_1(V, s, t, \varepsilon)\big]V_s + D(V, s, t, \varepsilon),$$

where C_0 and C_1 are 3×3 matrices and D is a 3-vector, all smooth in their arguments, such that,

$$C_0(s, t) \text{ has three distinct real eigenvalues for any } (s, t) \in \mathbb{R} \times (-1, 1), \tag{4.10}$$

and

$$C_1(\cdot, \cdot, \cdot, 0) = 0, \quad D(\cdot, \cdot, \cdot, 0) = 0.$$

In particular, $\mathcal{G}(V, 0) = 0$ is a strictly hyperbolic linear system in $\mathbb{R} \times (-1, 1)$.

For any integer $m \geq 5$ and any 3-vector $E \in H^m(\mathbb{R} \times (-1, 1))$, consider

$$\begin{aligned} V_t + C_0(s, t)V_s = E \quad & \text{in } \mathbb{R} \times (-1, 1), \\ V(\cdot, 0) = 0 \quad & \text{on } \mathbb{R}. \end{aligned} \tag{4.11}$$

Thanks to (4.10), this is the Cauchy problem for the strictly hyperbolic linear system $\mathcal{G}(\cdot, 0) = 0$. By the standard theory, there exists a solution $V \in H^m(\mathbb{R} \times (-1, 1))$ of (4.11) and

$$\|V\|_m \leq c\|E\|_m,$$

where c is a positive constant independent of E and V, and $\|\cdot\|_m$ is the H^m-norm in $\mathbb{R} \times (-1, 1)$.

For $\varepsilon > 0$ small, we consider the following Cauchy problem for $V = V(s, t)$:

$$\begin{aligned} \mathcal{G}(V, \varepsilon) = 0 \quad & \text{in } \mathbb{R} \times (-1, 1), \\ V(\cdot, 0) = 0 \quad & \text{on } \mathbb{R}. \end{aligned} \tag{4.12}$$

By a standard iteration or the contraction mapping principle, (4.12) admits a smooth solution in $\mathbb{R} \times (-1, 1)$ for any $\varepsilon > 0$ sufficiently small. Refer to the proof of Theorem 4.2.1 in [14] for details.

In conclusion, for any fixed $\varepsilon > 0$ sufficiently small, $\mathcal{F}(U) = 0$ admits a smooth solution U in $B_\varepsilon \subset \mathbb{R}^2$. □

Proof of Theorem 3. The case $\varepsilon = 1$ was proved in Theorem 4. For the case $\varepsilon = -1$, the proof is similar. The only difference is that (3.9) has a different sign, which results in minor changes. □

Remark. The local embedding of smooth surfaces into $\mathbb{R}^4$ obtained by Poznyak [20] is not Lagrangian. Nevertheless, it is an interesting question to ask whether one can modify his proof to give another proof of Theorem 2.

Remark. One can certainly consider the Hessian metric and the Lagrangian embedding problem for metrics g of class $C^{k,\alpha}$. We leave this problem to the interested reader.

Remark. With a similar method, we can also prove that any smooth surface can be locally embedded isometrically into the Heisenberg group $\mathbb{H}^2$ as a Legendrian surface. For the Legendrian surfaces, Heisenberg group $\mathbb{H}^2$ and related analysis, see [6].

Conflict of interest statement

No conflict of interest.

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