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Continuity and density results for a one-phase nonlocal free boundary problem ☆

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Abstract

We consider a one-phase nonlocal free boundary problem obtained by the superposition of a fractional Dirichlet energy plus a nonlocal perimeter functional. We prove that the minimizers are Hölder continuous and the free boundary has positive density from both sides.

For this, we also introduce a new notion of fractional harmonic replacement in the extended variables and we study its basic properties.

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1. Introduction

The recent research has payed a great attention to a class of nonlocal problems arising in both pure and applied mathematics. A natural setting in which nonlocal questions arise is given by the class of free boundary problems. Roughly speaking, many free boundary problems are built by the competition of two (or more) competing terms: for instance, an elastic (or ferromagnetic) energy can be combined with a tension effect (in this setting, the ferromagnetic

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energy favors the preservation of the values of a state parameter u, while the tension effect tends to make the interface given by the level sets of u as small as possible).

In order to take into account possible long-range interactions, some nonlocal energies have been considered in these types of free boundary problems. In particular, in [14], a new energy functional was considered, as the sum of a fractional Dirichlet energy, with fractional exponent $s \in (0,1)$, and a fractional perimeter, with fractional exponent $\sigma \in (0,1)$. When $s \to 1$, and when $\sigma \to 0$ or $\sigma \to 1$, the energy functional becomes the classical free boundary energy considered in [1–3]. An intermediate problem, with a local Dirichlet energy plus a fractional perimeter has been studied in [6].

Some results of classical flavor have been proved in [14], such as, among the others,[1] a monotonicity formula for the minimizers, some glueing lemmata, some uniform energy bounds, convergence results, a regularity theory for the planar cones and a trivialization result for the flat case. On the other hand, in [14] no result was proved concerning the regularity of the minimizers and the density properties of the free boundary. These type of results are indeed quite hard to obtain, due to the strong nonlocal feature of the problem: for instance, differently from the classical case, the nonlocal Dirichlet energy provides nontrivial interactions between the positivity and negativity sets of the functions, and a local modification of the free boundary produces global consequences in the fractional perimeter.

Goal of this paper is then to provide regularity and density results, at least in the case of the one-phase problem (i.e. when the boundary data are nonnegative).

The mathematical setting in which we work is the following. Let $s, \sigma \in (0,1)$, and $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Following [14], we define

$$\mathscr{F}_\Omega(u,E) := \iint_{Q_\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \mathrm{Per}_\sigma(E,\Omega),$$

where

$$Q_\Omega := (\Omega \times \Omega) \cup \big((\mathbb{R}^n \setminus \Omega) \times \Omega\big) \cup \big(\Omega \times (\mathbb{R}^n \setminus \Omega)\big)$$

and $\mathrm{Per}_\sigma(E,\Omega)$ denotes the fractional perimeter of E in Ω (see [5] or formulas (1.2) and (1.3) in [14]), that is

$$\mathrm{Per}_\sigma(E,\Omega) := L(E\cap\Omega, \Omega\setminus E) + L(E\cap\Omega, (\mathbb{R}^n\setminus E)\setminus\Omega) + L(E\setminus\Omega, \Omega\setminus E), \tag{1.1}$$

where, for any disjoint sets $A, B \subseteq \mathbb{R}^n$,

$$L(A,B) := \iint_{A\times B} \frac{dx\,dy}{|x-y|^{n+\sigma}}.$$

All sets and functions are implicitly assumed to be measurable from now on.

Let $E \subseteq \mathbb{R}^n$ and $u : \mathbb{R}^n \to \mathbb{R}$. We say that (u,E) is an admissible pair if $u \geqslant 0$ a.e. in E and $u \leqslant 0$ a.e. in $\mathbb{R}^n \setminus E$.

Also, we say that (u,E) is a minimizing pair in Ω if $\mathscr{F}_\Omega(u,E) < +\infty$ and

$$\mathscr{F}_\Omega(u,E) \leqslant \mathscr{F}_\Omega(v,F)$$

for any admissible pair (v,F) such that:

- $u - v \in H^s(\mathbb{R}^n)$,
- $u = v$ a.e. in $\mathbb{R}^n \setminus \Omega$, and
- $E \setminus \Omega = F \setminus \Omega$.

[1] We take this opportunity to amend some minor inconsistencies in [14].

First of all, in Theorem 1.2 and in Lemma 8.3, the condition "$\overline{u}_m$ is the extension of u_m" has to be intended "$\overline{u}_m$ is the extension of $u_m \in C(\mathbb{R}^n)$".

Then, in the statement of Lemma 3.2 "if $u \in C(\mathbb{R}^n)$" has to be placed in the beginning, and in the proof of Lemma 3.2 the expression "$\min_{B_r(x_o)} u$" needs to be replaced by "$\underline{\min}_{B_r(x_o)} u$".

Roughly speaking, a pair (u, E) is admissible if E is the positivity set of u, and it is minimizing if it has minimal energy among all the possible competing admissible pairs that coincide outside Ω. For the existence of minimizing pairs see Lemma 3.1 in [14].

We remark that this minimizing problem is nontrivial even in the one-phase case, i.e. when the boundary datum u is nonnegative, since the set E is not necessarily trivially prescribed outside Ω.

In this setting, our main result is the following:

Theorem 1.1 *(Density estimates and continuity for one-phase minimizers). Assume that (u, E) is minimizing in B_1, with $u \geqslant 0$ a.e. in $\mathbb{R}^n \setminus B_1$ and $0 \in \partial E$. Assume also that*

$$\int_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+2s}}\, dx \leqslant \Lambda, \tag{1.2}$$

for some $\Lambda > 0$.

Then, there exist c, $K > 0$, possibly depending on n, s, σ and Λ, such that for any $r \in (0, 1/2]$,

$$\min\Big\{|B_r \cap E|,\ |B_r \setminus E|\Big\} \geqslant cr^n \tag{1.3}$$

and

$$\|u\|_{L^\infty(B_{1/2})} \leqslant K. \tag{1.4}$$

In addition, if $s > \sigma/2$, then, given $r_0 \in (0, 1/4)$,

$$u \in C^{s-\frac{\sigma}{2}}(B_{r_0}), \text{ with } \|u\|_{C^{s-\frac{\sigma}{2}}(B_{r_0})} \leqslant C, \tag{1.5}$$

where $C > 0$ possibly depends on n, s, σ, r_0 and Λ.

We observe that both the growth condition (1.2) and the Hölder exponent in (1.5) are compatible with the degree of homogeneity of the minimizing cones, see Theorem 1.3 of [14]. Condition (1.2) is also a standard assumption to make sense of the fractional Laplace operator (though some very recent developments in [15] may also allow more general notions of suitable fractional Laplace operators for functions with more severe growth at infinity).

It is an open problem to investigate the optimal regularity of the solution (which could be possibly beyond the scaling arguments) and to classify (or trivialize) the minimizing cones: see also [6,14] for partial results and additional comments on these problems.

It is also an interesting question to study this type of free boundary problems for more general fractional operators (see e.g. [22,17] for a classical counterpart).

The rest of the paper is organized as follows. In Section 2 we introduce an extension problem which is useful to localize the Dirichlet energy (using a weighted space with an additional variable). This extended problem is different than the one considered in [14] since here the fractional perimeter functional is not modified by the extension procedure.

In Section 3, we introduce a fractional harmonic replacement in this weighed extended space. Fractional harmonic replacements are of course a classical topic in harmonic analysis and they have several applications to free boundary problems, see e.g. [3,6] and the references therein. In the literature, a fractional harmonic replacement was also studied in [16]. The setting of [16] is different than the one considered in Section 3 of this paper, since here we deal with the extended space and, in Section 4, we obtain localized energy estimates in the extended variable. These energy estimates play a crucial role in our subsequent density estimates (as a matter of fact, both the replacement of [16] and the one of Section 3 here will be used in this paper to prove density estimates from both sides).

In Section 5 we prove the density estimates. First we prove the density of the vanishing set around free boundary points, together with a uniform estimate on the size of the solution. Then we use this information to obtain density estimates of the positivity set as well, which completes the proof of the double-sided density estimate in (1.3).

By combining the density estimates with the uniform bound on the solution, one also obtains continuity of the minimizers, as claimed in (1.5).

2. An extended problem

In this section we introduce an extension problem in order to localize the Dirichlet energy, by adding one variable (see [7]).

We use the following setting. We consider variables $x \in \mathbb{R}^n$ and $z \in \mathbb{R}$, and we use the notation $X := (x, z) \in \mathbb{R}^{n+1}$. We consider the halfspace $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0, +\infty)$. The n-dimensional ball centered at $0 \in \mathbb{R}^n$ and of radius $r > 0$ is denoted by B_r.

Given $u : \mathbb{R}^n \to \mathbb{R}$, for any $(x, z) \in \mathbb{R}^{n+1}$ we define

$$\overline{u}(x,z) := \int_{\mathbb{R}^n} \frac{|z|^{2s}\, u(x-y)}{(|y|^2+z^2)^{\frac{n+2s}{2}}}\, dy = \int_{\mathbb{R}^n} \frac{|z|^{2s}\, u(y)}{(|x-y|^2+z^2)^{\frac{n+2s}{2}}}\, dy, \tag{2.1}$$

see e.g. [7], in particular Section 2.4 there (notice that in [7] in the definition of the extension function $\overline{u}$ there is also a normalizing constant, that we neglect here, since it will not play any role in our problem).

Next result states that if (u, E) is a minimal pair, then $(\overline{u}, E)$ is minimal for an extended problem:

Lemma 2.1. *Let (u, E) be a minimizing pair in B_r. Let $\mathscr{U}$ be a bounded and Lipschitz domain of $\mathbb{R}^{n+1}$ that is symmetric with respect to the z-coordinate, such that*

$$\mathscr{U} \cap \{z = 0\} \subset B_r \times \{0\}.$$

Then

$$\int_{\mathscr{U}} |z|^a |\nabla \overline{u}|^2\, dX + \mathrm{Per}_\sigma(E, B_r) \leqslant \int_{\mathscr{U}} |z|^a |\nabla \tilde{v}|^2\, dX + \mathrm{Per}_\sigma(F, B_r),$$

for every $(\tilde{v}, F)$ such that:

- $F \setminus B_r = E \setminus B_r$,
- $\tilde{v} - \overline{u}$ *is compactly supported inside* $\mathscr{U}$,
- $\tilde{v}(x, 0) \geqslant 0$ *a.e.* $x \in F$,
- $\tilde{v}(x, 0) \leqslant 0$ *a.e.* $x \in \mathbb{R}^n \setminus F$.

Proof. We take (u, E) and $(\tilde{v}, F)$ as in the statement of Lemma 2.1 and we define $v(x) := \tilde{v}(x, 0)$, for any $x \in \mathbb{R}^n$.

Notice that $(\mathbb{R}^n \setminus B_r) \times \{0\} \subseteq \mathbb{R}^{n+1} \setminus \mathscr{U}$, therefore $v(x) = \tilde{v}(x, 0) = \overline{u}(x, 0) = u(x)$ for a.e. $x \in \mathbb{R}^n \setminus B_r$. In addition, $v \geqslant 0$ a.e. on F and $v \leqslant 0$ a.e. on $\mathbb{R}^n \setminus F$. Therefore, the pair (v, F) is an admissible competitor for (u, E) and so, by the minimality of (u, E), we have that

$$\begin{aligned}
&\iint_{Q_{B_r}} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\, dx\, dy - \iint_{Q_{B_r}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dx\, dy \\
&\quad = \mathscr{F}_{B_r}(v, F) - \mathscr{F}_{B_r}(u, E) - \mathrm{Per}_\sigma(F, B_r) + \mathrm{Per}_\sigma(E, B_r) \\
&\quad \geqslant -\mathrm{Per}_\sigma(F, B_r) + \mathrm{Per}_\sigma(E, B_r).
\end{aligned} \tag{2.2}$$

On the other hand, by Lemma 7.2 of [5], up to a normalizing constant, we have that

$$\begin{aligned}
&\iint_{Q_{B_r}} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\, dx\, dy - \iint_{Q_{B_r}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dx\, dy \\
&\quad = \inf \int_{\mathscr{W}} |z|^a \big(|\nabla \tilde{w}|^2 - |\nabla \overline{u}|^2\big)\, dX,
\end{aligned}$$

where the infimum above is taken over all the couples $(\tilde{w}, \mathscr{W})$ satisfying the following properties:

- $\mathscr{W}$ is a bounded and Lipschitz domain of $\mathbb{R}^{n+1}$ that is symmetric with respect to the z-coordinate, such that

$$\mathscr{W} \cap \{z = 0\} \subset B_r \times \{0\},$$

- $\tilde{w} - \overline{u}$ is compactly supported inside $\mathscr{W}$,
- $\tilde{w}(x, 0) = v(x)$ for any $x \in \mathbb{R}^n$.

By construction, we can take $\tilde{w} := \tilde{v}$ and $\mathscr{W} := \mathscr{U}$ as candidates in the above infimum, and consequently

$$\iint_{Q_{B_r}} \frac{|v(x) - v(y)|^2}{|x - y|^{n+2s}}\, dx\, dy - \iint_{Q_{B_r}} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}}\, dx\, dy \leqslant \int_{\mathscr{U}} |z|^a \big(|\nabla \tilde{v}|^2 - |\nabla \overline{u}|^2\big)\, dX.$$

This and (2.2) give that

$$\int_{\mathscr{U}} |z|^a \big(|\nabla \tilde{v}|^2 - |\nabla \overline{u}|^2\big)\, dX \geqslant -\mathrm{Per}_\sigma(F, B_r) + \mathrm{Per}_\sigma(E, B_r),$$

that is the desired result. □

3. Fractional harmonic replacements in the extended variables

Goal of this section is to introduce a notion of fractional harmonic replacement in the extended variables and study its basic properties. In the classical case, a detailed study of the harmonic replacement was performed in [3,6]. See also [16] for the study of a related (but different) fractional harmonic replacement.

We set

$$\mathscr{B}_r := B_{\frac{9r}{10}} \times (-r, r). \tag{3.1}$$

It worth to link the norm in $\mathscr{B}_r$ for the extended function with the one on the trace, as pointed out by the following result:

Lemma 3.1. *Let u and $\overline{u}$ be as in (2.1). There exists $C_r > 0$ such that*

$$\|\overline{u}\|_{L^\infty(\mathscr{B}_r)} \leqslant C_r \left(\|u\|_{L^\infty(B_r)} + \int_{\mathbb{R}^n \setminus B_r} \frac{|u(y)|}{|y|^{n+2s}}\, dy \right).$$

Proof. Let $(x, z) \in \mathscr{B}_r$. Then $x \in B_{\frac{9r}{10}}$ and $|z| \leqslant r$. Therefore, if $y \in \mathbb{R}^n \setminus B_r$, we have that

$$|x - y| \geqslant |y| - |x| = \frac{|y|}{10} + \frac{9|y|}{10} - |x| \geqslant \frac{|y|}{10} + \frac{9r}{10} - \frac{9r}{10} = \frac{|y|}{10}.$$

Hence, if $y \in \mathbb{R}^n \setminus B_r$,

$$\frac{|z|^{2s}\, |u(y)|}{(|x - y|^2 + z^2)^{\frac{n+2s}{2}}} \leqslant \frac{r^{2s}\, |u(y)|}{|x - y|^{n+2s}} \leqslant C_r\, \frac{|u(y)|}{|y|^{n+2s}},$$

for some $C_r > 0$. As a consequence

$$\int_{\mathbb{R}^n \setminus B_r} \frac{|z|^{2s}\, |u(y)|}{(|x - y|^2 + z^2)^{\frac{n+2s}{2}}}\, dy \leqslant C_r \int_{\mathbb{R}^n \setminus B_r} \frac{|u(y)|}{|y|^{n+2s}}\, dy. \tag{3.2}$$

Moreover,

$$\int_{B_r} \frac{|z|^{2s}\,|u(y)|}{(|x-y|^2+z^2)^{\frac{n+2s}{2}}}\,dy \leqslant \|u\|_{L^\infty(B_r)} \int_{B_r} \frac{|z|^{2s}}{(|x-y|^2+z^2)^{\frac{n+2s}{2}}}\,dy$$
$$\leqslant \|u\|_{L^\infty(B_r)} \int_{\mathbb{R}^n} \frac{|z|^{2s}}{(|x-y|^2+z^2)^{\frac{n+2s}{2}}}\,dy = C\,\|u\|_{L^\infty(B_r)},$$

for some $C > 0$. The latter estimate and (3.2) imply the desired result, up to renaming the constants. □

3.1. *Functional spaces*

Given $r > 0$, we consider the seminorm

$$[v]_{\mathbb{H}^s(\mathscr{B}_r)} := \sqrt{\int_{\mathscr{B}_r} |z|^a |\nabla v|^2\,dX},$$

with $a := 1 - 2s \in (-1, 1)$. We denote by $\mathbb{H}^s(\mathscr{B}_r)$ the closure of $C^\infty(\mathscr{B}_r)$ with respect to the norm

$$\|v\|_{\mathbb{H}^s(\mathscr{B}_r)} := [v]_{\mathbb{H}^s(\mathscr{B}_r)} + \sqrt{\int_{\mathscr{B}_r} |z|^a |v|^2\,dX}.$$

We also set $\mathbb{H}^s_0(\mathscr{B}_r)$ to be the closure of $C^\infty_0(\mathscr{B}_r)$ with respect to the norm above.

For completeness, we point out that the seminorm $[\cdot]_{\mathbb{H}^s(\mathscr{B}_r)}$ is indeed a norm on $\mathbb{H}^s_0(\mathscr{B}_r)$:

Lemma 3.2. *If $v \in \mathbb{H}^s_0(\mathscr{B}_r)$ and $[v]_{\mathbb{H}^s(\mathscr{B}_r)} = 0$ then $v = 0$ a.e. in $\mathscr{B}_r$.*

Proof. Let $v_k \in C^\infty_0(\mathscr{B}_r)$ be such that $\|v_k - v\|_{\mathbb{H}^s(\mathscr{B}_r)} \to 0$ as $k \to +\infty$. Up to subsequences, we may suppose that

$$v_k \to v \text{ a.e. in } \mathscr{B}_r. \tag{3.3}$$

Also, by Proposition 2.1.1 in [12],

$$\left(\int_{\mathscr{B}_r} |z|^a |v_k|^{2\gamma}\,dX \right)^{\frac{1}{2\gamma}} \leqslant \hat{S}\,[v_k]_{\mathbb{H}^s(\mathscr{B}_r)},$$

for some $\gamma > 1$ and $\hat{S} > 0$. Therefore

$$\left(\int_{\mathscr{B}_r} |z|^a |v_k|^{2\gamma}\,dX \right)^{\frac{1}{2\gamma}} \leqslant \hat{S}\left([v_k - v]_{\mathbb{H}^s(\mathscr{B}_r)} + [v]_{\mathbb{H}^s(\mathscr{B}_r)} \right)$$
$$= \hat{S}\,[v_k - v]_{\mathbb{H}^s(\mathscr{B}_r)} \to 0$$

as $k \to +\infty$. This implies that $v_k \to 0$ a.e. in $\mathscr{B}_r$, up to subsequences, and therefore $v = 0$ a.e. in $\mathscr{B}_r$, thanks to (3.3). □

Given $\varphi \in \mathbb{H}^s(\mathscr{B}_2)$, we define

$$\mathscr{D}^\varphi := \left\{ v \in \mathbb{H}^s(\mathscr{B}_1) \text{ s.t. } v - \varphi \in \mathbb{H}^s_0(\mathscr{B}_1) \right\}.$$

Now we observe that functions in $\mathscr{D}^\varphi$ possess a trace along $\{z = 0\}$. The expert reader may skip this part and go directly to formula (3.6). To give an elementary proof of this fact (which is rather well known in general, see e.g. Lemma 3.1 of [23] or the references therein), we make this preliminary observation:

Lemma 3.3. *For any $v \in C_0^\infty(\mathscr{B}_r)$ and any $x \in B_{\frac{9r}{10}}$, we define $T_v(x) := v(x,0)$. Then, there exists $C>0$ such that*

$$\|T_v\|_{L^2(B_{9r/10})} \leqslant C\|v\|_{\mathbb{H}^s(\mathscr{B}_r)}.$$

Proof. For any $x \in B_{\frac{9r}{10}}$,

$$\begin{aligned}|T_v(x)| &= |v(x,0)-v(x,r)| \\ &\leqslant \int_0^r |\partial_z v(x,z)|\,dz \leqslant \int_0^r |z|^{-\frac{a}{2}}|z|^{\frac{a}{2}}|\nabla v(x,z)|\,dz.\end{aligned}$$

So, by Cauchy–Schwarz inequality, for any $x \in B_{\frac{9r}{10}}$,

$$|T_v(x)|^2 \leqslant \int_0^r |z|^{-a}\,dz \int_0^r |z|^a|\nabla v(x,z)|^2\,dz = C\int_0^r |z|^a|\nabla v(x,z)|^2\,dz.$$

Hence we integrate over $x \in B_{\frac{9r}{10}}$ and the desired result easily follows. □

Now, for any $w \in \mathbb{H}_0^s(\mathscr{B}_r)$, we know from the definition of $\mathbb{H}_0^s(\mathscr{B}_r)$ that there exists a sequence of functions $w_k \in C_0^\infty(\mathscr{B}_r)$ such that $\|w-w_k\|_{\mathbb{H}^s(\mathscr{B}_r)} \to 0$ as $k \to +\infty$. By Lemma 3.3, we have that

$$T_{w_k-w_h}(x) = w_k(x,0)-w_h(x,0) = T_{w_k}(x)-T_{w_h}(x)$$

and so

$$\|T_{w_k}-T_{w_h}\|_{L^2(B_{9r/10})} \leqslant C\|w_k-w_h\|_{\mathbb{H}^s(\mathscr{B}_r)}.$$

This means that the sequence T_{w_k} is Cauchy in $L^2(B_{9r/10})$, hence it converges to some function, denoted as T_w, in $L^2(B_{9r/10})$, which we call the trace of w along $\{z=0\}$. Of course, the trace T_w is defined up to sets of zero n-dimensional Lebesgue measure, and a different approximating sequence does produce the same trace: to check this, take an approximating sequence $\tilde{w}_k$ and use again Lemma 3.3 to see that

$$\begin{aligned}\|T_{w_k}-T_{\tilde{w}_k}\|_{L^2(B_{9r/10})} &\leqslant C\|w_k-\tilde{w}_k\|_{\mathbb{H}^s(\mathscr{B}_r)} \\ &\leqslant C\|w_k-w\|_{\mathbb{H}^s(\mathscr{B}_r)} + C\|\tilde{w}_k-w\|_{\mathbb{H}^s(\mathscr{B}_r)},\end{aligned}$$

hence T_{w_k} and $T_{\tilde{w}_k}$ have the same limit in $L^2(B_{9r/10})$.

Our next goal is to show that we can trace also $\varphi \in \mathbb{H}^s(\mathscr{B}_2)$ along $B_{9/10}$. This is not completely obvious since $\varphi \notin \mathbb{H}_0^s(\mathscr{B}_2)$, so the above construction does not apply. For this, we observe that:

Lemma 3.4. *If $\varphi_1, \varphi_2 \in \mathbb{H}^s(\mathscr{B}_2)$ and $\varphi_1 = \varphi_2$ a.e. in $\mathscr{B}_{5/4}$, then $\mathscr{D}^{\varphi_1} = \mathscr{D}^{\varphi_2}$.*

Proof. Let $v \in \mathscr{D}^{\varphi_1}$. Then $v-\varphi_1 \in \mathbb{H}_0^s(\mathscr{B}_1)$. Hence there exists a sequence $w_k \in C_0^\infty(\mathscr{B}_1)$ such that $\|v-\varphi_1-w_k\|_{\mathbb{H}^s(\mathscr{B}_1)} \to 0$ as $k \to +\infty$. Since $\varphi_1 = \varphi_2$ a.e. in $\mathscr{B}_{5/4}$, we have that $\|v-\varphi_1-w_k\|_{\mathbb{H}^s(\mathscr{B}_1)} = \|v-\varphi_2-w_k\|_{\mathbb{H}^s(\mathscr{B}_1)}$. As a consequence, $\|v-\varphi_2-w_k\|_{\mathbb{H}^s(\mathscr{B}_1)} \to 0$ as $k \to +\infty$, which shows that $v \in \mathscr{D}^{\varphi_2}$.

The reverse inclusion is completely analogous. □

Now, given $\varphi \in \mathbb{H}^s(\mathscr{B}_2)$, we can take $\tau \in C_0^\infty(\mathscr{B}_{3/2})$ with $\tau = 1$ in $\mathscr{B}_{5/4}$ and consider $\varphi_o := \tau\varphi$. By the trace construction in $\mathbb{H}_0^s(\mathscr{B}_2)$, we can define the trace T_{φ_o} as a function in $L^2(B_{2\cdot 9/10})$. So we define the trace of φ in $B_{9/10}$ as $T_\varphi := T_{\varphi_o}$. By construction, $T_\varphi \in L^2(B_{9/10})$. Next observation shows that this definition is independent on the particular cut-off chosen:

Lemma 3.5. *If $\varphi_1, \varphi_2 \in \mathbb{H}_0^s(\mathscr{B}_2)$, with $\varphi_1 = \varphi_2$ a.e. in $\mathscr{B}_{5/4}$, then $T_{\varphi_1} = T_{\varphi_2}$ a.e. in $B_{9/10}$.*

Proof. By construction, for any $i \in \{1,2\}$, there are sequences $\varphi_{i,k} \in C_0^\infty(\mathscr{B}_2)$ such that $\|\varphi_i - \varphi_{i,k}\|_{\mathbb{H}^s(\mathscr{B}_2)} \to 0$ as $k \to +\infty$. Let $\Theta \in C_0^\infty(\mathscr{B}_{5/4})$ with $\Theta = 1$ in $\mathscr{B}_{11/10}$. Let also

$$\tilde{\varphi}_{1,k} := \varphi_{1,k} + \Theta(\varphi_{2,k} - \varphi_{1,k}).$$

We claim that

$$\lim_{k \to +\infty} \|\varphi_1 - \tilde{\varphi}_{1,k}\|_{\mathbb{H}^s(\mathscr{B}_2)} = 0. \tag{3.4}$$

To prove this, we observe that

$$\begin{aligned}
|\varphi_1 - \tilde{\varphi}_{1,k}|^2 &= \left|\varphi_1 - \varphi_{1,k} - \Theta(\varphi_{2,k} - \varphi_{1,k})\right|^2 \\
&\leqslant C\Big(|\varphi_1 - \varphi_{1,k}|^2 + |\Theta|^2 |\varphi_{2,k} - \varphi_{1,k}|^2\Big) \\
&\leqslant C\Big(|\varphi_1 - \varphi_{1,k}|^2 + + \chi_{\mathscr{B}_{5/4}} |\varphi_{2,k} - \varphi_{1,k}|^2\Big),
\end{aligned}$$

up to renaming $C > 0$. Hence, since $\varphi_1 = \varphi_2$ a.e. in $\mathscr{B}_{5/4}$,

$$\chi_{\mathscr{B}_{5/4}} |\varphi_{2,k} - \varphi_{1,k}|^2 = \chi_{\mathscr{B}_{5/4}} |\varphi_{2,k} - \varphi_2 + \varphi_1 - \varphi_{1,k}|^2.$$

Therefore

$$|\varphi_1 - \tilde{\varphi}_{1,k}|^2 \leqslant C\Big(|\varphi_1 - \varphi_{1,k}|^2 + |\varphi_{2,k} - \varphi_2|^2\Big).$$

As a consequence,

$$\lim_{k \to +\infty} \int_{\mathscr{B}_2} |z|^a |\varphi_i - \tilde{\varphi}_{i,k}|^2 \, dX = 0. \tag{3.5}$$

Moreover, we observe that

$$\begin{aligned}
|\nabla(\varphi_1 - \tilde{\varphi}_{1,k})|^2 &= \left|\nabla(\varphi_1 - \varphi_{1,k}) - \nabla(\Theta(\varphi_{2,k} - \varphi_{1,k}))\right|^2 \\
&\leqslant C\Big(|\nabla(\varphi_1 - \varphi_{1,k})|^2 + |\nabla\Theta|^2 |\varphi_{2,k} - \varphi_{1,k}|^2 + |\Theta|^2 |\nabla(\varphi_{2,k} - \varphi_{1,k})|^2\Big) \\
&\leqslant C\Big(|\nabla(\varphi_1 - \varphi_{1,k})|^2 + \chi_{\mathscr{B}_{5/4} \setminus \mathscr{B}_{11/10}} |\varphi_{2,k} - \varphi_{1,k}|^2 + \chi_{\mathscr{B}_{5/4}} |\nabla(\varphi_{2,k} - \varphi_{1,k})|^2\Big),
\end{aligned}$$

up to renaming $C > 0$. Hence, since $\varphi_1 = \varphi_2$ a.e. in $\mathscr{B}_{5/4}$,

$$\chi_{\mathscr{B}_{5/4} \setminus \mathscr{B}_{11/10}} |\varphi_{2,k} - \varphi_{1,k}|^2 = \chi_{\mathscr{B}_{5/4} \setminus \mathscr{B}_{11/10}} |\varphi_{2,k} - \varphi_2 + \varphi_1 - \varphi_{1,k}|^2$$

and

$$\chi_{\mathscr{B}_{5/4}} |\nabla(\varphi_{2,k} - \varphi_{1,k})|^2 = \chi_{\mathscr{B}_{5/4}} |\nabla(\varphi_{2,k} - \varphi_2 + \varphi_1 - \varphi_{1,k})|^2.$$

Therefore

$$\begin{aligned}
&|\nabla(\varphi_1 - \tilde{\varphi}_{1,k})|^2 \\
&\leqslant C\Big(|\nabla(\varphi_1 - \varphi_{1,k})|^2 + |\varphi_{2,k} - \varphi_2|^2 + |\varphi_1 - \varphi_{1,k}|^2 \\
&\qquad + |\nabla(\varphi_{2,k} - \varphi_2)|^2 + |\nabla(\varphi_1 - \varphi_{1,k})|^2\Big),
\end{aligned}$$

which, after an integration, implies that

$$\lim_{k \to +\infty} \int_{\mathscr{B}_2} |z|^a |\nabla(\varphi_i - \tilde{\varphi}_{i,k})|^2 \, dX = 0.$$

This and (3.5) give (3.4).

With this, and setting $\tilde\varphi_{2,k} := \varphi_{2,k}$, we have that $\tilde\varphi_{i,k} \in C_0^\infty(\mathscr{B}_2)$, $\|\varphi_i - \tilde\varphi_{i,k}\|_{\mathbb{H}^s(\mathscr{B}_2)} \to 0$ as $k \to +\infty$, and, additionally, if $X \in \mathscr{B}_{11/10}$ then

$$\tilde\varphi_{1,k}(X) = \tilde\varphi_{2,k}(X).$$

Since T_{φ_i} is the limit in $L^2(B_{2\cdot(9/10)})$ (and so a.e. in $B_{2\cdot(9/10)}$, up to subsequences) of $T_{\tilde\varphi_{i,k}}$ as $k \to +\infty$, we have, for a.e. $x \in B_{9/10} \subseteq \mathscr{B}_{11/10} \cap \{z = 0\}$,

$$\begin{aligned} T_{\varphi_1}(x) &= \lim_{k\to+\infty} T_{\tilde\varphi_{1,k}}(x) = \lim_{k\to+\infty} \tilde\varphi_{1,k}(x,0) \\ &= \lim_{k\to+\infty} \tilde\varphi_{2,k}(x,0) = \lim_{k\to+\infty} T_{\tilde\varphi_{2,k}}(x) = T_{\varphi_2}(x), \end{aligned}$$

as desired. □

Having defined T_w for any $w \in \mathbb{H}_0^s(\mathscr{B}_1)$ and T_φ for any $\varphi \in \mathbb{H}^s(\mathscr{B}_2)$, we now define the trace of any function $v \in \mathscr{D}^\varphi$, by setting

$$T_v := T_{v-\varphi} + T_\varphi.$$

To simplify the notation, given a set $K \subseteq \mathscr{B}_1 \cap \{z = 0\}$, we say that $v = 0$ a.e. in K to mean that $T_v = 0$ a.e. in K (i.e. $v(x,0) = 0$ for a.e. $x \in K$, in the sense of traces). We set

$$\mathscr{D}_K^\varphi := \left\{ v \in \mathscr{D}^\varphi \text{ s.t. } v = 0 \text{ a.e. in } K \right\}. \tag{3.6}$$

In some intermediate results, we also need a slightly more general definition in which the values attained at K are not necessarily zero. For this, given $\gamma : K \to \mathbb{R}$, we also define

$$\mathscr{D}_{K,\gamma}^\varphi := \left\{ v \in \mathscr{D}^\varphi \text{ s.t. } v = \gamma \text{ a.e. in } K \right\}. \tag{3.7}$$

Notice that $\mathscr{D}_{K,\gamma}^\varphi$ reduces to $\mathscr{D}_K^\varphi$ when $\gamma \equiv 0$. The functional structure of $\mathscr{D}_{K,\gamma}^\varphi$ that is needed for our purposes is given by the following result:

Lemma 3.6. *Let $w_j \in \mathscr{D}_{K,\gamma}^\varphi$ be such that*

$$\sup_{j\in\mathbb{N}} \int_{\mathscr{B}_1} |z|^a |\nabla w_j|^2 \, dX < +\infty.$$

Then there exists $w \in \mathscr{D}_{K,\gamma}^\varphi$ such that, up to a subsequence,

$$\lim_{j\to+\infty} \int_{\mathscr{B}_1} |z|^a |w - w_j|^2 \, dX = 0 \tag{3.8}$$

and, for any $\phi \in \mathscr{D}_{K,\gamma}^\varphi$,

$$\lim_{j\to+\infty} \int_{\mathscr{B}_1} |z|^a \nabla w_j \cdot \nabla\phi \, dX = \int_{\mathscr{B}_1} |z|^a \nabla w \cdot \nabla\phi \, dX. \tag{3.9}$$

Proof. First, we use Lemma 2.1.2 in [12] and we obtain that there exists w (with finite weighted Lebesgue norm) such that (3.8) holds true. Then, by Theorem 1.31 in [19], we obtain (3.9). It remains to show that

$$w \in \mathscr{D}_{K,\gamma}^\varphi. \tag{3.10}$$

To this goal, we first observe that $\mathbb{H}_0^s(\mathscr{B}_1)$ is closed (with respect to $\|\cdot\|_{\mathbb{H}^s(\mathscr{B}_1)}$) and convex. Hence $\mathscr{D}^\varphi$ is also closed and convex, and then so is $\mathscr{D}_{K,\gamma}^\varphi$. Therefore (3.10) follows from (3.8), (3.9) and Theorem 1.30 in [19] (applied here with $\mathscr{K} := \mathscr{D}_{K,\gamma}^\varphi$). □

Now we define

$$\mathscr{E}(v) := \int_{\mathscr{B}_1} |z|^a |\nabla v|^2 \, dX.$$

Then we have:

Theorem 3.7. *Assume that*

$$\mathscr{D}^{\varphi}_{K,\gamma} \neq \varnothing. \tag{3.11}$$

Then there exists a unique $\Phi^{\varphi}_{K,\gamma} \in \mathscr{D}^{\varphi}_{K,\gamma}$ *such that*

$$\mathscr{E}(\Phi^{\varphi}_{K,\gamma}) = \min_{v \in \mathscr{D}^{\varphi}_{K,\gamma}} \mathscr{E}(v).$$

In particular, taking $\gamma \equiv 0$*, we have that if* $\mathscr{D}^{\varphi}_{K} \neq \varnothing$ *then there exists a unique* $\Phi^{\varphi}_{K} \in \mathscr{D}^{\varphi}_{K}$ *such that*

$$\mathscr{E}(\Phi^{\varphi}_{K}) = \min_{v \in \mathscr{D}^{\varphi}_{K}} \mathscr{E}(v).$$

Proof. Let

$$\iota := \inf_{v \in \mathscr{D}^{\varphi}_{K,\gamma}} \mathscr{E}(v).$$

We take a minimizing sequence $w_j \in \mathscr{D}^{\varphi}_{K,\gamma}$ such that

$$\mathscr{E}(w_j) \leqslant \iota + e^{-j}. \tag{3.12}$$

By Lemma 3.6, up to a subsequence we have that there exists $w \in \mathscr{D}^{\varphi}_{K,\gamma}$ such that

$$\lim_{j \to +\infty} \int_{\mathscr{B}_1} |z|^a \nabla w_j \cdot \nabla \phi \, dX = \int_{\mathscr{B}_1} |z|^a \nabla w \cdot \nabla \phi \, dX,$$

for every $\phi \in \mathscr{D}^{\varphi}_{K,\gamma}$. In particular,

$$\begin{aligned}
0 &\leqslant \liminf_{j \to +\infty} \int_{\mathscr{B}_1} |z|^a |\nabla (w_j - w)|^2 \, dX \\
&= \liminf_{j \to +\infty} \int_{\mathscr{B}_1} |z|^a |\nabla w_j|^2 \, dX + \int_{\mathscr{B}_1} |z|^a |\nabla w|^2 \, dX - 2 \int_{\mathscr{B}_1} |z|^a \nabla w_j \cdot \nabla w \, dX \\
&= \liminf_{j \to +\infty} \int_{\mathscr{B}_1} |z|^a |\nabla w_j|^2 \, dX - \int_{\mathscr{B}_1} |z|^a |\nabla w|^2 \, dX \\
&= \liminf_{j \to +\infty} \mathscr{E}(w_j) - \mathscr{E}(w).
\end{aligned}$$

By inserting this into (3.12) we obtain that

$$\mathscr{E}(w) \leqslant \liminf_{j \to +\infty} \mathscr{E}(w_j) \leqslant \liminf_{j \to +\infty} \iota + e^{-j} = \iota.$$

This shows that w is the desired minimizer.

Now we show that the minimizer is unique. The proof relies on a standard convexity argument, we give the details for the facility of the reader. Suppose that we have two minimizers w_1 and w_2, and let $w := (w_1 + w_2)/2$. Notice that $w \in \mathscr{D}^{\varphi}_{K,\gamma}$ by the convexity of the space, hence

$$\mathscr{E}(w_1) = \mathscr{E}(w_2) \leqslant \mathscr{E}(w).$$

Also $w_1 - w_2 \in \mathbb{H}^s_0(\mathscr{B}_1)$, thus

$$\begin{aligned}[w_1 - w_2]^2_{\mathbb{H}^s(\mathscr{B}_1)} &= \int_{\mathscr{B}_1} |z|^a |\nabla(w_1 - w_2)|^2\, dX \\ &= \int_{\mathscr{B}_1} |z|^a \big(|\nabla w_1|^2 + |\nabla w_2|^2 - 2\nabla w_1 \cdot \nabla w_2\big)\, dX \\ &= \int_{\mathscr{B}_1} |z|^a \big(2|\nabla w_1|^2 + 2|\nabla w_2|^2 - |\nabla(w_1 + w_2)|^2\big)\, dX \\ &= 2\mathscr{E}(w_1) + 2\mathscr{E}(w_2) - 4\mathscr{E}(w) \\ &\leqslant 0.\end{aligned}$$

This, together with Lemma 3.2, shows that $w_1 = w_2$ and so it completes the proof of the uniqueness claim. □

From now on, we will implicitly assume that $\mathscr{D}^\varphi_K \neq \varnothing$. Then, the minimizer Φ^φ_K introduced in Theorem 3.7 is the fractional harmonic replacement that we consider in this paper. Roughly speaking, it is a minimizer with boundary datum φ of a fractional energy in the extended variables under the additional condition of vanishing in the set K.

3.2. *Basic properties of the fractional harmonic replacement*

In this subsection, we prove some simple, but useful, properties of the fractional harmonic replacement, such as symmetry and harmonicity properties and maximum principles.

We remark that the fractional harmonic replacement is defined in a whole $(n+1)$-dimensional set. This can be translated into subset of the halfspace $\mathbb{R}^{n+1}_+$ if the boundary datum is even in z, as the forthcoming Lemma 3.8 will point out.

Lemma 3.8. *If* $\varphi(x, -z) = \varphi(x, z)$ *then* $\Phi^\varphi_{K,\gamma}(x, -z) = \Phi^\varphi_{K,\gamma}(x, z)$.

Proof. We let $\Psi(x, z) := \Phi^\varphi_{K,\gamma}(x, -z)$. Then $\Psi \in \mathscr{D}^\varphi_{K,\gamma}$. Furthermore

$$\int_{\mathscr{B}_1} |z|^a |\nabla\Psi|^2\, dX = \int_{\mathscr{B}_1} |z|^a |\nabla\Phi^\varphi_{K,\gamma}(x, -z)|^2\, dX = \int_{\mathscr{B}_1} |z|^a |\nabla\Phi^\varphi_{K,\gamma}(x, z)|^2\, dX,$$

hence Ψ is also a minimizer for $\mathscr{E}$ in $\mathscr{D}^\varphi_{K,\gamma}$. By the uniqueness result in Theorem 3.7, we conclude that $\Psi = \Phi^\varphi_{K,\gamma}$. □

Now we write $\mathscr{D}^0_K$ to mean the functional space $\mathscr{D}^\varphi_K$ when $\varphi \equiv 0$. In this notation, we have that the fractional harmonic replacement is orthogonal to $\mathscr{D}^0_K$, as stated in the following result:

Lemma 3.9. *For every* $\psi \in \mathscr{D}^0_K$,

$$\int_{\mathscr{B}_1} |z|^a \nabla\Phi^\varphi_{K,\gamma} \cdot \nabla\psi\, dX = 0 \tag{3.13}$$

and

$$\mathscr{E}(\Phi^\varphi_{K,\gamma} \pm \psi) = \mathscr{E}(\Phi^\varphi_{K,\gamma}) + \mathscr{E}(\psi). \tag{3.14}$$

Proof. Notice that for every $\varepsilon \in (-1, 1)$, we have that $\Phi^\varphi_{K,\gamma} + \varepsilon\psi \in \mathscr{D}^\varphi_{K,\gamma}$, therefore $\mathscr{E}(\Phi^\varphi_{K,\gamma} + \varepsilon\psi) - \mathscr{E}(\Phi^\varphi_{K,\gamma}) \geqslant 0$ and then (3.13) follows.

Then, using (3.13),

$$\begin{aligned}&\mathscr{E}(\Phi^{\varphi}_{K,\gamma}\pm\psi)-\mathscr{E}(\Phi^{\varphi}_{K,\gamma})-\mathscr{E}(\psi)\\&=\int_{\mathscr{B}_1}|z|^a\big[|\nabla\Phi^{\varphi}_{K,\gamma}|^2+|\nabla\psi|^2\pm 2\nabla\Phi^{\varphi}_{K,\gamma}\cdot\nabla\psi\big]\,dX\\&\qquad-\int_{\mathscr{B}_1}|z|^a|\nabla\Phi^{\varphi}_{K,\gamma}|^2\,dX-\int_{\mathscr{B}_1}|z|^a|\nabla\psi|^2\,dX\\&=0,\end{aligned}$$

that establishes (3.14). □

Now we show that the fractional harmonic extension is indeed "harmonic" outside the constrain, i.e. it satisfies a weighted elliptic equation in the interior of $\mathscr{B}_1\setminus K$. The precise statement goes as follows:

Lemma 3.10. *We have that*

$$\operatorname{div}(|z|^a\nabla\Phi^{\varphi}_{K,\gamma})=0 \tag{3.15}$$

in the interior of $\mathscr{B}_1\setminus K$, in the distributional sense.

Proof. Let $\mathscr{N}$ be an open set contained in $\mathscr{B}_1\setminus K$. Let $\psi\in C^\infty_0(\mathscr{N})$. Then $\psi=0$ in K and so $\psi\in\mathscr{D}^0_K$. Accordingly, by (3.13),

$$\int_{\mathscr{B}_1}|z|^a\nabla\Phi^{\varphi}_{K,\gamma}\cdot\nabla\psi\,dX=0,$$

which establishes (3.15) in the distributional sense. □

The forthcoming two results in Lemmata 3.16 and 3.17 provide uniform bounds on Φ^{φ}_K by Maximum Principle. To this goal, we need the ancillary observations in the following Lemmata 3.11–3.15:

Lemma 3.11. *Let $c\in\mathbb{R}$ and $\phi\in\mathbb{H}^s(\mathscr{B}_1)$. Let $\phi_k\in\mathbb{H}^s(\mathscr{B}_1)$ be a sequence such that*

$$\lim_{k\to+\infty}\|\phi-\phi_k\|_{\mathbb{H}^s(\mathscr{B}_1)}=0. \tag{3.16}$$

Let $\psi:=(\phi-c)^+$ and $\psi_k:=(\phi_k-c)^+$. Then, up to a subsequence,

$$\lim_{k\to+\infty}\|\psi-\psi_k\|_{\mathbb{H}^s(\mathscr{B}_1)}=0.$$

Proof. First, we observe that, up to a subsequence, $\phi_k\to\phi$ a.e. in $\mathscr{B}_1$. Accordingly

$$\limsup_{k\to+\infty}\chi_{\{\phi_k>c\geqslant\phi\}}\leqslant\chi_{\{\phi=c\}}\quad\text{and}\quad\limsup_{k\to+\infty}\chi_{\{\phi>c\geqslant\phi_k\}}\leqslant\chi_{\{\phi=c\}} \tag{3.17}$$

a.e. in $\mathscr{B}_1$. Also, for any domain $\mathscr{N}$ compactly contained in $\mathscr{B}_1\setminus\{z=0\}$, we have that $\phi\in W^{1,1}_{\rm loc}(\mathscr{N})$ and so, by Stampacchia's Theorem (see e.g. Theorem 6.19 in [21]), it follows that $\nabla\phi=0$ a.e. in $\{\phi=c\}$, and so

$$|z|^a|\nabla\phi|^2\chi_{\{\phi=c\}}=0\text{ a.e. in }\mathscr{B}_1.$$

Therefore, by (3.17),

$$\lim_{k\to+\infty}|z|^a|\nabla\phi|^2\chi_{\{\phi_k>c\geqslant\phi\}}=0$$
$$\text{and}\quad\lim_{k\to+\infty}|z|^a|\nabla\phi|^2\chi_{\{\phi>c\geqslant\phi_k\}}=0.$$

Consequently, by the Dominated Convergence Theorem,

$$\begin{aligned}&\lim_{k\to+\infty}\int\limits_{\mathscr{B}_1\cap\{\phi>c\geqslant\phi_k\}}|z|^a|\nabla\phi|^2\,dX=0\\ \text{and}\quad&\lim_{k\to+\infty}\int\limits_{\mathscr{B}_1\cap\{\phi_k>c\geqslant\phi\}}|z|^a|\nabla\phi|^2\,dX=0.\end{aligned}\tag{3.18}$$

Moreover, by Corollary 2.1 in [18],

$$\begin{aligned}[\psi-\psi_k]^2_{\mathbb{H}^s(\mathscr{B}_1)}&=\int\limits_{\mathscr{B}_1}|z|^a|\nabla\psi-\nabla\psi_k|^2\,dX\\ &=\int\limits_{\mathscr{B}_1}|z|^a|\nabla(\phi-c)^+-\nabla(\phi_k-c)^+|^2\,dX\\ &=\int\limits_{\mathscr{B}_1\cap\{\phi>c\}\cap\{\phi_k>c\}}|z|^a|\nabla\phi-\nabla\phi_k|^2\,dX\\ &\quad+\int\limits_{\mathscr{B}_1\cap\{\phi>c\geqslant\phi_k\}}|z|^a|\nabla\phi|^2\,dX\\ &\quad+\int\limits_{\mathscr{B}_1\cap\{\phi_k>c\geqslant\phi\}}|z|^a|\nabla\phi_k|^2\,dX.\end{aligned}$$

We also observe that

$$|\nabla\phi_k|^2\leqslant 2\Big(|\nabla\phi_k-\nabla\phi|^2+|\nabla\phi|^2\Big)$$

and therefore

$$\begin{aligned}[\psi-\psi_k]^2_{\mathbb{H}^s(\mathscr{B}_1)}&\leqslant 3\int\limits_{\mathscr{B}_1}|z|^a|\nabla\phi-\nabla\phi_k|^2\,dX\\ &+\int\limits_{\mathscr{B}_1\cap\{\phi>c\geqslant\phi_k\}}|z|^a|\nabla\phi|^2\,dX+2\int\limits_{\mathscr{B}_1\cap\{\phi_k>c\geqslant\phi\}}|z|^a|\nabla\phi|^2\,dX.\end{aligned}$$

From this, (3.16) and (3.18), we get

$$\lim_{k\to+\infty}[\psi-\psi_k]^2_{\mathbb{H}^s(\mathscr{B}_1)}\leqslant 0.\tag{3.19}$$

Now we observe that $|z|^a|\phi-\phi_k|^2\to 0$ in $L^1(\mathscr{B}_1)$, thanks to (3.16). Therefore (see e.g. Theorem 4.9(b) in [4]), we know that, up to a subsequence,

$$|z|^a|\phi-\phi_k|^2\leqslant h,$$

for every $k\in\mathbb{N}$, with $h\in L^1(\mathscr{B}_1)$. As a consequence,

$$|z|^{\frac{a}{2}}|\phi_k|\leqslant|z|^{\frac{a}{2}}|\phi-\phi_k|+|z|^{\frac{a}{2}}|\phi|\leqslant\sqrt{h}+|z|^{\frac{a}{2}}|\phi|.$$

Consequently,

$$|z|^{\frac{a}{2}}|\psi-\psi_k|\leqslant|z|^{\frac{a}{2}}\Big(|\phi|+|\phi_k|+2|c|\Big)\leqslant 2|z|^{\frac{a}{2}}\Big(|\phi|+|c|\Big)+\sqrt{h}$$

and thus

$$|z|^a|\psi-\psi_k|^2 \leqslant C\left[|z|^a\left(|\phi|^2+c^2\right)+h\right]=: g,$$

with $g \in L^1(\mathscr{B}_1)$. So, by the Dominated Convergence Theorem,

$$\lim_{k\to+\infty}\int_{\mathscr{B}_1}|z|^a|\psi-\psi_k|^2\,dX=0.$$

This formula and (3.19) imply the desired result. □

We need now a technical modification of Lemma 3.11. Namely, given $\phi\in\mathbb{H}^s(\mathscr{B}_1)$, in order to approximate ϕ^+ in $\mathbb{H}^s(\mathscr{B}_1)$ it is not always convenient to consider the positive parts of the approximating sequence (as done in Lemma 3.11), since taking positive parts may decrease the regularity of the smooth functions. To avoid this, we introduce a smooth modification of an approximating sequence, which still converges to the positive part in the limit. The key step in this procedure is given by the following result:

Lemma 3.12. *Let $\phi\in\mathbb{H}^s(\mathscr{B}_1)$ and fix $\varepsilon>0$. Then, there exist $\overline{\theta}_\varepsilon$, $\underline{\theta}_\varepsilon\in C^\infty(\mathbb{R})$ such that $\underline{\theta}_\varepsilon(t)\leqslant t^+\leqslant\overline{\theta}_\varepsilon(t)$ for any $t\in\mathbb{R}$ and*

$$\|\phi^+-\overline{\theta}_\varepsilon(\phi)\|_{\mathbb{H}^s(\mathscr{B}_1)}+\|\phi^+-\underline{\theta}_\varepsilon(\phi)\|_{\mathbb{H}^s(\mathscr{B}_1)}\leqslant\varepsilon. \tag{3.20}$$

Proof. Let $\tau\in C^\infty(\mathbb{R},[0,1])$ such that $\tau(t)=0$ for any $t\leqslant 1/2$, and $\tau(t)=1$ for any $t\geqslant 3/4$. Let also $\Theta(t):=t\,\tau(t)$ and

$$\underline{\theta}_\varepsilon(t):=\varepsilon\Theta\left(\frac{t}{\varepsilon}\right).$$

By construction, $\Theta(t)\leqslant t^+$ and so $\underline{\theta}_\varepsilon(t)\leqslant t^+$ for any $t\in\mathbb{R}$.

Moreover,

$$|\underline{\theta}'_\varepsilon|\leqslant C, \tag{3.21}$$

for some $C>0$, and

$$\underline{\theta}_\varepsilon(t)=t^+ \text{ for any } |t|\geqslant\varepsilon. \tag{3.22}$$

Now we take a nondecreasing function $\mu\in C^\infty(\mathbb{R})$ such that $\mu(t)=0$ if $t\leqslant -1/100$, $\mu(t)\in(0,1)$ for any $t\in(-1/100,1/100)$ and $\mu(t)=1$ for any $t\geqslant 1/100$. Notice that

$$\iota:=\int_{-\infty}^{\frac{1}{100}}\mu(t)\,dt\leqslant\frac{1}{50}. \tag{3.23}$$

For any $r>0$, we define

$$\mu_r(t):=\mu\left(t-\frac{99}{100}+r\right).$$

We observe that $\mu_r(t)=0$ if $t\leqslant(98/100)-r$, $\mu_r(t)\in(0,1)$ for any $t\in((98/100)-r,1-r)$ and $\mu_r(t)=1$ for any $t\geqslant 1-r$.

We claim that

$$\text{there exists } r\in[0,1] \text{ such that } \int_{-\infty}^{1}\mu_r(t)\,dt=1. \tag{3.24}$$

To prove this, notice that, using the change of variable $\tilde{t}=t-\frac{99}{100}+r$ and recalling (3.23),

$$\int_{-\infty}^{1} \mu_r(t)\,dt = \int_{-\infty}^{1} \mu\left(t - \frac{99}{100} + r\right)\,dt$$
$$= \int_{-\infty}^{\frac{1}{100}+r} \mu(\tilde t)\,d\tilde t = \int_{-\infty}^{\frac{1}{100}} \mu(\tilde t)\,d\tilde t + \int_{\frac{1}{100}}^{\frac{1}{100}+r} \mu(\tilde t)\,d\tilde t$$
$$= \iota + \int_{\frac{1}{100}}^{\frac{1}{100}+r} 1\,d\tilde t = \iota + r.$$

Now, if $r = 0$ then $\iota \leqslant 1/50$, thanks to (3.23), and if $r = 1$ then $\iota + 1 \geqslant 1$, since $\iota \geqslant 0$. So, by continuity, we obtain the claim in (3.24).

Notice that the parameter r given by (3.24) will be considered as fixed from now on. We define

$$T(t) := \int_{-\infty}^{t} \mu_r(\rho)\,d\rho.$$

We claim that

$$T(t) = t^+ \text{ for any } |t| \geqslant 1. \tag{3.25}$$

Indeed, if $t \leqslant -1$ then $t \leqslant (98/100) - r$ and so we have that $T(t) = 0 = t^+$, since the integrand vanishes. Also, if $t \geqslant 1$ then

$$T(t) = \int_{-\infty}^{1} \mu_r(\rho)\,d\rho + \int_{1}^{t} \mu_r(\rho)\,d\rho = 1 + \int_{1}^{t} 1\,d\rho = t,$$

where (3.24) was used. This proves (3.25).

We also claim that

$$T(t) \geqslant t^+ \text{ for any } t \in \mathbb{R}. \tag{3.26}$$

To prove it, we notice that it is enough to consider the case $t \in (-1, 1)$, in view of (3.25). Moreover, $T(t) \geqslant 0 = t^+$ for any $t \leqslant 0$, so we can focus on the case $t \in (0, 1)$. For this, for any $t \in (0, 1)$, we let $H(t) := T(t) - t^+ = T(t) - t$. Then

$$H'(t) = T'(t) - 1 = \mu_r(t) - 1 \leqslant 0.$$

Therefore, for any $t \in (0, 1)$,

$$T(t) - t^+ = H(t) \geqslant H(1) = T(1) - 1 = 0,$$

due to (3.25), and this completes the proof of (3.26).

Now we define

$$\overline{\theta}_\varepsilon(t) := \varepsilon T\left(\frac{t}{\varepsilon}\right).$$

From (3.26), we know that $\overline{\theta}_\varepsilon(t) \geqslant t^+$ for any $t \in \mathbb{R}$. Also,

$$|\overline{\theta}'_\varepsilon| \leqslant C, \tag{3.27}$$

for some $C > 0$, and we deduce from (3.25) that

$$\overline{\theta}_\varepsilon(t) = t^+ \text{ for any } |t| \geqslant \varepsilon. \tag{3.28}$$

Having completed the construction of $\overline{\theta}_\varepsilon$ and $\underline{\theta}_\varepsilon$, we now prove (3.20). To this goal, by Lemma 2.1 in [18], we have that $\nabla(\underline{\theta}_\varepsilon(\phi)) = \underline{\theta}'_\varepsilon(\phi)\nabla\phi$, therefore

$$\begin{aligned} \|\phi^+ - \underline{\theta}_\varepsilon(\phi)\|^2_{\mathbb{H}^s(\mathscr{B}_1)} &= \int_{\mathscr{B}_1} |z|^a \left|\nabla\phi^+ - \underline{\theta}'_\varepsilon(\phi)\nabla\phi\right|^2 dX \\ &= \int_{\mathscr{B}_1\cap\{|\phi|<\varepsilon\}} |z|^a \left|\nabla\phi^+ - \underline{\theta}'_\varepsilon(\phi)\nabla\phi\right|^2 dX, \end{aligned} \tag{3.29}$$

since the other contributions cancel, thanks to (3.22).

We also use (3.21) to see that $|z|^a\left|\nabla\phi^+ - \underline{\theta}'_\varepsilon(\phi)\nabla\phi\right|^2 \chi_{\{|\phi|<\varepsilon\}} \leqslant C\,|z|^a|\nabla\phi|^2 \in L^1(\mathscr{B}_1)$, since $\phi\in\mathbb{H}^s(\mathscr{B}_1)$, therefore, by the Dominated Convergence Theorem and the Theorem of Stampacchia (see e.g. Theorem 6.19 in [21]), we have

$$\lim_{\varepsilon\to 0} \int_{\mathscr{B}_1\cap\{|\phi|<\varepsilon\}} |z|^a \left|\nabla\phi^+ - \underline{\theta}'_\varepsilon(\phi)\nabla\phi\right|^2 dX \leqslant C \int_{\mathscr{B}_1\cap\{\phi=0\}} |z|^a|\nabla\phi|^2\, dX = 0.$$

This and (3.29) give that

$$\lim_{\varepsilon\to 0}[\phi^+ - \underline{\theta}_\varepsilon(\phi)]^2_{\mathbb{H}^s(\mathscr{B}_1)} = 0. \tag{3.30}$$

Now we observe that $|\underline{\theta}_\varepsilon(t)| \leqslant C(1+|t|)$, due to (3.21), and therefore, by the Dominated Convergence Theorem,

$$\lim_{\varepsilon\to 0}\int_{\mathscr{B}_1} |z|^a|\phi^+ - \underline{\theta}_\varepsilon(\phi)|^2\, dX = 0.$$

This and (3.30) imply that

$$\lim_{\varepsilon\to 0}\|\phi^+ - \underline{\theta}_\varepsilon(\phi)\|^2_{\mathbb{H}^s(\mathscr{B}_1)} = 0. \tag{3.31}$$

In a similar way (using (3.27) and (3.28) instead of (3.21) and (3.22)), we obtain that

$$\lim_{\varepsilon\to 0}\|\phi^+ - \overline{\theta}_\varepsilon(\phi)\|^2_{\mathbb{H}^s(\mathscr{B}_1)} = 0.$$

This and (3.31) give (3.20) (up to renaming ε). $\square$

As a consequence of Lemmata 3.11 and 3.12 we have the following smooth approximation result for the positive part:

Corollary 3.13. *Let $c\in\mathbb{R}$ and $\phi\in\mathbb{H}^s(\mathscr{B}_1)$. Let $\phi_k\in\mathbb{H}^s(\mathscr{B}_1)$ be a sequence such that*

$$\lim_{k\to+\infty}\|\phi - \phi_k\|_{\mathbb{H}^s(\mathscr{B}_1)} = 0.$$

Then, there exist sequences of functions $\overline{\theta}_k$, $\underline{\theta}_k \in C^\infty(\mathbb{R})$ such that $\underline{\theta}_k(t) \leqslant t^+ \leqslant \overline{\theta}_k(t)$ for any $t\in\mathbb{R}$ and

$$\lim_{k\to+\infty}\|(\phi-c)^+ - \underline{\theta}_k(\phi_k - c)\|_{\mathbb{H}^s(\mathscr{B}_1)} = 0 \tag{3.32}$$

and

$$\lim_{k\to+\infty}\|(\phi-c)^+ - \overline{\theta}_k(\phi_k - c)\|_{\mathbb{H}^s(\mathscr{B}_1)} = 0. \tag{3.33}$$

Proof. First we use Lemma 3.11 to say that

$$\lim_{k\to+\infty}\|(\phi-c)^+ - (\phi_k - c)^+\|_{\mathbb{H}^s(\mathscr{B}_1)} = 0.$$

Now, fixed $k\in\mathbb{N}$, we use Lemma 3.12 to find $\overline{\theta}_k$, $\underline{\theta}_k\in C^\infty(\mathbb{R})$ such that $\underline{\theta}_k(t)\leqslant t^+\leqslant\overline{\theta}_k(t)$ for any $t\in\mathbb{R}$ and

$$\|(\phi_k - c)^+ - \overline{\theta}_k(\phi_k - c)\|_{\mathbb{H}^s(\mathscr{B}_1)} + \|(\phi_k - c)^+ - \underline{\theta}_k(\phi_k - c)\|_{\mathbb{H}^s(\mathscr{B}_1)} \leqslant e^{-k}.$$

These considerations and the triangle inequality imply (3.32) and (3.33), as desired. □

With this, we can now prove the following result:

Lemma 3.14. *Let $g, \varphi \in \mathbb{H}^s(\mathscr{B}_2)$ with $g - \varphi \in \mathbb{H}^s_0(\mathscr{B}_1)$. Let also $c \geqslant \sup_{\mathscr{B}_1} \varphi$. Then $(g-c)^+ \in \mathbb{H}^s_0(\mathscr{B}_1)$.*

Proof. By construction $g - c \in \mathbb{H}^s(\mathscr{B}_1)$. Thus, by Corollary 2.1 in [18], we have that $(g-c)^+ \in \mathbb{H}^s(\mathscr{B}_1)$. Moreover, there exist sequences $f_k \in C^\infty_0(\mathscr{B}_1)$ and $\varphi_k \in C^\infty(\mathscr{B}_1)$ such that $f_k \to g - \varphi$ and $\varphi_k \to \varphi$ in $\mathbb{H}^s(\mathscr{B}_1)$ as $k \to +\infty$, respectively.

Now, we define $\tilde{\varphi}_k := \varphi_k - \overline{\theta}_k(\varphi_k - c)$, where $\overline{\theta}_k$ is the smooth function given by Corollary 3.13. Notice that $\tilde{\varphi}_k \in C^\infty(\mathscr{B}_1)$. Also, by Corollary 3.13, we have that $\overline{\theta}_k(\varphi_k - c) \to (\varphi - c)^+ = 0$ in $\mathbb{H}^s(\mathscr{B}_1)$, therefore $\tilde{\varphi}_k \to \varphi$ in $\mathbb{H}^s(\mathscr{B}_1)$, as $k \to +\infty$.

Now we define $h_k := \underline{\theta}_k(f_k + \tilde{\varphi}_k - c - e^{-k})$, where $\underline{\theta}_k$ is given by Corollary 3.13. Notice that $h_k \in C^\infty(\mathscr{B}_1)$. Also, the support of h_k is compactly contained inside $\mathscr{B}_1$, since $\tilde{\varphi}_k \leqslant \varphi_k - (\varphi_k - c)^+ = \min\{\varphi_k, c\} \leqslant c$ (recall that $\overline{\theta}_k(t) \geqslant t^+$ for any $t \in \mathbb{R}$) and the support of f_k is compactly contained inside $\mathscr{B}_1$. Therefore, we have that $h_k \in C^\infty_0(\mathscr{B}_1)$. Also, by Corollary 3.13, we have that $h_k \to ((g-\varphi) + \varphi - c)^+ = (g-c)^+$ in $\mathbb{H}^s(\mathscr{B}_1)$. This implies that $(g-c)^+ \in \mathbb{H}^s_0(\mathscr{B}_1)$. □

For further reference, we point out that a statement analogous to Lemma 3.14 holds when the positive part is replaced with the negative part of the functions:

Lemma 3.15. *Let $g, \varphi \in \mathbb{H}^s(\mathscr{B}_2)$ with $g - \varphi \in \mathbb{H}^s_0(\mathscr{B}_1)$. Let also $c \leqslant \inf_{\mathscr{B}_1} \varphi$. Then $(g-c)^- \in \mathbb{H}^s_0(\mathscr{B}_1)$.*

Now we establish pointwise bounds, from above and below, of the fractional harmonic replacement:

Lemma 3.16. *We have that*

$$\Phi^\varphi_{K,\gamma} \leqslant \max\left\{\sup_{\mathscr{B}_1} \varphi,\ \sup_K \gamma\right\}.$$

Proof. Let

$$c := \max\left\{\sup_{\mathscr{B}_1} \varphi,\ \sup_K \gamma\right\}$$

and $\psi := (\Phi^\varphi_{K,\gamma} - c)^+$. By Lemma 3.14, we know that $\psi \in \mathbb{H}^s_0(\mathscr{B}_1)$. Also, a.e. in K,

$$\psi = (\Phi^\varphi_{K,\gamma} - c)^+ = (\gamma - c)^+ = 0$$

in the sense of traces, hence $\psi \in \mathscr{D}^0_K$. As a consequence, using (3.13),

$$0 = \int_{\mathscr{B}_1} |z|^a \nabla \Phi^\varphi_{K,\gamma} \cdot \nabla \psi \, dX = \int_{\mathscr{B}_1 \cap \{\Phi^\varphi_{K,\gamma} > c\}} |z|^a |\nabla \Phi^\varphi_{K,\gamma}|^2 \, dX,$$

which gives the desired result. □

Lemma 3.17. *If $\varphi \geqslant 0$ and $\gamma \geqslant 0$, then $\Phi^\varphi_{K,\gamma} \geqslant 0$.*

Proof. Let $\psi := (-\Phi^\varphi_{K,\gamma})^+ = (\Phi^\varphi_{K,\gamma})^-$. By Corollary 2.1 in [18] we have that $\psi \in \mathbb{H}^s(\mathscr{B}_1)$, and, using Lemma 3.15 with $c := 0$, we have that $\psi \in \mathbb{H}^s_0(\mathscr{B}_1)$. Also, a.e. in K, we have that $\psi = (\Phi^\varphi_{K,\gamma})^- = (\gamma)^- = 0$ in the trace sense. As a consequence, $\psi \in \mathscr{D}^0_K$, thus we can use (3.13) and conclude that

$$0=\int_{\mathscr{B}_1}|z|^a\nabla\Phi^{\varphi}_{K,\gamma}\cdot\nabla\psi\,dX=-\int_{\mathscr{B}_1\cap\{\Phi^{\varphi}_{K,\gamma}<0\}}|z|^a|\nabla\Phi^{\varphi}_{K,\gamma}|^2\,dX,$$

which gives the desired result. □

3.3. Relaxation of the functional spaces and subharmonicity properties

The purpose of this subsection is to relax the functional prescription in the space $\mathscr{D}^{\varphi}_K$ by allowing approximating sequences to take also negative values in K. This observation will be exploited to deduce subharmonicity properties of Φ^{φ}_K and it will also play a role in the proof of the monotonicity statement of Theorem 3.20. For this scope, we define

$$\tilde{\mathscr{D}}^{\varphi}_K:=\Big\{v\in\mathscr{D}^{\varphi}\text{ s.t. }v\leqslant 0\text{ a.e. in }K\Big\}. \tag{3.34}$$

The reader may compare this definition with (3.6): the only difference is that in (3.6) the function is forced to vanish on K, while in the latter setting it can also attain negative values on K. Of course, $\tilde{\mathscr{D}}^{\varphi}_K\supseteq\mathscr{D}^{\varphi}_K$, therefore

$$\inf_{v\in\tilde{\mathscr{D}}^{\varphi}_K}\mathscr{E}(v)\leqslant\min_{v\in\mathscr{D}^{\varphi}_K}\mathscr{E}(v)=\mathscr{E}(\Phi^{\varphi}_K).$$

We will show that in fact equality holds if $\varphi\geqslant 0$:

Lemma 3.18. *If $\varphi\geqslant 0$, then*

$$\min_{v\in\tilde{\mathscr{D}}^{\varphi}_K}\mathscr{E}(v)=\min_{v\in\mathscr{D}^{\varphi}_K}\mathscr{E}(v)=\mathscr{E}(\Phi^{\varphi}_K).$$

Proof. Let $v\in\tilde{\mathscr{D}}^{\varphi}_K$. Since $|\nabla v^+|\leqslant|\nabla v|$, we have that $\mathscr{E}(v^+)\leqslant\mathscr{E}(v)$. So, to prove the desired result, we only have to show that

$$v^+\in\mathscr{D}^{\varphi}_K. \tag{3.35}$$

For this, we note that $v^+\in\mathbb{H}^s(\mathscr{B}_1)$, thanks to Corollary 2.1 in [18]. Now we claim that

$$v^+-\varphi\in\mathbb{H}^s_0(\mathscr{B}_1). \tag{3.36}$$

For this, we use the sequences $f_k\in C^\infty_0(\mathscr{B}_1)$ and $\varphi_k\in C^\infty(\mathscr{B}_1)$ that converge, respectively, to $v-\varphi$ and φ in $\mathbb{H}^s(\mathscr{B}_1)$, as $k\to+\infty$.

We define $g_k:=f_k+\overline{\theta}_k(\varphi_k)$, where $\overline{\theta}_k$ is given by Corollary 3.13. Hence, by Corollary 3.13, we know that $\overline{\theta}_k(\varphi_k)\to\varphi^+=\varphi$ in $\mathbb{H}^s(\mathscr{B}_1)$. Therefore $g_k\to(v-\varphi)+\varphi=v$ in $\mathbb{H}^s(\mathscr{B}_1)$.

As a consequence, using again Corollary 3.13, we obtain that $\overline{\theta}_k(g_k)\to v^+$ in $\mathbb{H}^s(\mathscr{B}_1)$.

Let now $h_k:=\overline{\theta}_k(g_k)-\overline{\theta}_k\big(\overline{\theta}_k(\varphi_k)\big)$. We have that $h_k\to v^+-\varphi$. We also notice that $f_k=0$ outside a compact subset $\mathscr{K}_k$ contained inside $\mathscr{B}_1$. Hence $g_k=\overline{\theta}_k(\varphi_k)$ outside $\mathscr{K}_k$. Therefore $h_k=\overline{\theta}_k(g_k)-\overline{\theta}_k\big(\overline{\theta}_k(\varphi_k)\big)=\overline{\theta}_k\big(\overline{\theta}_k(\varphi_k)\big)-\overline{\theta}_k\big(\overline{\theta}_k(\varphi_k)\big)=0$ outside $\mathscr{K}_k$. This shows that $h_k\in C^\infty_0(\mathscr{B}_1)$ and it completes the proof of (3.36).

Now we observe that $v^+=0$ a.e. in K in the trace sense. This and (3.36) complete the proof of (3.35) and so of Lemma 3.18. □

While Lemma 3.10 gives that the harmonic replacement is "harmonic" apart from K, next result states that it is "subharmonic" in the whole of the domain if the boundary datum is nonnegative:

Lemma 3.19. *If $\varphi\geqslant 0$, then for every $\psi\in\mathbb{H}^s_0(\mathscr{B}_1)$ with $\psi\geqslant 0$ a.e. in $\mathscr{B}_1$, we have that*

$$\int_{\mathscr{B}_1}|z|^a\nabla\Phi^{\varphi}_K\cdot\nabla\psi\,dX\leqslant 0.$$

Proof. Given $\varepsilon > 0$, we set $\psi_\varepsilon := \Phi_K^\varphi - \varepsilon\psi$. Since $\Phi_K^\varphi - \varphi \in \mathbb{H}_0^s(\mathscr{B}_1)$ and $\psi \in \mathbb{H}_0^s(\mathscr{B}_1)$, we have that $\psi_\varepsilon - \varphi \in \mathbb{H}_0^s(\mathscr{B}_1)$. Furthermore, a.e. in K, we have that $\psi_\varepsilon = -\varepsilon\psi \leqslant 0$ in the trace sense, therefore $\psi_\varepsilon \in \tilde{\mathscr{D}}_K^\varphi$.

From this and Lemma 3.18, it follows that $\mathscr{E}(\psi_\varepsilon) - \mathscr{E}(\Phi_K^\varphi) \geqslant 0$ and this gives the desired result. □

For our purposes we will never use Lemma 3.19, but we stated and proved it since it can be a useful consequence of the theory developed so far in Section 3.

3.4. A monotonicity property for the fractional harmonic replacement

Now we show that the fractional harmonic replacement enjoys a monotonicity property with respect to its boundary data and the constrain:

Theorem 3.20. *Let $\mathbb{H}^s(\mathscr{B}_1) \ni \varphi_2 \geqslant \varphi_1 \geqslant 0$. Let also $K_2 \subseteq K_1 \subseteq B_{\frac{9}{10}}$ and $A_1 \subseteq A_2 \Subset B_{\frac{9}{10}}$. Then*

$$\mathscr{E}(\Phi^{\varphi_1}_{K_1\cup A_1}) - \mathscr{E}(\Phi^{\varphi_1}_{K_1}) \leqslant \mathscr{E}(\Phi^{\varphi_2}_{K_2\cup A_2}) - \mathscr{E}(\Phi^{\varphi_2}_{K_2}).$$

Proof. We consider the minimization problem in $\mathscr{D}^{\varphi_2}_{K_1,\,\Phi^{\varphi_2}_{K_2\cup A_2}}$. In the notation of Theorem 3.7, the associated minimizer will be denoted by $\Phi^{\varphi_2}_{K_1,\,\Phi^{\varphi_2}_{K_2\cup A_2}}$.

We claim that

$$\Phi^{\varphi_1}_{K_1} \leqslant \Phi^{\varphi_2}_{K_1,\,\Phi^{\varphi_2}_{K_2\cup A_2}}. \tag{3.37}$$

To prove this, we let $g := \Phi^{\varphi_1}_{K_1} - \Phi^{\varphi_2}_{K_1,\,\Phi^{\varphi_2}_{K_2\cup A_2}}$ and $\varphi := \varphi_1 - \varphi_2$. Notice that $\sup_{\mathscr{B}_1} \varphi \leqslant 0$, thus we can use Lemma 3.14 (with $c := 0$) and conclude that $h := g^+ \in \mathbb{H}_0^s(\mathscr{B}_1)$. Furthermore, in the trace sense, a.e. in K_1 we have that $h = (0 - \Phi^{\varphi_2}_{K_1,\,\Phi^{\varphi_2}_{K_2\cup A_2}})^+ \leqslant 0$, thanks to Lemma 3.17, and so

$$h \in \mathscr{D}^0_{K_1} \tag{3.38}$$

Consequently, for every $\delta \in (-1, 1)$, we conclude that $\Phi^{\varphi_2}_{K_1,\,\Phi^{\varphi_2}_{K_2\cup A_2}} + \delta h \in \mathscr{D}^{\varphi_2}_{K_1,\,\Phi^{\varphi_2}_{K_2\cup A_2}}$ and then, by the minimizing property of $\Phi^{\varphi_2}_{K_1,\,\Phi^{\varphi_2}_{K_2\cup A_2}}$, it follows that $\mathscr{E}(\Phi^{\varphi_2}_{K_1,\,\Phi^{\varphi_2}_{K_2\cup A_2}}) \leqslant \mathscr{E}(\Phi^{\varphi_2}_{K_1,\,\Phi^{\varphi_2}_{K_2\cup A_2}} + \delta h)$.

This implies that

$$\int_{\mathscr{B}_1} |z|^a \nabla \Phi^{\varphi_2}_{K_1,\,\Phi^{\varphi_2}_{K_2\cup A_2}} \cdot \nabla h \, dX = 0.$$

Hence, we have

$$\begin{aligned}\mathscr{E}(h) &= \int_{\mathscr{B}_1} |z|^a \nabla(\Phi^{\varphi_1}_{K_1} - \Phi^{\varphi_2}_{K_1,\,\Phi^{\varphi_2}_{K_2\cup A_2}}) \cdot \nabla h \, dX \\ &= \int_{\mathscr{B}_1} |z|^a \nabla \Phi^{\varphi_1}_{K_1} \cdot \nabla h \, dX.\end{aligned}$$

Thus, recalling (3.38) and (3.13), we obtain that $\mathscr{E}(h) = 0$. This, together with Lemma 3.2, implies that h vanishes and establishes (3.37).

Now we set

$$\eta := \Phi^{\varphi_2}_{K_1,\,\Phi^{\varphi_2}_{K_2\cup A_2}} - \Phi^{\varphi_2}_{K_2\cup A_2}. \tag{3.39}$$

Notice that $\Phi^{\varphi_2}_{K_1,\,\Phi^{\varphi_2}_{K_2\cup A_2}} - \varphi_2$ and $\Phi^{\varphi_2}_{K_2\cup A_2} - \varphi_2$ belong to $\mathbb{H}_0^s(\mathscr{B}_1)$, hence so does η. Moreover, a.e. in K_1, in the sense of traces, we have that $\eta = \Phi^{\varphi_2}_{K_2\cup A_2} - \Phi^{\varphi_2}_{K_2\cup A_2} = 0$. This says that

$$\eta \in \mathscr{D}^0_{K_1} \tag{3.40}$$

and so we can use (3.14) (with $\psi := \eta$ here) and conclude that

$$\mathscr{E}(\Phi^{\varphi_2}_{K_1,\, \Phi^{\varphi_2}_{K_2\cup A_2}} - \eta) = \mathscr{E}(\Phi^{\varphi_2}_{K_1,\, \Phi^{\varphi_2}_{K_2\cup A_2}}) + \mathscr{E}(\eta).$$

Thus, from (3.39),

$$\begin{aligned} \mathscr{E}(\eta) &= \mathscr{E}(\Phi^{\varphi_2}_{K_1,\, \Phi^{\varphi_2}_{K_2\cup A_2}} - \eta) - \mathscr{E}(\Phi^{\varphi_2}_{K_1,\, \Phi^{\varphi_2}_{K_2\cup A_2}}) \\ &= \mathscr{E}(\Phi^{\varphi_2}_{K_2\cup A_2}) - \mathscr{E}(\Phi^{\varphi_2}_{K_1,\, \Phi^{\varphi_2}_{K_2\cup A_2}}). \end{aligned} \tag{3.41}$$

Now, since $K_1 \supseteq K_2$ and $\Phi^{\varphi_2}_{K_2\cup A_2} = 0$ a.e. in K_2, we have that

$$\Phi^{\varphi_2}_{K_1,\, \Phi^{\varphi_2}_{K_2\cup A_2}} \in \mathscr{D}^{\varphi_2}_{K_1,\, \Phi^{\varphi_2}_{K_2\cup A_2}} \subseteq \mathscr{D}^{\varphi_2}_{K_2,\, \Phi^{\varphi_2}_{K_2\cup A_2}} = \mathscr{D}^{\varphi_2}_{K_2,\,0} = \mathscr{D}^{\varphi_2}_{K_2}$$

and so

$$\mathscr{E}(\Phi^{\varphi_2}_{K_1,\, \Phi^{\varphi_2}_{K_2\cup A_2}}) \geqslant \mathscr{E}(\Phi^{\varphi_2}_{K_2}),$$

thanks to the minimality of $\Phi^{\varphi_2}_{K_2}$.

This and (3.41) imply that

$$\mathscr{E}(\eta) \leqslant \mathscr{E}(\Phi^{\varphi_2}_{K_2\cup A_2}) - \mathscr{E}(\Phi^{\varphi_2}_{K_2}). \tag{3.42}$$

On the other hand, from Lemma 3.18, we know that

$$\mathscr{E}(\Phi^{\varphi_1}_{K_1\cup A_1}) = \min_{v \in \tilde{\mathscr{D}}^{\varphi_1}_{K_1\cup A_1}} \mathscr{E}(v).$$

Therefore, calling $\psi := \Phi^{\varphi_1}_{K_1} - v$, we have that

$$\mathscr{E}(\Phi^{\varphi_1}_{K_1\cup A_1}) = \min_{\psi \in \Phi^{\varphi_1}_{K_1} - \tilde{\mathscr{D}}^{\varphi_1}_{K_1\cup A_1}} \mathscr{E}(\Phi^{\varphi_1}_{K_1} - \psi). \tag{3.43}$$

Now we claim that

$$\eta \in \Phi^{\varphi_1}_{K_1} - \tilde{\mathscr{D}}^{\varphi_1}_{K_1\cup A_1}. \tag{3.44}$$

For this, we recall (3.39), and we have that

$$\tilde{\eta} := \Phi^{\varphi_1}_{K_1} - \eta = \Phi^{\varphi_2}_{K_2\cup A_2} - \Phi^{\varphi_2}_{K_1,\, \Phi^{\varphi_2}_{K_2\cup A_2}} + \Phi^{\varphi_1}_{K_1}.$$

From this, it follows that $\tilde{\eta} - \varphi_1 \in \mathbb{H}^s_0(\mathscr{B}_1)$. Also, a.e. in K_1, we have that $\tilde{\eta} = \Phi^{\varphi_2}_{K_2\cup A_2} - \Phi^{\varphi_2}_{K_2\cup A_2} + 0 = 0$, in the trace sense. Moreover, a.e. in $A_1 \subseteq A_2$, we have that $\tilde{\eta} = 0 - \Phi^{\varphi_2}_{K_1,\, \Phi^{\varphi_2}_{K_2\cup A_2}} + \Phi^{\varphi_1}_{K_1} \leqslant 0$, where (3.37) has been exploited. These observations imply that $\tilde{\eta} \in \tilde{\mathscr{D}}^{\varphi_1}_{K_1\cup A_1}$, which in turn implies (3.44).

From (3.43) and (3.44), we obtain that

$$\mathscr{E}(\Phi^{\varphi_1}_{K_1\cup A_1}) \leqslant \mathscr{E}(\Phi^{\varphi_1}_{K_1} - \eta). \tag{3.45}$$

Moreover, by (3.40) and (3.14) (used here with $\psi := \eta$), we have that

$$\mathscr{E}(\Phi^{\varphi_1}_{K_1} - \eta) = \mathscr{E}(\Phi^{\varphi_1}_{K_1}) + \mathscr{E}(\eta).$$

Thus, formula (3.45) becomes

$$\mathscr{E}(\Phi^{\varphi_1}_{K_1\cup A_1}) - \mathscr{E}(\Phi^{\varphi_1}_{K_1}) \leqslant \mathscr{E}(\Phi^{\varphi_1}_{K_1} - \eta) - \mathscr{E}(\Phi^{\varphi_1}_{K_1}) = \mathscr{E}(\eta).$$

Therefore, recalling (3.42),

$$\mathscr{E}(\Phi^{\varphi_1}_{K_1\cup A_1}) - \mathscr{E}(\Phi^{\varphi_1}_{K_1}) \leqslant \mathscr{E}(\Phi^{\varphi_2}_{K_2\cup A_2}) - \mathscr{E}(\Phi^{\varphi_2}_{K_2}).$$

This concludes the proof of Theorem 3.20. □

4. Energy estimates for the fractional harmonic replacement

The goal of this section is to prove that the energy of the fractional harmonic replacement in $K \cup A$ is controlled by the energy of the fractional harmonic replacement in K, plus a term of the order of the n-dimensional measure of the additional set A. The precise statement of this result goes as follows:

Theorem 4.1. *Let $\varphi \geqslant 0$ and $\rho \in [1/4, 3/4]$. Let $K \subseteq \mathscr{B}_1 \cap \{z = 0\}$ and $A := B_\rho \setminus K$. Then*

$$\mathscr{E}(\Phi^{\varphi}_{K\cup A}) - \mathscr{E}(\Phi^{\varphi}_{K}) \leqslant C\,|A|\,\|\varphi\|^2_{L^\infty(\mathscr{B}_1)},$$

for some $C > 0$ that depends on n and s.

In the local case of the classical harmonic replacement, a statement similar to the one in Theorem 4.1 was obtained in Lemma 2.3 of [6]. Also, a fractional case in a different setting was dealt with in Theorem 1.3 of [16] (as a matter of fact, the right hand side of the estimate obtained here is more precise than the one in [16] since it only depends on the values of φ in a fixed ball, and this plays an important role in the blow-up analysis of the problem).

To proof Theorem 4.1, we will reduce to the radial case. For this, we will first show that a suitable radial rearrangement decreases the energy and then estimate the energy in the radial case. An important step of the proof is also obtained by using the monotonicity property of Theorem 3.20, in order to reduce to the case of constant Dirichlet datum. The following subsections contain the details of this strategy.

4.1. Symmetric rearrangements

In this subsection, we will consider the symmetric rearrangement in the variable $x \in \mathbb{R}^n$, for a fixed $z \in \mathbb{R}$. In the forthcoming Theorem 4.3 we will show that this rearrangement decreases the energy.

To this goal, we first state a useful density property of polynomials in the space we work with.

Lemma 4.2. *Let $v \in \mathbb{H}^s(\mathscr{B}_1)$ and $\varepsilon > 0$. Then there exists a polynomial p_ε such that*

$$\|v - p_\varepsilon\|_{\mathbb{H}^s(\mathscr{B}_1)} \leqslant \varepsilon.$$

Proof. By the definition of $\mathbb{H}^s(\mathscr{B}_1)$ given in Subsection 3.1, we have that there exists $w_\varepsilon \in C^\infty(\mathscr{B}_1)$ such that $\|v - w_\varepsilon\|_{\mathbb{H}^s(\mathscr{B}_1)} \leqslant \varepsilon$. Moreover, by the Stone–Weierstraß Theorem (see e.g. Lemma 2.1 in [13]), we have that there exists a polynomial p_ε such that $\|w_\varepsilon - p_\varepsilon\|_{C^1(\mathscr{B}_1)} \leqslant \varepsilon$. Therefore

$$\begin{aligned}\|w_\varepsilon - p_\varepsilon\|_{\mathbb{H}^s(\mathscr{B}_1)} &= \sqrt{\int_{\mathscr{B}_1} |z|^a |w_\varepsilon - p_\varepsilon|^2\,dX} + \sqrt{\int_{\mathscr{B}_1} |z|^a |\nabla w_\varepsilon - \nabla p_\varepsilon|^2\,dX} \\ &\leqslant \|w_\varepsilon - p_\varepsilon\|_{C^1(\mathscr{B}_1)} \sqrt{\int_{\mathscr{B}_1} |z|^a\,dX} \leqslant C\varepsilon,\end{aligned}$$

for some $C > 0$. As a consequence,

$$\|v - p_\varepsilon\|_{\mathbb{H}^s(\mathscr{B}_1)} \leqslant \|v - w_\varepsilon\|_{\mathbb{H}^s(\mathscr{B}_1)} + \|w_\varepsilon - p_\varepsilon\|_{\mathbb{H}^s(\mathscr{B}_1)} \leqslant \varepsilon + C\,\varepsilon,$$

which implies the desired result after renaming ε. □

Now, given $v \in L^\infty(\mathscr{B}_1)$, and fixed any $z \in \mathbb{R}$, we consider the Steiner symmetric rearrangement $v^\sigma(\cdot, z)$ of $v(\cdot, z)$ (see e.g. Section 2 of [9]). With this notation, we are ready to establish the main result of this subsection, that states that the symmetric rearrangement in the x variables decreases energy:

Theorem 4.3. *For any* $v \in \mathbb{H}^s_0(\mathscr{B}_1)$,

$$\int_{\mathscr{B}_1} |z|^a |\nabla v^\sigma|^2 \, dX \leqslant \int_{\mathscr{B}_1} |z|^a |\nabla v|^2 \, dX.$$

Proof. The idea of the proof is to first prove the desired claim for polynomials using some results in [9] and then pass to the limit. The details go as follows. By Lemma 4.2, we can take a sequence of polynomials p_j such that

$$\lim_{j\to+\infty} \|v - p_j\|_{\mathbb{H}^s(\mathscr{B}_1)} = 0. \tag{4.1}$$

Consequently,

$$\lim_{j\to+\infty} \int_{\mathscr{B}_1} |z|^a |\nabla p_j|^2 \, dX = \int_{\mathscr{B}_1} |z|^a |\nabla v|^2 \, dX. \tag{4.2}$$

Now, for any $(\eta, \zeta) \in \mathbb{R}^n \times \mathbb{R}$, we set

$$f(\eta,\zeta) := |\eta|^2 + |\zeta|^2 = |(\eta,\zeta)|^2.$$

Also, for any fixed $z \in \mathbb{R}$, we set

$$\mathscr{B}_1^z := \{x \in \mathbb{R}^n \text{ s.t. } (x,z) \in \mathscr{B}_1\}.$$

Notice that the Steiner symmetric rearrangement of $\mathscr{B}_1^z$ coincides with $\mathscr{B}_1^z$ itself, thanks to (3.1). By formula (4.20) in [9], we have that

$$\int_{\partial^*\{x\in\mathscr{B}_1^z \text{ s.t. } p_j^\sigma(x,z)>t\}} \frac{f(\nabla p_j^\sigma)}{|\nabla_x p_j^\sigma|} \, dx \leqslant \int_{\partial^*\{x\in\mathscr{B}_1^z \text{ s.t. } p_j(x,z)>t\}} \frac{f(\nabla p_j)}{|\nabla_x p_j|} \, dx,$$

for any $t \in \mathbb{R}$, where ∂^* denotes, as usual, the reduced boundary in the sense of geometric measure theory. Thus, by the Coarea Formula,

$$\begin{aligned}
\int_{\mathscr{B}_1^z} |\nabla p_j^\sigma|^2 \, dx &= \int_{\mathscr{B}_1^z} f(\nabla p_j^\sigma) \, dx \\
&= \int_{\mathbb{R}} \left[\int_{\partial^*\{x\in\mathscr{B}_1^z \text{ s.t. } p_j^\sigma(x,z)>t\}} \frac{f(\nabla p_j^\sigma)}{|\nabla_x p_j^\sigma|} \, dx \right] dt \\
&\leqslant \int_{\mathbb{R}} \left[\int_{\partial^*\{x\in\mathscr{B}_1^z \text{ s.t. } p_j(x,z)>t\}} \frac{f(\nabla p_j)}{|\nabla_x p_j|} \, dx \right] dt \\
&= \int_{\mathscr{B}_1^z} f(\nabla p_j) \, dx = \int_{\mathscr{B}_1^z} |\nabla p_j|^2 \, dx,
\end{aligned}$$

for any fixed $z \in \mathbb{R}$.

Hence, we multiply by $|z|^a$ and integrate, to obtain

$$\int_{\mathscr{B}_1} |z|^a |\nabla p_j^\sigma|^2 \, dX \leqslant \int_{\mathscr{B}_1} |z|^a |\nabla p_j|^2 \, dX. \tag{4.3}$$

Our objective is now to pass to the limit (4.3). The right hand side of (4.3) will pass to the limit thanks to (4.2), so we discuss now the left hand side. Since the Schwarz rearrangement is nonexpansive (see e.g. Theorem 3.5 of [21]), we have that, for any fixed $z \in \mathbb{R}$,

$$\int_{\mathscr{B}_1^z} |v^\sigma - p_j^\sigma|^2\,dx \leqslant \int_{\mathscr{B}_1^z} |v - p_j|^2\,dx.$$

So, we multiply by $|z|^a$ and we integrate over z, and we see that

$$\int_{\mathscr{B}_1} |z|^a |v^\sigma - p_j^\sigma|^2\,dX \leqslant \int_{\mathscr{B}_1} |z|^a |v - p_j|^2\,dX.$$

This and (4.1) give that

$$\lim_{j\to+\infty} \int_{\mathscr{B}_1} |z|^a |v^\sigma - p_j^\sigma|^2\,dX = 0. \tag{4.4}$$

Now, by (4.3) and (4.2), we have that

$$\sup_{j\in\mathbb{N}} \int_{\mathscr{B}_1} |z|^a |\nabla p_j^\sigma|^2\,dX < +\infty.$$

Accordingly, by Lemma 3.6 (see also Theorem 1.30 in [19]), we obtain that

$$\lim_{j\to+\infty} \int_{\mathscr{B}_1} |z|^a \nabla p_j^\sigma \cdot \nabla\phi\,dX = \int_{\mathscr{B}_1} |z|^a \nabla v^\sigma \cdot \nabla\phi\,dX,$$

for any $\phi \in \mathbb{H}^s(\mathscr{B}_1)$. As a consequence,

$$\begin{aligned}
0 &\leqslant \liminf_{j\to+\infty} \int_{\mathscr{B}_1} |z|^a |\nabla p_j^\sigma - \nabla v^\sigma|^2\,dX \\
&= \liminf_{j\to+\infty} \int_{\mathscr{B}_1} |z|^a \big(|\nabla p_j^\sigma|^2 + |\nabla v^\sigma|^2 - 2\nabla p_j^\sigma \cdot \nabla v^\sigma\big)\,dX \\
&= \liminf_{j\to+\infty} \int_{\mathscr{B}_1} |z|^a |\nabla p_j^\sigma|^2\,dX - \int_{\mathscr{B}_1} |z|^a |\nabla v^\sigma|^2\,dX.
\end{aligned}$$

This, (4.3) and (4.2) yield that

$$\begin{aligned}
\int_{\mathscr{B}_1} |z|^a |\nabla v^\sigma|^2\,dX &\leqslant \liminf_{j\to+\infty} \int_{\mathscr{B}_1} |z|^a |\nabla p_j^\sigma|^2\,dX \\
&\leqslant \liminf_{j\to+\infty} \int_{\mathscr{B}_1} |z|^a |\nabla p_j|^2\,dX = \int_{\mathscr{B}_1} |z|^a |\nabla v^\sigma|^2\,dX,
\end{aligned}$$

as desired. □

4.2. The radial case

The goal of this subsection is to prove Theorem 4.1 in the radial case, that is when the Dirichlet datum is constant, K is a ball and A is a ring. More precisely, we prove that:

Lemma 4.4. *Let $\rho \in [1/4, 3/4]$, $r \in (0, \rho)$ and $c \in [0, +\infty)$. Then*

$$\mathscr{E}(\Phi_{B_\rho}^c) - \mathscr{E}(\Phi_{B_r}^c) \leqslant C\,c\,|B_\rho \setminus B_r|,$$

for some $C > 0$ that depends on n and s.

Proof. If $c = 0$, then $\Phi^c_{B_\rho} \equiv 0$ and $\Phi^c_{B_r} \equiv 0$, in virtue of Lemmata 3.16 and 3.17. Thus we may assume that $c \neq 0$. In fact, by dividing by $c \neq 0$, we may assume that $c = 1$.

We let $\mu := \rho - r$ and we observe that

$$\begin{aligned} |B_\rho \setminus B_r| &= |B_1|\,(\rho^n - r^n) = |B_1|\,(\rho - r) \sum_{j=1}^{n} \rho^{n-j} r^{j-1} \\ &\geqslant |B_1|\,(\rho - r)\,\rho^{n-1} \geqslant \frac{|B_1|}{4^{n-1}}\,(\rho - r) = \frac{|B_1|\,\mu}{4^{n-1}}. \end{aligned} \tag{4.5}$$

Now we fix $\phi \in C^\infty(\mathbb{R}^{n+1})$ such that $\phi = 1 = c$ in $\mathbb{R}^{n+1} \setminus \mathscr{B}_1$ and $\phi = 0$ in $B_{3/4} \times \{0\}$. We let $C_0 := \mathscr{E}(\phi)$. By construction ϕ vanishes in $B_\rho \times \{0\} \supseteq B_r \times \{0\}$, therefore, by the minimality properties of $\Phi^c_{B_\rho}$ and $\Phi^c_{B_r}$, we have that

$$\max\{\mathscr{E}(\Phi^c_{B_\rho}),\, \mathscr{E}(\Phi^c_{B_r})\} \leqslant \mathscr{E}(\phi) = C_0 = C_0\, c. \tag{4.6}$$

We define

$$\mathscr{C}_+ := B_{5/6} \times \left(-\frac{1}{2}, \frac{1}{2}\right)$$
$$\text{and} \quad \mathscr{C}_- := B_{4/5} \times \left(-\frac{1}{4}, \frac{1}{4}\right).$$

By (3.1),

$$B_\rho \times \{0\} \Subset \mathscr{C}_- \Subset \mathscr{C}_+ \Subset \mathscr{B}_1.$$

Therefore, there exists $\tau \in C^\infty(\mathbb{R}^{n+1}, [0,1])$ such that $\tau = 1$ in $\mathscr{C}_-$ and $\tau = 0$ in $\mathbb{R}^{n+1} \setminus \mathscr{C}_+$.

For any $X \in \mathbb{R}^{n+1}$ we define

$$\alpha(X) := \left(1 - \frac{\mu}{\rho}\,\tau(X)\right) X = X - \frac{\mu}{\rho}\,\tau(X)\,X.$$

Let also 1_{n+1} be the identity $(n+1)$-dimensional matrix. Notice that $X \mapsto \tau(X)\,X$ is a smooth and compactly supported function, and so

$$|D\alpha(X) - 1_{n+1}| = \frac{\mu}{\rho}\,\left|D\big(\tau(X)\,X\big)\right| \leqslant C_1 \mu, \tag{4.7}$$

for some $C_1 > 0$. Accordingly

$$|\det D\alpha(X)| \geqslant 1 - C_2 \mu, \tag{4.8}$$

as long as μ is small enough.

Now we observe that

$$\alpha(B_\rho \times \{0\}) \subseteq B_r \times \{0\}. \tag{4.9}$$

Indeed, if $x \in B_\rho$, then $(x, 0) \in \mathscr{C}_-$, thus $\tau(x, 0) = 1$, which gives

$$\alpha(x, 0) = \left(1 - \frac{\mu}{\rho}\right)(x, 0) = \frac{r}{\rho}\,(x, 0),$$

proving (4.9).

We also notice that

$$\alpha(\mathbb{R}^{n+1} \setminus \mathscr{B}_1) \subseteq \mathbb{R}^{n+1} \setminus \mathscr{B}_1. \tag{4.10}$$

Indeed, if $X \in \mathbb{R}^{n+1} \setminus \mathscr{B}_1$, then in particular $X \in \mathbb{R}^{n+1} \setminus \mathscr{C}_+$, which gives that $\tau(X) = 0$ and so $\alpha(X) = X \in \mathbb{R}^{n+1} \setminus \mathscr{B}_1$, establishing (4.10).

Now we claim that

$$\alpha(\mathscr{B}_1) \subseteq \mathscr{B}_1. \tag{4.11}$$

To prove this, let $X \in \mathscr{B}_1$. If $X \in \mathscr{B}_1 \setminus \mathscr{C}_+$, we have that $\tau(X) = 0$, thus $\alpha(X) = X \in \mathscr{B}_1$ and we are done. If instead $X \in \mathscr{C}_+ = B_{5/6} \times [-1/2, 1/2]$, then $\alpha(X) = \theta(X)\, X$, for some $\theta(X) \in [0, 1]$, thus $\alpha(X)$ also lies in $B_{5/6} \times [-1/2, 1/2] = \mathscr{C}_+ \subseteq \mathscr{B}_1$, and this completes the proof of (4.11).

Now we observe that

$$\text{if } \tilde{X} = (\tilde{x}, \tilde{z}) = \alpha(X) = \alpha(x, z), \text{ then } \frac{|z|}{1 + C_3\mu} \leqslant |\tilde{z}| \leqslant (1 + C_3\mu)|z|, \tag{4.12}$$

for some $C_3 > 0$, as long as μ is sufficiently small. To prove this, we observe that

$$\tilde{z} = \left(1 - \frac{\mu}{\rho}\, \tau(X)\right) z,$$

and this gives (4.12).

Now we define $\phi^\star(X) := \Phi^c_{B_r}(\alpha(X))$. From (4.9) and (4.10), we have that $\phi^\star \in \mathscr{D}^c_{B_\rho}$, therefore the minimizing property of $\Phi^c_{B_\rho}$ gives that

$$\mathscr{E}(\Phi^c_{B_\rho}) \leqslant \mathscr{E}(\phi^\star). \tag{4.13}$$

On the other hand, by (4.7), (4.8), (4.11) and (4.12),

$$\begin{aligned}
\mathscr{E}(\phi^\star) &= \int_{\mathscr{B}_1} |z|^a \left|\nabla\left(\Phi^c_{B_r}(\alpha(X))\right)\right|^2 dX \\
&\leqslant (1 + C_1\mu)^2 \int_{\mathscr{B}_1} |z|^a \left|\nabla \Phi^c_{B_r}(\alpha(X))\right|^2 dX \\
&\leqslant (1 + C_4\mu) \int_{\alpha(\mathscr{B}_1)} |\tilde{z}|^a \left|\nabla \Phi^c_{B_r}(\tilde{X})\right|^2 d\tilde{X} \\
&\leqslant (1 + C_4\mu) \int_{\mathscr{B}_1} |\tilde{z}|^a \left|\nabla \Phi^c_{B_r}(\tilde{X})\right|^2 d\tilde{X} \\
&= (1 + C_5\mu)\, \mathscr{E}(\Phi^c_{B_r}),
\end{aligned}$$

for some $C_4, C_5 > 0$, where the change of variable $\tilde{X} := \alpha(X)$ was exploited.

Hence, recalling (4.13), we obtain that

$$\mathscr{E}(\Phi^c_{B_\rho}) \leqslant \mathscr{E}(\phi^\star) \leqslant (1 + C_5\mu)\, \mathscr{E}(\Phi^c_{B_r}),$$

provided that μ is small enough. As a consequence, from (4.5) and (4.6),

$$\mathscr{E}(\Phi^c_{B_\rho}) - \mathscr{E}(\Phi^c_{B_r}) \leqslant C_5\mu\mathscr{E}(\Phi^c_{B_r}) \leqslant C_6\,|B_\rho \setminus B_r|\, \mathscr{E}(\Phi^c_{B_r}) \leqslant C_7\, c\, |B_\rho \setminus B_r|,$$

for some $C_6, C_7 > 0$, provided that μ is small enough.

This completes the proof of Lemma 4.4 for small μ, say $\mu \leqslant \mu_0$ for a suitable $\mu_0 > 0$.

Conversely, when $\mu > \mu_0$, we have that

$$\mathscr{E}(\Phi^c_{B_\rho}) - \mathscr{E}(\Phi^c_{B_r}) \leqslant \mathscr{E}(\Phi^c_{B_\rho}) \leqslant C_0\, c \leqslant C_0\, c\, \mu_0^{-1}\, \mu \leqslant C_8\, c\, |B_\rho \setminus B_r|,$$

for some $C_8 > 0$, thanks to (4.5) and (4.6), which establishes Lemma 4.4 also when $\mu > \mu_0$. □

Now we generalize Lemma 4.4 to the case in which the Dirichlet datum is still constant, but the supporting sets K and A are not necessarily radially symmetric. In this framework, we have:

Lemma 4.5. *Let $\rho \in [1/4, 3/4]$ and $c \in [0, +\infty)$. Let $K \subseteq \mathscr{B}_\rho \cap \{z = 0\}$ and $A := B_\rho \setminus K$. Then*

$$\mathscr{E}(\Phi^c_{K \cup A}) - \mathscr{E}(\Phi^c_K) \leqslant C\, c\, |A|,$$

for some $C > 0$ that depends on n and s.

Proof. We point out that Lemma 4.5 reduces to Lemma 4.4 in the special case in which $K := B_r$, with $r \in (0, \rho)$. In the general case, we argue as follows. We take r such that $|B_r| = |K|$. Then

$$|A| = |B_\rho \setminus K| = |B_\rho| - |K| = |B_\rho| - |B_r| = |B_\rho \setminus B_r|. \tag{4.14}$$

Also, we define $\psi := c - \Phi^c_K$. Notice that $0 \leqslant \psi \leqslant c$, due to Lemmata 3.16 and 3.17 and $\psi \in \mathbb{H}^s_0(\mathscr{B}_1)$. Thus $\psi \in \mathscr{D}^0_{K,c}$ and so its symmetric rearrangement ψ^σ in the variable $x \in \mathbb{R}^n$ (as defined in Subsection 4.1) satisfies $\psi^\sigma \in \mathscr{D}^0_{B_r,c}$.

Let $\psi^\star := c - \psi^\sigma$. Then $\psi^\star \in \mathscr{D}^c_{B_r}$, therefore, by the minimality of $\Phi^c_{B_r}$, we have that

$$\mathscr{E}(\Phi^c_{B_r}) \leqslant \mathscr{E}(\psi^\star) = \mathscr{E}(\psi^\sigma).$$

On the other hand, by Theorem 4.3, we know that $\mathscr{E}(\psi^\sigma) \leqslant \mathscr{E}(\psi)$. As a consequence

$$\mathscr{E}(\Phi^c_{B_r}) \leqslant \mathscr{E}(\psi) = \mathscr{E}(\Phi^c_K).$$

Now we remark that $K \cup A = B_\rho$, therefore

$$\mathscr{E}(\Phi^c_{K\cup A}) - \mathscr{E}(\Phi^c_K) = \mathscr{E}(\Phi^c_{B_\rho}) - \mathscr{E}(\Phi^c_K) \leqslant \mathscr{E}(\Phi^c_{B_\rho}) - \mathscr{E}(\Phi^c_{B_r}).$$

Then, using Lemma 4.4,

$$\mathscr{E}(\Phi^c_{K\cup A}) - \mathscr{E}(\Phi^c_K) \leqslant C\, c\, |B_\rho \setminus B_r|.$$

This and (4.14) complete the proof of Lemma 4.5. □

4.3. *Completion of the proof of Theorem 4.1*

With the arguments introduced till now, we can complete the proof of Theorem 4.1. The idea is that, by the monotonicity property in Theorem 3.20, one can reduce to the case of constant boundary data and then use Lemma 4.5. The details of the proof go as follows.

Proof of Theorem 4.1. We define $c^\star := \|\varphi\|_{L^\infty(\mathscr{B}_1)}$, $K^\star := K \cap B_\rho$ and

$$A^\star := B_\rho \setminus K^\star = B_\rho \setminus (K \cap B_\rho) = B_\rho \setminus K = A.$$

From Lemma 4.5, we have

$$\mathscr{E}(\Phi^{c^\star}_{K^\star\cup A^\star}) - \mathscr{E}(\Phi^{c^\star}_{K^\star}) \leqslant C\, c^\star\, |A^\star| = C\, \|\varphi\|_{L^\infty(\mathscr{B}_1)}\, |A|. \tag{4.15}$$

On the other hand, we see that $c^\star \geqslant \varphi \geqslant 0$ a.e. in $\mathscr{B}_2$, $K^\star \subseteq K \subseteq B_{\frac{9}{10}}$ and $A^\star = A \Subset B_{\frac{9}{10}}$, therefore, by Theorem 3.20,

$$\mathscr{E}(\Phi^\varphi_{K\cup A}) - \mathscr{E}(\Phi^\varphi_K) \leqslant \mathscr{E}(\Phi^{c^\star}_{K^\star\cup A^\star}) - \mathscr{E}(\Phi^{c^\star}_{K^\star}).$$

Combining this with (4.15), we obtain the desired result. □

5. Density estimates

In this section, we deal with density estimates. A crucial ingredient of our argument will be the estimate previously obtained in Theorem 4.1.

5.1. *Density estimates from one side*

We start by proving a density estimate from one side and a uniform bound on the minimizers.

Lemma 5.1. *Assume that (u, E) is minimizing in B_1, with $u \geqslant 0$ a.e. in $\mathbb{R}^n \setminus B_1$ and $0 \in \partial E$.*

Then, there exist δ, $K > 0$, possibly depending on n, s, and σ such that

$$|B_{1/2} \setminus E| \geqslant \delta \tag{5.1}$$

and

$$\|u\|_{L^\infty(B_{1/2})} \leqslant K. \tag{5.2}$$

Proof. The proof is an appropriate modification of the one in Lemma 3.1 of [6], combined with some results in [16]. First we prove (5.1). For this, for any $r \in [1/4, 3/4]$, we define

$$V_r := |B_r \setminus E| \ \text{ and } \ a(r) := \mathscr{H}^{n-1}\big((\partial B_r) \setminus E\big). \tag{5.3}$$

The terms V_r and $a(r)$ play the role of volume and area terms, respectively.

We suppose, by contradiction, that

$$V_{1/2} < \delta \tag{5.4}$$

(of course we are free to choose δ suitably small, and then we will obtain a contradiction for such fixed δ). We set

$$A := B_r \setminus E. \tag{5.5}$$

By Lemma 3.3 in [14] we have that

$$u \geqslant 0 \text{ a.e.}, \tag{5.6}$$

hence $u \geqslant 0$ a.e. in E and $u = 0$ a.e. in $\mathbb{R}^n \setminus E$.

In particular, $u \geqslant 0$ a.e. in $E \cup A$ and $u = 0$ a.e. in $(\mathbb{R}^n \setminus E) \setminus A = \mathbb{R}^n \setminus (E \cup A)$. As a consequence, the pair $(u, E \cup A)$ is admissible.

Accordingly, from the minimality of (u, E), we obtain that $\mathscr{F}_{B_1}(u, E) \leqslant \mathscr{F}_{B_1}(u, E \cup A)$, that is

$$\operatorname{Per}_\sigma(E, B_1) - \operatorname{Per}_\sigma(E \cup A, B_1) \leqslant 0. \tag{5.7}$$

Also, by (1.1),

$$\operatorname{Per}_\sigma(E, B_1) - \operatorname{Per}_\sigma(E \cup A, B_1) = L(A, E) - L(A, \mathbb{R}^n \setminus (E \cup A)). \tag{5.8}$$

Hence, recalling (5.7), we conclude that

$$\begin{aligned} L(A, \mathbb{R}^n \setminus A) &= L(A, E) + L(A, \mathbb{R}^n \setminus (E \cup A)) \\ &= 2L(A, \mathbb{R}^n \setminus (E \cup A)) + \operatorname{Per}_\sigma(E, B_1) - \operatorname{Per}_\sigma(E \cup A, B_1) \\ &\leqslant 2L(A, \mathbb{R}^n \setminus (E \cup A)) \\ &\leqslant 2L(A, \mathbb{R}^n \setminus B_r). \end{aligned} \tag{5.9}$$

Furthermore, using the fractional Sobolev inequality (see e.g. [10]), we have that

$$\|\chi_A\|^2_{L^{\frac{2n}{n-\sigma}}(\mathbb{R}^n)} \leqslant C \iint_{\mathbb{R}^{2n}} \frac{|\chi_A(x) - \chi_A(y)|^2}{|x-y|^{n+\sigma}}\, dx\, dy,$$

for some $C > 0$, which may be written as

$$V_r^{\frac{n-\sigma}{n}} \leqslant 2C\, L(A, \mathbb{R}^n \setminus A). \tag{5.10}$$

From this and (5.9), possibly renaming constants, we deduce that

$$V_r^{\frac{n-\sigma}{n}} \leqslant C\, L(A, \mathbb{R}^n \setminus B_r). \tag{5.11}$$

Now, using polar coordinates, we see that, for any $x \in A \subseteq B_r$,

$$\int_{\mathbb{R}^n \setminus B_r} \frac{dy}{|x-y|^{n+\sigma}} \leqslant \int_{\mathbb{R}^n \setminus B_{r-|x|}} \frac{dz}{|z|^{n+\sigma}} \leqslant C \int_{r-|x|}^{+\infty} \rho^{-1-\sigma}\, d\rho \leqslant C\,(r - |x|)^{-\sigma},$$

up to renaming constants. Therefore, integrating over $x \in A = B_r \setminus E$, we obtain that

$$\begin{aligned} L(A, \mathbb{R}^n \setminus B_r) &\leqslant C \int_{B_r \setminus E} (r - |x|)^{-\sigma}\, dx \\ &\leqslant C \int_0^r a(\rho)\,(r-\rho)^{-\sigma} \rho^{n-1}\, d\rho \leqslant C \int_0^r a(\rho)\,(r-\rho)^{-\sigma}\, d\rho. \end{aligned}$$

So, we plug this into (5.11) and we conclude that

$$V_r^{\frac{n-\sigma}{n}} \leqslant C \int_0^r a(\rho)\,(r-\rho)^{-\sigma}\,d\rho.$$

Now we fix $t \in [1/4, 1/2]$ and we integrate this estimate in $r \in [1/4, t]$: we conclude that

$$\int_{1/4}^{t} V_r^{\frac{n-\sigma}{n}}\,dr \leqslant C \int_0^t \left[\int_\rho^t a(\rho)\,(r-\rho)^{-\sigma}\,dr \right] d\rho \leqslant C\,t^{1-\sigma} \int_0^t a(\rho)\,d\rho \leqslant C\,V_t, \tag{5.12}$$

again up to renaming the constants. Now we iterate this estimate by setting, for any $k \geqslant 2$,

$$t_k := \frac{1}{4} + \frac{1}{2^k} \quad \text{and} \quad v_k := V_{t_k}.$$

Since V_r is monotone in r, we have that

$$\int_{1/4}^{t_k} V_r^{\frac{n-\sigma}{n}}\,dr \geqslant \int_{t_{k+1}}^{t_k} V_r^{\frac{n-\sigma}{n}}\,dr \geqslant V_{t_{k+1}}^{\frac{n-\sigma}{n}}\,(t_k - t_{k+1}) = 2^{-(k+1)}\,v_{k+1}^{\frac{n-\sigma}{n}}.$$

Hence, if we write (5.12) with $t := t_k$ we obtain that

$$v_{k+1}^{\frac{n-\sigma}{n}} \leqslant C^k\,v_k,$$

up to renaming the constants. Also, by (5.4), $v_2 < \delta$, which is assumed to be conveniently small. Then, it is easy to see that $v_k \leqslant C\eta^k$, for some $C > 0$ and $\eta \in (0,1)$ (see e.g. formula (8.18) in [11]) and so

$$0 = \lim_{k \to +\infty} v_k = V_{1/4}. \tag{5.13}$$

As a consequence, $|B_{1/4} \setminus E| = 0$, which is in contradiction with the fact that $0 \in \partial E$ (in the measure theoretic sense) and so it establishes (5.1).

Now we show the validity of (5.2). To this scope, we take $r = 3/4$ in (5.5) and we consider the s-harmonic replacement of u in $E \cup B_{3/4} = E \cup A$, according to Definition 1.1 in [16] (notice that the replacement considered in [16] is defined in a setting different than the one introduced in Section 3 in this paper; as a matter of fact, the framework introduced in Section 3 only plays a role in the forthcoming Subsection 5.2). Namely, we define $u_\star$ the function that minimizes the fractional Dirichlet energy

$$\iint_{Q_{B_1}} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy$$

among all the functions $v : \mathbb{R}^n \to \mathbb{R}$ such that $v - u \in L^2(\mathbb{R}^n)$, $v = u$ a.e. in $\mathbb{R}^n \setminus B_1$ and $v = 0$ a.e. in $(\mathbb{R}^n \setminus E) \setminus B_{3/4} = \mathbb{R}^n \setminus (E \cup A)$.

The existence (and, as a matter of fact, uniqueness) of such $u_\star$ is ensured by Lemma 2.1 of [16].

We set $\psi := u_\star - u$. Notice that $\psi = 0$ a.e. in $(\mathbb{R}^n \setminus B_1) \cup (\mathbb{R}^n \setminus (E \cup A))$. Hence, by formula (2.8) of [16] (applied here with $K := \mathbb{R}^n \setminus (E \cup A)$),

$$\begin{aligned} &\iint_{Q_{B_1}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \iint_{Q_{B_1}} \frac{|u_\star(x)-u_\star(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\ &\qquad = \iint_{Q_{B_1}} \frac{|\psi(x)-\psi(y)|^2}{|x-y|^{n+2s}}\,dx\,dy. \end{aligned} \tag{5.14}$$

Also

$$u_\star \geqslant 0 \text{ a.e.,} \tag{5.15}$$

thanks to (5.6) and Lemma 2.4 in [16]. So, since $u_\star = 0$ a.e. in $\mathbb{R}^n \setminus (E \cup A)$, we see that the pair $(u_\star, E \cup A)$ is admissible.

Therefore, by the minimality of (u, E), we have that

$$\mathscr{F}_{B_1}(u, E) \leqslant \mathscr{F}_{B_1}(u_\star, E \cup A).$$

This and (5.14) give that

$$\begin{aligned}
&\iint_{Q_{B_1}} \frac{|\psi(x) - \psi(y)|^2}{|x-y|^{n+2s}}\, dx\, dy \\
&= \iint_{Q_{B_1}} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}}\, dx\, dy - \iint_{Q_{B_1}} \frac{|u_\star(x) - u_\star(y)|^2}{|x-y|^{n+2s}}\, dx\, dy \\
&= \mathscr{F}_{B_1}(u, E) - \mathscr{F}_{B_1}(u_\star, E \cup A) + \mathrm{Per}_\sigma(E \cup A, B_1) - \mathrm{Per}_\sigma(E, B_1) \\
&\leqslant \mathrm{Per}_\sigma(E \cup A, B_1) - \mathrm{Per}_\sigma(E, B_1).
\end{aligned} \tag{5.16}$$

Now we recall that $(-\Delta)^s u_\star = 0$ in $B_{3/4} \subseteq E \cup A$, due to Lemma 2.3 in [16]. Therefore, recalling (5.15), we can use the fractional Harnack inequality (see e.g. Theorem 2.1 in [20]) and obtain that

$$\sup_{B_{1/2}} u_\star \leqslant C \inf_{B_{1/2}} u_\star. \tag{5.17}$$

Now we claim that

$$\|u_\star\|^2_{L^2(B_{1/2}\setminus E)} \geqslant c_0 \Big(\sup_{B_{1/2}} u_\star \Big)^2, \tag{5.18}$$

for some $c_0 > 0$. To prove this, we use (5.17) to see that

$$\begin{aligned}
\|u_\star\|^2_{L^2(B_{1/2}\setminus E)} &= \int_{B_{1/2}\setminus E} u_\star^2\, dx \\
&\geqslant \Big(\inf_{B_{1/2}} u_\star \Big)^2 |B_{1/2} \setminus E| \geqslant \Big(C^{-1} \sup_{B_{1/2}} u_\star \Big)^2 |B_{1/2} \setminus E|
\end{aligned}$$

and this proves (5.18), thanks to (5.1).

Furthermore, since $\psi = 0$ a.e. in $\mathbb{R}^n \setminus B_1$, we have that

$$\begin{aligned}
\iint_{\mathbb{R}^{2n}} \frac{|\psi(x) - \psi(y)|^2}{|x-y|^{n+2s}}\, dx\, dy &\geqslant \int_{B_{3/4}} \left[\int_{\mathbb{R}^n \setminus B_1} \frac{|\psi(x) - \psi(y)|^2}{|x-y|^{n+2s}} dy \right] dx \\
&= \int_{B_{3/4}} \left[\int_{\mathbb{R}^n \setminus B_1} \frac{|\psi(x)|^2}{|x-y|^{n+2s}} dy \right] dx \\
&\geqslant \int_{B_{3/4}} \left[\int_{\mathbb{R}^n \setminus B_2} \frac{|\psi(x)|^2}{|z|^{n+2s}} dz \right] dx \\
&= c\, \|\psi\|^2_{L^2(B_{3/4})},
\end{aligned} \tag{5.19}$$

for some $c > 0$. Then, since $\psi = u_\star$ in $B_{1/2} \setminus E$, we deduce from (5.18) and (5.19) that

$$\begin{aligned}\left(\sup_{B_{1/2}} u_\star\right)^2 &\leqslant c_0^{-1}\|\psi\|^2_{L^2(B_{1/2}\setminus E)}\\ &\leqslant c_0^{-1}\|\psi\|^2_{L^2(B_{3/4})}\\ &\leqslant C\iint_{\mathbb{R}^{2n}} \frac{|\psi(x)-\psi(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\\ &= C\iint_{Q_{B_1}} \frac{|\psi(x)-\psi(y)|^2}{|x-y|^{n+2s}}\,dx\,dy,\end{aligned} \tag{5.20}$$

for some $C > 0$. Now we claim that

$$u_\star \geqslant u \text{ a.e. in } \mathbb{R}^n. \tag{5.21}$$

To prove it, we set $\beta := (u - u_\star)$ and we remark that

$$\beta^+ = 0 \text{ a.e. in } (\mathbb{R}^n \setminus B_1) \cup (\mathbb{R}^n \setminus E). \tag{5.22}$$

Thus, from formula (2.7) in [16], we have that

$$\iint_{Q_{B_1}} \frac{\big(u_\star(x)-u_\star(y)\big)\big(\beta^+(x)-\beta^+(y)\big)}{|x-y|^{n+2s}}\,dx\,dy = 0. \tag{5.23}$$

Moreover, fixed $\varepsilon \in (0, 1)$, we define $u_\varepsilon := u - \varepsilon\beta^+$. We notice that

$$u_\varepsilon \geqslant 0 \text{ a.e. in } E. \tag{5.24}$$

Indeed, let $x \in E$: if $\beta^+(x) = 0$ then $u_\varepsilon(x) = u(x) \geqslant 0$ (up to negligible sets); if instead $\beta^+(x) > 0$, then $\beta^+(x) = u(x) - u_\star(x)$, thus $u_\varepsilon(x) = (1-\varepsilon)u(x) + \varepsilon u_\star(x) \geqslant 0$, thanks to (5.15). This proves (5.24).

From (5.22) and (5.24), we obtain that (u_ε, E) is an admissible competitor for (u, E), therefore, by the minimality of (u, E), we see that

$$\iint_{Q_{B_1}} \frac{\big(u(x)-u(y)\big)\big(\beta^+(x)-\beta^+(y)\big)}{|x-y|^{n+2s}}\,dx\,dy \leqslant 0.$$

This and (5.23) give that

$$\iint_{Q_{B_1}} \frac{\big(\beta(x)-\beta(y)\big)\big(\beta^+(x)-\beta^+(y)\big)}{|x-y|^{n+2s}}\,dx\,dy \leqslant 0.$$

On the other hand (see e.g. formula (8.10) in [11]), we have that

$$\big(\beta(x)-\beta(y)\big)\big(\beta^+(x)-\beta^+(y)\big) \geqslant \big|\beta^+(x)-\beta^+(y)\big|^2,$$

so we deduce that

$$\iint_{Q_{B_1}} \frac{\big|\beta^+(x)-\beta^+(y)\big|^2}{|x-y|^{n+2s}}\,dx\,dy \leqslant 0.$$

This says that $\beta^+ = 0$ a.e. in $\mathbb{R}^n$, which in turn implies (5.21).

From (5.20) and (5.21) we obtain that

$$\left(\sup_{B_{1/2}} u\right)^2 \leqslant C\iint_{Q_{B_1}} \frac{|\psi(x)-\psi(y)|^2}{|x-y|^{n+2s}}\,dx\,dy.$$

By plugging this into (5.16) we conclude that

$$\left(\sup_{B_{1/2}} u\right)^2 \leqslant C\left(\mathrm{Per}_\sigma\,(E\cup A, B_1) - \mathrm{Per}_\sigma\,(E, B_1)\right).$$

Hence, recalling (5.8), we deduce that

$$\begin{aligned}\left(\sup_{B_{1/2}} u\right)^2 &\leqslant C\left(L(A, \mathbb{R}^n\setminus(E\cup A)) - L(A,E)\right)\\ &\leqslant C\,L(A,\mathbb{R}^n\setminus(E\cup A))\\ &\leqslant C\,L(B_{3/4},\mathbb{R}^n\setminus B_{3/4})\\ &\leqslant C,\end{aligned}$$

up to relabeling the constants. This completes the proof of (5.2). □

Now, recalling the definition in (2.1), we prove a uniform bound also for the extension function of minimizers. This will play a crucial role in the proof of Lemma 5.3, in order to obtain that the constant δ does not depend on the quantity Λ (see formula (5.25) below). Indeed, differently from the "local" case (see [6]), the energy estimate for the fractional harmonic replacement provided by Theorem 4.1 depends on the L^∞-norm of the extension function, and this makes the blow-up analysis more delicate.

More precisely, we have:

Lemma 5.2. *Let (u, E) be a minimizing pair in B_1, with $u \geqslant 0$ a.e. in $\mathbb{R}^n\setminus B_1$ and $0\in\partial E$. Suppose that*

$$\int_{\mathbb{R}^n} \frac{|u(y)|}{1+|y|^{n+2s}}\,dy \leqslant \Lambda, \tag{5.25}$$

for some $\Lambda > 0$. Let also $\overline{u}$ be the as in (2.1).

Then, there exists $K > 0$, possibly depending on n, s, and σ such that

$$\|\overline{u}\|_{L^\infty(\mathscr{B}_{5/9})} \leqslant K. \tag{5.26}$$

Proof. The proof of (5.26) is a suitable variation of the one of (5.2). The difference here is that we will consider the harmonic replacement that we constructed in Section 3. For this, we consider the extension function $\overline{u}$ of u, as defined in (2.1).

Moreover, we set

$$A := B_{3/4}\setminus E \tag{5.27}$$

and we observe that $E\cup B_{3/4} = E\cup A$. We consider the fractional harmonic replacement of $\overline{u}$, as introduced in Section 3, by prescribing $B_{9/10}\setminus(E\cup A)$ as vanishing set. More precisely, with the notation of Theorem 3.7, we consider $\Phi^{\overline{u}}_{B_{9/10}\setminus(E\cup A)}$. For shortness of notation, we define

$$\tilde{w} := \Phi^{\overline{u}}_{B_{9/10}\setminus(E\cup A)}. \tag{5.28}$$

Notice that $\tilde{w} = \overline{u}$ in $\mathscr{B}_2\setminus\mathscr{B}_1$, therefore (up to extending $\tilde{w}$ outside $\mathscr{B}_2$) we can say that

$$\tilde{w} = \overline{u} \quad \text{in } \mathbb{R}^{n+1}\setminus\mathscr{B}_1. \tag{5.29}$$

Notice also that

$$\tilde{w} \geqslant 0 \quad \text{in } \mathbb{R}^{n+1}. \tag{5.30}$$

Indeed, $u \geqslant 0$ thanks to Lemma 3.3 in [14], and so $\overline{u} \geqslant 0$, in view of (2.1). Therefore (5.30) follows from Lemma 3.17.

Now we set

$$F := E\cup A \tag{5.31}$$

and we claim that

$$\tilde{w}(x,0) \geqslant 0 \text{ a.e. } x \in F, \text{ and } \tilde{w}(x,0) \leqslant 0 \text{ a.e. } x \in \mathbb{R}^n \setminus F. \tag{5.32}$$

For this, we first observe that we only need to prove that $\tilde{w}(x,0) = 0$ a.e. $x \in \mathbb{R}^n \setminus F$, thanks to (5.30). Now, if $x \in B_{9/10} \setminus F$ then $\tilde{w}(x,0) = 0$ by the definition of harmonic replacement in (5.28). Moreover, if $x \in (\mathbb{R}^n \setminus B_{9/10}) \setminus F$ then $(x,0) \in \mathbb{R}^{n+1} \setminus \mathscr{B}_1$ (recall (3.1)), and so, by (5.29), we have that $\tilde{w}(x,0) = \overline{u}(x,0) = u(x) = 0$, since $x \in \mathbb{R}^n \setminus E$. This shows (5.32).

Now we define $\mathscr{U} := \mathscr{B}_{\frac{11}{10}}$ and we observe that

$$\mathscr{U} \cap \{z = 0\} = B_{\frac{99}{100}} \times \{0\} \subset B_1 \times \{0\},$$

thanks to (3.1). This, (5.31) and (5.32) imply that the assumptions of Lemma 2.1 are satisfied (with $r := 1$), and so

$$\int_{\mathscr{U}} |z|^a |\nabla \overline{u}|^2 \, dX + \mathrm{Per}_\sigma(E, B_1) \leqslant \int_{\mathscr{U}} |z|^a |\nabla \tilde{w}|^2 \, dX + \mathrm{Per}_\sigma(F, B_1).$$

Recalling (5.29), we can rewrite the formula above as

$$\int_{\mathscr{B}_1} |z|^a |\nabla \overline{u}|^2 \, dX + \mathrm{Per}_\sigma(E, B_1) \leqslant \int_{\mathscr{B}_1} |z|^a |\nabla \tilde{w}|^2 \, dX + \mathrm{Per}_\sigma(F, B_1). \tag{5.33}$$

Now we observe that, by (1.1),

$$\mathrm{Per}_\sigma(E \cup A, B_1) - \mathrm{Per}_\sigma(E, B_1) = L(A, \mathbb{R}^n \setminus (E \cup A)) - L(A, E),$$

therefore, recalling (5.31), we have that

$$\begin{aligned} \mathrm{Per}_\sigma(F, B_1) - \mathrm{Per}_\sigma(E, B_1) &= L(A, \mathbb{R}^n \setminus (E \cup A)) - L(A, E) \\ &\leqslant L(A, \mathbb{R}^n \setminus (E \cup A)) \leqslant L(B_{3/4}, \mathbb{R}^n \setminus B_{3/4}) \leqslant C, \end{aligned}$$

for some $C > 0$. From this and (5.33), we conclude that

$$\int_{\mathscr{B}_1} |z|^a |\nabla \overline{u}|^2 \, dX - \int_{\mathscr{B}_1} |z|^a |\nabla \tilde{w}|^2 \, dX \leqslant C. \tag{5.34}$$

Now we set $\Psi := \tilde{w} - \overline{u}$. Notice that

$$\Psi = 0 \text{ in } \mathbb{R}^{n+1} \setminus \mathscr{B}_1, \tag{5.35}$$

thanks to (5.29). Moreover, if $x \in B_{9/10} \setminus (E \cup A)$ then (in the sense of traces)

$$\Psi(x,0) = \tilde{w}(x,0) - \overline{u}(x,0) = -\overline{u}(x,0) = -u(x) = 0,$$

since $x \in \mathbb{R}^n \setminus E$ (recall also the definition of fractional harmonic replacement in (5.28)). Hence,[2] $\Psi \in \mathscr{D}^0_K$, where $K := B_{9/10} \setminus (E \cup A)$. Therefore, (3.14) gives that

$$\int_{\mathscr{B}_1} |z|^a |\nabla \overline{u}|^2 \, dX = \int_{\mathscr{B}_1} |z|^a |\nabla \tilde{w}|^2 \, dX + \int_{\mathscr{B}_1} |z|^a |\nabla \Psi|^2 \, dX.$$

Plugging this information into (5.34), we obtain

$$\int_{\mathscr{B}_1} |z|^a |\nabla \Psi|^2 \, dX \leqslant C. \tag{5.36}$$

Also, from (5.35) and Lemma 3.8 (recall that $\overline{u}$ is even in z by definition), we have

[2] The careful reader may have noticed that K here recalls the set notation in the fractional harmonic replacement framework and of course cannot be confused with the constant K in (5.26).

$$\int_{\mathscr{B}_1} |z|^a |\nabla\Psi|^2\, dX = \int_{\mathbb{R}^{n+1}} |z|^a |\nabla\Psi|^2\, dX = 2 \int_{\mathbb{R}^{n+1}_+} |z|^a |\nabla\Psi|^2\, dX,$$

where $\mathbb{R}^{n+1}_+ := \mathbb{R}^{n+1} \cap \{z > 0\}$. So we set $2^*_s := \frac{2n}{n-2s}$ and we use Proposition 1.2.1 in [12], obtaining that

$$\begin{aligned}\int_{\mathscr{B}_1} |z|^a |\nabla\Psi|^2\, dX &\geqslant C \left(\int_{\mathbb{R}^n} |\Psi(x,0)|^{2^*_s}\, dx \right)^{2/2^*_s} \\ &\geqslant C \left(\int_{B_{1/2}\setminus E} |\Psi(x,0)|^{2^*_s}\, dx \right)^{2/2^*_s} \geqslant |B_{1/2} \setminus E|^{\frac{2}{2^*_s}-1} \int_{B_{1/2}\setminus E} |\Psi(x,0)|^2\, dx \\ &\geqslant |B_{1/2} \setminus E|^{\frac{2}{2^*_s}} \left(\inf_{B_{1/2}\setminus E} |\Psi| \right)^2 \geqslant C \left(\inf_{B_{1/2}\setminus E} |\Psi| \right)^2,\end{aligned} \tag{5.37}$$

for some $C > 0$, thanks to (5.1) (notice that the Hölder inequality was also used).

Now we notice that, if $x \in B_{1/2} \setminus E$, then $\Psi(x, 0) = \tilde{w}(x, 0) - \overline{u}(x, 0) = \tilde{w}(x, 0) - u(x) = \tilde{w}(x, 0)$ in the sense of traces. Using this information into (5.37) we get

$$\int_{\mathscr{B}_1} |z|^a |\nabla\Psi|^2\, dX \geqslant C \left(\inf_{B_{1/2}\setminus E} |\tilde{w}| \right)^2. \tag{5.38}$$

Now we observe that $\mathscr{B}_{5/9}$ is compactly contained in $\mathscr{B}_1 \setminus K$, where $K = B_{9/10} \setminus (E \cup A)$. Indeed, by (3.1) and (5.27),

$$\begin{aligned}\mathscr{B}_{5/9} &= B_{\frac{1}{2}} \times \left(-\frac{5}{9}, \frac{5}{9} \right) \Subset B_{\frac{3}{4}} \times (-1, 1) \\ &\subset \big((E \cup A) \cap B_{9/10} \big) \times (-1, 1) \subset \mathscr{B}_1 \setminus K.\end{aligned}$$

Also, Lemma 3.10 says that

$$\operatorname{div}(|z|^a \nabla \tilde{w}) = 0$$

in the interior of $\mathscr{B}_1 \setminus K$, in the distributional sense. Therefore (recalling also (5.30)) we can use the Harnack inequality (see e.g. Theorem 2.3.8 in [18]) and we obtain that

$$\sup_{\mathscr{B}_{5/9}} \tilde{w} \leqslant C \inf_{\mathscr{B}_{5/9}} \tilde{w},$$

for some constant $C > 0$ independent of $\tilde{w}$. This, together with (5.38) and (5.30), gives that

$$\int_{\mathscr{B}_1} |z|^a |\nabla\Psi|^2\, dX \geqslant C \left(\inf_{B_{1/2}\setminus E} \tilde{w} \right)^2 \geqslant C \left(\inf_{\mathscr{B}_{5/9}} \tilde{w} \right)^2 \geqslant C \left(\sup_{\mathscr{B}_{5/9}} \tilde{w} \right)^2,$$

up to relabeling C. From this and (5.36) we obtain that

$$\sup_{\mathscr{B}_{5/9}} \tilde{w} \leqslant C, \tag{5.39}$$

for some $C > 0$, depending only on n, s and σ.

Now we claim that

$$\tilde{w} \geqslant \overline{u} \quad \text{in } \mathbb{R}^{n+1}. \tag{5.40}$$

To prove this, we set $\beta := \overline{u} - \tilde{w}$, and we observe that $\beta^+ = 0$ in $\mathbb{R}^{n+1} \setminus \mathscr{B}_1$, due to (5.29). Furthermore, if $x \in \mathbb{R}^n \setminus \big(B_{9/10} \setminus (E \cup A) \big)$, then $\beta^+(x, 0) = (\overline{u}(x, 0) - \tilde{w}(x, 0))^+ = 0$ in the sense of traces. Therefore, by (3.13),

$$\int_{\mathscr{B}_1} |z|^a \nabla\tilde{w} \cdot \nabla\beta^+ \, dX = 0. \tag{5.41}$$

Now we fix $\varepsilon \in (0,1)$ and we define $\overline{u}_\varepsilon := \overline{u} - \varepsilon\beta^+$. Notice that

$$\overline{u}_\varepsilon = \overline{u} \quad \text{in } \mathbb{R}^{n+1} \setminus \mathscr{B}_1. \tag{5.42}$$

Also, we observe that

$$\overline{u}_\varepsilon \geqslant 0. \tag{5.43}$$

Indeed, if $\beta^+ = 0$ then $\overline{u}_\varepsilon = \overline{u} \geqslant 0$; if instead $\beta^+ > 0$ then $\overline{u}_\varepsilon = (1-\varepsilon)\overline{u} + \tilde{w} \geqslant 0$. This proves (5.43).

Moreover, if $x \in \mathbb{R}^n \setminus E$,

$$\overline{u}_\varepsilon(x,0) = \overline{u}(x,0) - \varepsilon\,(\overline{u}(x,0) - \tilde{w}(x,0))^+ = -\varepsilon\,(-\tilde{w}(x,0))^+ = 0,$$

thanks to (5.30). This, (5.42) and (5.43) imply that the assumptions of Lemma 2.1 are satisfied with $\mathscr{U} := \mathscr{B}_{\frac{11}{20}}$, and so

$$\int_{\mathscr{U}} |z|^a |\nabla\overline{u}|^2 \, dX \leqslant \int_{\mathscr{U}} |z|^a |\nabla\overline{u}_\varepsilon|^2 \, dX.$$

Using (5.42), we can write

$$\int_{\mathscr{B}_1} |z|^a |\nabla\overline{u}|^2 \, dX \leqslant \int_{\mathscr{B}_1} |z|^a |\nabla\overline{u}_\varepsilon|^2 \, dX,$$

which, recalling the definition of $\overline{u}_\varepsilon$, implies that

$$\int_{\mathscr{B}_1} |z|^a \nabla\overline{u} \cdot \nabla\beta^+ \, dX \leqslant 0.$$

This and (5.41) give that

$$\int_{\mathscr{B}_1} |z|^a |\nabla\beta^+|^2 \, dX = \int_{\mathscr{B}_1} |z|^a \nabla\beta \cdot \nabla\beta^+ \, dX \leqslant 0.$$

Therefore, we have that $\beta^+ = 0$ in $\mathscr{B}_1$. This and (5.29) imply (5.40).

From (5.39) and (5.40) we obtain that

$$\sup_{\mathscr{B}_{5/9}} \overline{u} \leqslant C,$$

for some $C > 0$, possibly depending on n, s and σ. This shows (5.26). □

We remark that the finiteness of Λ in (5.25) is only used in order to have a well-defined extended function $\overline{u}$. In particular, the quantity K in Lemma 5.2 does not depend on Λ.

5.2. *Density estimates from the other side*

In Lemma 5.1, a density estimate from one side was obtained, namely we proved that the complement of E has positive density near the free boundary.

The purpose of this subsection is to prove that also the set E has positive density near the free boundary.

To this goal, we need to modify appropriately the argument in Lemma 5.1, by using the machinery developed in the previous sections. With respect to the argument developed in the proof of Lemma 5.1, in this subsection the sets in (5.3) and (5.5) are replaced by the similar quantities in which the intersection with E (rather than with the complement of E) is taken into account (see (5.44) and (5.45) below).

This apparently minor difference causes a conceptual difficulty in terms of harmonic replacements: indeed, in the proof of Lemma 5.1, the competitor was built by extending the positivity set of the minimizer u, while, in the case considered here, the positivity set gets reduced in the competitor, i.e. the competitor is forced to attain zero value on a larger set, and this makes its Dirichlet energy possibly bigger. For this reason, one needs to estimate the error in the Dirichlet energy produced by this further constrain on the zero set. This is the point in which Theorem 4.1 and Lemma 5.2 come into play. Indeed, for this estimate, we need to control the energy difference with a term only involving the measure of the additional zero set and the local size of the data (this is the reason for introducing the fractional harmonic replacement in the extended variables in Section 3 and for considering the extended problem in Lemma 2.1).

Lemma 5.3. *Let (u, E) be a minimizing pair in B_2, with $u \geqslant 0$ a.e. in $\mathbb{R}^n \setminus B_2$. Suppose that*

$$\int_{\mathbb{R}^n} \frac{|u(y)|}{1+|y|^{n+2s}}\, dy \leqslant \Lambda,$$

for some $\Lambda > 0$.

Assume also that $0 \in \partial E$.

Then, there exists $\delta > 0$, possibly depending on n, s and σ such that

$$|B_{1/2} \cap E| \geqslant \delta.$$

Proof. First of all, we notice that $u \geqslant 0$ in the whole of $\mathbb{R}^n$, thanks to Lemma 3.3 of [14]. Also, for any $r \in [1/4, 3/4]$, we define

$$V_r := |B_r \cap E| \ \text{ and } \ a(r) := \mathscr{H}^{n-1}\big((\partial B_r) \cap E\big). \tag{5.44}$$

The desired result will follow by arguing by contradiction. Suppose that the desired result does not hold. Then $V_{1/2} < \delta$. We will find a contradiction by taking δ conveniently small. To this goal, we set

$$A := B_r \cap E. \tag{5.45}$$

We let $\overline{u} : \mathbb{R}^{n+1} \to \mathbb{R}$ be the extension of u, according to (2.1).

We consider the fractional harmonic replacement of $\overline{u}$, as introduced in Section 3, prescribing $(B_{9/10} \setminus E) \cup A$ as supporting sets, i.e., in the notation of Theorem 3.7, we consider $\Phi^{\overline{u}}_{(B_{9/10}\setminus E)\cup A}$, and we define, for short,

$$\tilde{v} := \Phi^{\overline{u}}_{(B_{9/10}\setminus E)\cup A}. \tag{5.46}$$

Notice that $\tilde{v} = \overline{u}$ in $\mathscr{B}_2 \setminus \mathscr{B}_1$, so up to extending $\tilde{v}$ outside $\mathscr{B}_2$, we can write

$$\tilde{v} = \overline{u} \text{ in } \mathbb{R}^{n+1} \setminus \mathscr{B}_1. \tag{5.47}$$

We also set

$$F := E \setminus A. \tag{5.48}$$

We notice that

$$\tilde{v}(x,0) \geqslant 0 \text{ a.e. } x \in F, \text{ and } \tilde{v}(x,0) \leqslant 0 \text{ a.e. } x \in \mathbb{R}^n \setminus F. \tag{5.49}$$

Indeed, $u \geqslant 0$ due to Lemma 3.3 in [14], hence $\overline{u} \geqslant 0$, in view of (2.1). Therefore $\tilde{v} \geqslant 0$, thanks to Lemma 3.17. As a consequence $\tilde{v}(x,0) \geqslant 0$ in the trace sense. So it only remains to prove that $\tilde{v}(x,0) = 0$ a.e. $x \in \mathbb{R}^n \setminus F$. For this, notice that

$$\mathbb{R}^n \setminus F = \mathbb{R}^n \setminus (E \setminus A) = (\mathbb{R}^n \setminus E) \cup A.$$

So, if $x \in (B_{9/10} \setminus E) \cup A$, we have that $\tilde{v}(x,0) = 0$ by definition of fractional replacement. Also, if $x \in (\mathbb{R}^n \setminus B_{9/10}) \setminus E$, then $(x,0) \in \mathbb{R}^{n+1} \setminus \mathscr{B}_1$, due to (3.1), and so, by (5.47), in this case we have $\tilde{v}(x,0) = \overline{u}(x,0) = u(x) = 0$, since $x \in \mathbb{R}^n \setminus E$. This proves (5.49).

Now we define $\mathscr{U} := \mathscr{B}_{\frac{11}{10}}$ and we observe that

$$\mathscr{U} \cap \{z = 0\} = B_{\frac{99}{100}} \times \{0\} \subset B_1 \times \{0\},$$

due to (3.1). From this, (5.48) and (5.49), we see that the assumptions of Lemma 2.1 are satisfied (with $r := 1$): so we obtain that

$$\int_{\mathscr{U}} |z|^a |\nabla \overline{u}|^2 \, dX + \mathrm{Per}_\sigma(E, B_1) \leqslant \int_{\mathscr{U}} |z|^a |\nabla \tilde{v}|^2 \, dX + \mathrm{Per}_\sigma(F, B_1).$$

Thus, using again (5.47),

$$\int_{\mathscr{B}_1} |z|^a |\nabla \overline{u}|^2 \, dX + \mathrm{Per}_\sigma(E, B_1) \leqslant \int_{\mathscr{B}_1} |z|^a |\nabla \tilde{v}|^2 \, dX + \mathrm{Per}_\sigma(F, B_1). \tag{5.50}$$

Now, by (5.48),

$$\begin{aligned} &\mathrm{Per}_\sigma(F, B_1) - \mathrm{Per}_\sigma(E, B_1) = \mathrm{Per}_\sigma(E \setminus A, B_1) - \mathrm{Per}_\sigma(E, B_1) \\ &\quad = L(A, E \setminus A) - L(A, \mathbb{R}^n \setminus E). \end{aligned}$$

By inserting this information into (5.50) and recalling (5.46) we obtain that

$$\begin{aligned} L(A, \mathbb{R}^n \setminus E) - L(A, E \setminus A) &\leqslant \int_{\mathscr{B}_1} |z|^a |\nabla \tilde{v}|^2 \, dX - \int_{\mathscr{B}_1} |z|^a |\nabla \overline{u}|^2 \, dX \\ &= \mathscr{E}(\Phi^{\overline{u}}_{(B_{9/10} \setminus E) \cup A}) - \mathscr{E}(\overline{u}). \end{aligned} \tag{5.51}$$

On the other hand, $\overline{u}$ vanishes on $(B_{9/10} \setminus E) \times \{0\}$, thus, by the minimality of $\Phi^{\overline{u}}_{B_{9/10} \setminus E}$, we have that

$$\mathscr{E}(\Phi^{\overline{u}}_{B_{9/10} \setminus E}) \leqslant \mathscr{E}(\overline{u}).$$

Using this inequality into (5.51) and recalling Theorem 4.1, we obtain

$$\begin{aligned} L(A, \mathbb{R}^n \setminus E) - L(A, E \setminus A) &\leqslant \mathscr{E}(\Phi^{\overline{u}}_{(B_{9/10} \setminus E) \cup A}) - \mathscr{E}(\Phi^{\overline{u}}_{B_{9/10} \setminus E}) \\ &\leqslant C\, |A| \, \|\overline{u}\|^2_{L^\infty(\mathscr{B}_1)}. \end{aligned} \tag{5.52}$$

From Lemma 5.2, we have a uniform bound on $\|\overline{u}\|_{L^\infty(\mathscr{B}_1)}$. This and (5.52) give

$$L(A, \mathbb{R}^n \setminus E) - L(A, E \setminus A) \leqslant C\, |A|.$$

Then, the argument in [6] can be repeated verbatim (see in particular from the first formula in display after (3.2) to the fifth line below (3.4)) and one obtains that $V_{1/4} = 0$. This contradicts the fact that $0 \in \partial E$ and so it completes the proof of Lemma 5.3. □

By putting together Lemmata 5.1 and 5.3 we obtain:

Corollary 5.4. *Assume that (u, E) is minimizing in B_1, with $u \geqslant 0$ a.e. in $\mathbb{R}^n \setminus B_1$ and $0 \in \partial E$. Suppose that*

$$\int_{\mathbb{R}^n} \frac{|u(y)|}{1 + |y|^{n+2s}} \, dy \leqslant \Lambda,$$

for some $\Lambda > 0$.

Then, there exist δ, $K > 0$, possibly depending on n, s and σ such that

$$\min \Big\{ |B_{1/2} \setminus E|, \, |B_{1/2} \cap E| \Big\} \geqslant \delta \tag{5.53}$$

and

$$\|u\|_{L^\infty(B_{1/2})} \leqslant K. \tag{5.54}$$

We remark that the quantity K appearing in (5.54) does not depend on Λ. This fact will allow us to rescale the picture and deduce from (5.54) a universal growth from the free boundary, as stated in the following result:

Corollary 5.5. *Assume that (u, E) is minimizing in B_6, with $u \geqslant 0$ a.e. in $\mathbb{R}^n \setminus B_6$ and $0 \in \partial E$. Suppose that*

$$\int_{\mathbb{R}^n} \frac{|u(y)|}{1+|y|^{n+2s}}\,dy \leqslant \Lambda,$$

for some $\Lambda > 0$.

Also, for any $x \in B_1 \cap E$, we define $d(x) := \operatorname{dist}(x, \partial E)$ to be the distance of a point x from the free boundary. Then, there exists $K > 0$, possibly depending on n, s and σ such that

$$|u(x)| \leqslant K(d(x))^{s-\frac{\sigma}{2}} \quad \text{for any } x \in B_1 \cap E. \tag{5.55}$$

Furthermore,

$$|u(x)| \leqslant K|x|^{s-\frac{\sigma}{2}} \quad \text{for any } x \in B_1 \cap E. \tag{5.56}$$

Proof. We fix $x_0 \in B_1 \cap E$ and we set $r_0 := d(x_0)$. Let also $p_0 \in \partial E \cap \partial B_{r_0}(x_0)$ be such that $r_0 = d(x_0) = |x_0 - p_0|$. Notice that $r_0 \leqslant 1$, since $0 \in \partial E$, and so $|p_0| \leqslant 2$.

Now, we define

$$u_{r_0}(x) := r_0^{\frac{\sigma}{2}-s} u(r_0 x + p_0) \ \text{ and } \ E_{r_0} := \frac{1}{r_0}(E - p_0), \tag{5.57}$$

and we observe that (u_{r_0}, E_{r_0}) is a minimizing pair in $B_{6/r_0}(p_0/r_0)$. Also, notice that $B_4 \subset B_{6/r_0}(p_0/r_0)$. Indeed, if $x \in B_4$ then

$$\left| x - \frac{p_0}{r_0} \right| \leqslant |x| + \frac{|p_0|}{r_0} \leqslant 4 + \frac{2}{r_0} \leqslant \frac{4}{r_0} + \frac{2}{p_0} = \frac{6}{r_0},$$

and so $x \in B_{6/r_0}(p_0/r_0)$. Therefore, we can say that (u_{r_0}, E_{r_0}) is a minimizing pair in B_4. Moreover, by construction, $0 \in \partial E_{r_0}$. Furthermore, we see that

$$\int_{\mathbb{R}^n} \frac{|u_{r_0}(y)|}{1+|y|^{n+2s}}\,dy = r_0^{s+\frac{\sigma}{2}} \int_{\mathbb{R}^n} \frac{|u(y)|}{r_0^{n+2s}+|y|^{n+2s}}\,dy \leqslant \Lambda_{r_0},$$

for some $\Lambda_{r_0} > 0$ depending on r_0. So we can apply Corollary 5.4 to (u_{r_0}, E_{r_0}) obtaining that

$$\|u_{r_0}\|_{L^\infty(B_2)} \leqslant K, \tag{5.58}$$

for some K that depends only on n, s and σ.

Now we set $\omega := \frac{p_0 - x_0}{r_0}$, and we observe that $\omega \in B_2$ (and so $-\omega \in B_2$). Therefore, from this and (5.58) we get

$$|u_{r_0}(-\omega)| \leqslant K.$$

On the other hand, recalling (5.57),

$$u_{r_0}(-\omega) = r_0^{\frac{\sigma}{2}-s} u(-r_0\omega + p_0) = r_0^{\frac{\sigma}{2}-s} u(x_0).$$

The last two formulas imply that

$$|u(x_0)| \leqslant K r_0^{s-\frac{\sigma}{2}} = K(d(x_0))^{s-\frac{\sigma}{2}},$$

which shows (5.55).

Finally, (5.56) follows from (5.55) and the fact that $0 \in \partial E$. This concludes the proof of Corollary 5.5. □

5.3. Completion of the proof of Theorem 1.1

In order to end the proof of Theorem 1.1, we recall a Hölder continuity property for nonlocal solutions:

Lemma 5.6. *Assume that $(-\Delta)^s u = 0$ in B_1, with $u \in L^\infty(B_1)$. Then $u \in C^1(B_{1/2})$ and*

$$\|u\|_{C^1(B_{1/2})} \leqslant C \left(\|u\|_{L^\infty(B_1)} + \int_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+2s}}\,dx \right),$$

for some $C > 0$.

Proof. First of all, by Theorem 2.6 of [8], we know that $u \in C^\alpha(B_{9/10})$. Then we can apply Theorem 2.7 of [8] (say, in $B_{3/4}$) and obtain the desired result. □

Now we provide a rescaled version of Lemma 5.6.

Corollary 5.7. *Assume that $(-\Delta)^s u = 0$ in $B_t(q)$, for some $t > 0$ and $q \in \mathbb{R}^n$, with $u \in L^\infty(B_t(q))$. Then $u \in C^1(B_{t/2}(q))$ and*

$$\|\nabla u\|_{L^\infty(B_{t/2}(q))} \leqslant C\,t^{-1} \left(\|u\|_{L^\infty(B_t(q))} + t^{2s} \int_{\mathbb{R}^n \setminus B_t(q)} \frac{|u(x)|}{|x-q|^{n+2s}}\,dx \right),$$

for some $C > 0$.

Proof. For any $x \in B_1$, we define $v(x) := u(tx+q)$. Notice that, by construction, $(-\Delta)^s v = 0$ in B_1, and $v \in L^\infty(B_1)$. Hence, we are in position to apply Lemma 5.6 to the function v, obtaining that $v \in C^1(B_{1/2})$ and

$$\|\nabla v\|_{L^\infty(B_{1/2})} \leqslant C \left(\|v\|_{L^\infty(B_1)} + \int_{\mathbb{R}^n} \frac{|v(x)|}{1+|x|^{n+2s}}\,dx \right), \tag{5.59}$$

for some $C > 0$. Now we observe that

$$\|v\|_{L^\infty(B_1)} = \|u\|_{L^\infty(B_t(q))} \quad \text{and} \quad \|\nabla v\|_{L^\infty(B_{1/2})} = t\|\nabla u\|_{L^\infty(B_{t/2}(q))}. \tag{5.60}$$

Moreover, using the change of variable $y = tx + q$,

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{|v(x)|}{1+|x|^{n+2s}}\,dx &= \int_{\mathbb{R}^n} \frac{|u(tx+q)|}{1+|x|^{n+2s}}\,dx \\ &= t^{2s} \int_{\mathbb{R}^n} \frac{|u(y)|}{t^{n+2s}+|y-q|^{n+2s}}\,dy. \end{aligned} \tag{5.61}$$

We observe that

$$\begin{aligned} t^{2s} \int_{B_t(q)} \frac{|u(y)|}{t^{n+2s}+|y-q|^{n+2s}}\,dy &\leqslant t^{2s}\|u\|_{L^\infty(B_t(q))} \int_{B_t(q)} \frac{dy}{t^{n+2s}} \\ &\leqslant Ct^{2s}\|u\|_{L^\infty(B_t(q))} t^n t^{-n-2s} \leqslant C\|u\|_{L^\infty(B_t(q))}, \end{aligned} \tag{5.62}$$

up to renaming C. Also,

$$\int_{\mathbb{R}^n \setminus B_t(q)} \frac{|u(y)|}{t^{n+2s}+|y-q|^{n+2s}}\,dy \leqslant \int_{\mathbb{R}^n \setminus B_t(q)} \frac{|u(y)|}{|y-q|^{n+2s}}\,dy.$$

Using this and (5.62) into (5.61) we obtain that

$$\int_{\mathbb{R}^n} \frac{|v(x)|}{1+|x|^{n+2s}}\,dx \leqslant C\|u\|_{L^\infty(B_t(q))} + t^{2s} \int_{\mathbb{R}^n\setminus B_t(q)} \frac{|u(y)|}{|y-q|^{n+2s}}\,dy.$$

Plugging this and (5.60) into (5.59) we get the desired result. □

Now we can complete the proof of Theorem 1.1.

Proof of Theorem 1.1. We define, for any $r>0$,

$$u_r(x) := r^{\frac{\sigma}{2}-s} u(rx) \text{ and } E_r := \frac{1}{r}E \tag{5.63}$$

and we apply Corollary 5.4 to the minimizing pair (u_r, E_r), with $r \in (0, 1/2]$. For this, notice that

$$\int_{\mathbb{R}^n} \frac{|u_r(y)|}{1+|y|^{n+2s}}\,dy = r^{s+\frac{\sigma}{2}} \int_{\mathbb{R}^n} \frac{|u(y)|}{r^{n+2s}+|y|^{n+2s}}\,dy \leqslant \Lambda_r,$$

for some $\Lambda_r > 0$ depending on r. Then, (1.3) follows from (5.53). Also, (1.4) is a consequence of (5.54).

Now we prove (1.5). For this, since Theorem 1.1 deals with interior estimates, we may suppose that

$$\text{the minimizing property of } (u, E) \text{ holds in } B_{72} \text{ instead of } B_1. \tag{5.64}$$

Now we assume that $s > \sigma/2$ and we fix $x, y \in B_{1/2}$. We claim that

$$|u(x) - u(y)| \leqslant C\,|x-y|^{s-\frac{\sigma}{2}}. \tag{5.65}$$

Let $p := (x+y)/2$ and $r := |x-y|$. Notice that we may suppose that

$$r \leqslant \frac{1}{100}, \tag{5.66}$$

otherwise the fact that $|u(x)-u(y)| \leqslant 2\|u\|_{L^\infty(B_{1/2})}$ would give (5.65). Then, there are three possibilities:

$$B_{5r}(p)\setminus E = \varnothing, \tag{5.67}$$

$$B_{5r}(p)\setminus E \neq \varnothing \text{ and } u(x) = u(y) = 0, \tag{5.68}$$

$$B_{5r}(p)\setminus E \neq \varnothing \text{ and either } u(x) > 0 \text{ or } u(y) > 0. \tag{5.69}$$

If (5.68) holds true then (5.65) is obvious, therefore we consider only the possibilities (5.67) and (5.69).

If (5.67) holds, we consider $R > 0$ such that $d(p) = R$, where we recall that $d(x) = \operatorname{dist}(x, \partial E)$ denotes the distance of x from ∂E. By (5.67), we have that $5r < R < 2$.

Also, we have that

$$B_{10}(p) \subset B_{12}, \tag{5.70}$$

indeed if $x \in B_{10}(p)$ then

$$|x| \leqslant |x-p| + |p| < 10 + 2 = 12,$$

and so $x \in B_{12}$. This proves (5.70). Therefore, recalling (5.64) and using Corollary 5.4, we see that

$$|u(x)| \leqslant K(d(x))^{s-\frac{\sigma}{2}}, \quad \text{for any } x \in B_{10}(p). \tag{5.71}$$

Also, if $x \in B_R(p)$,

$$d(x) \leqslant |x-p| + d(p) \leqslant R + R = 2R,$$

therefore, from (5.71) (recall that $R < 2$), we obtain that

$$|u(x)| \leqslant K R^{s-\frac{\sigma}{2}}, \quad \text{for any } x \in B_R(p), \tag{5.72}$$

up to renaming K.

Now we use Lemma 2.3 in [16] and we obtain that $(-\Delta)^s u = 0$ in $B_{R/2}(p)$. Moreover, from Corollary 5.4 and recalling (5.64) and (5.70), we have that $\|u\|_{L^\infty(B_{R/2}(p))} \leqslant K$. So we are in position to apply Corollary 5.7 (with $t := R/2$ and $q := p$), thus obtaining that

$$\|\nabla u\|_{L^\infty(B_{R/4}(p))} \leqslant C\, R^{-1} \left(\|u\|_{L^\infty(B_{R/2}(p))} + R^{2s} \int\limits_{\mathbb{R}^n \setminus B_{R/2}(p)} \frac{|u(x)|}{|x-p|^{n+2s}}\, dx \right). \tag{5.73}$$

We notice that, by (5.72),

$$\|u\|_{L^\infty(B_{R/2}(p))} \leqslant K\, R^{s-\frac{\sigma}{2}}. \tag{5.74}$$

Moreover,

$$\begin{aligned} &\int\limits_{\mathbb{R}^n \setminus B_{R/2}(p)} \frac{|u(x)|}{|x-p|^{n+2s}}\, dx \\ &= \int\limits_{\mathbb{R}^n \setminus B_{10}(p)} \frac{|u(x)|}{|x-p|^{n+2s}}\, dx + \int\limits_{B_{10}(p) \setminus B_{R/2}(p)} \frac{|u(x)|}{|x-p|^{n+2s}}\, dx. \end{aligned} \tag{5.75}$$

Now we observe that, if $x \in \mathbb{R}^n \setminus B_{10}(p)$, then $10 < |x-p| \leqslant |x| + |p| \leqslant |x| + 2$, and so $|x| > 8$. Hence,

$$|x-p| \geqslant |x| - 2 = \frac{3}{4}|x| + \frac{1}{4}|x| - 2 \geqslant \frac{3}{4}|x| + 2 - 2 = \frac{3}{4}|x|.$$

Therefore,

$$\int\limits_{\mathbb{R}^n \setminus B_{10}(p)} \frac{|u(x)|}{|x-p|^{n+2s}}\, dx \leqslant C \int\limits_{\mathbb{R}^n \setminus B_{10}(p)} \frac{|u(x)|}{|x|^{n+2s}}\, dx \leqslant C\, \Lambda. \tag{5.76}$$

Furthermore, using (5.71), we obtain that, if $x \in B_{10}(p) \setminus B_{R/2}(p)$, then

$$|u(x)| \leqslant K\,(d(x))^{s-\frac{\sigma}{2}} \leqslant K\,(|x-p| + d(p))^{s-\frac{\sigma}{2}} = K\,(|x-p| + R)^{s-\frac{\sigma}{2}}.$$

As a consequence,

$$\int\limits_{B_{10}(p) \setminus B_{R/2}(p)} \frac{|u(x)|}{|x-p|^{n+2s}}\, dx \leqslant K \int\limits_{B_{10}(p) \setminus B_{R/2}(p)} \frac{(|x-p| + R)^{s-\frac{\sigma}{2}}}{|x-p|^{n+2s}}\, dx.$$

So, by making the change of variable $y = (x-p)/R$, we obtain

$$\begin{aligned} \int\limits_{B_{10}(p) \setminus B_{R/2}(p)} \frac{|u(x)|}{|x-p|^{n+2s}}\, dx &\leqslant K \int\limits_{B_{10/R} \setminus B_{1/2}} \frac{(R|y| + R)^{s-\frac{\sigma}{2}}}{R^{n+2s}|y|^{n+2s}} R^n\, dy \\ &\leqslant K\, R^{s-\frac{\sigma}{2}-2s} \int\limits_{\mathbb{R}^n \setminus B_{1/2}} \frac{(|y|+1)^{s-\frac{\sigma}{2}}}{|y|^{n+2s}}\, dy \leqslant C K\, R^{s-\frac{\sigma}{2}-2s}, \end{aligned}$$

for some $C > 0$. This information, (5.76) and (5.75) give

$$\int\limits_{\mathbb{R}^n \setminus B_{R/2}(p)} \frac{|u(x)|}{|x-p|^{n+2s}}\, dx \leqslant C\, \Lambda + C K\, R^{s-\frac{\sigma}{2}-2s}.$$

Plugging this and (5.74) into (5.73), we conclude that

$$\|\nabla u\|_{L^\infty(B_{R/4}(p))} \leqslant C\, R^{-1} \left(K\, R^{s-\frac{\sigma}{2}} + C\, \Lambda\, R^{2s} + C K\, R^{s-\frac{\sigma}{2}} \right) \leqslant C\, R^{s-\frac{\sigma}{2}-1}, \tag{5.77}$$

up to relabeling C (recall that $R < 2$).

From (5.77) we obtain that, for any $a, b \in B_{R/4}(p)$,

$$|u(a) - u(b)| \leqslant C\, R^{s-\frac{\sigma}{2}-1}|a-b|.$$

Since $R > 5r$, we have that $x, y \in B_{R/4}(p)$, and so

$$\begin{aligned}|u(x) - u(y)| &\leqslant C\, R^{s-\frac{\sigma}{2}-1}|x-y| \\ &= C\, R^{s-\frac{\sigma}{2}-1} r^{1-s+\frac{\sigma}{2}}|x-y|^{s-\frac{\sigma}{2}} \leqslant C|x-y|^{s-\frac{\sigma}{2}},\end{aligned}$$

where C may change from step to step. This says that (5.65) holds true in this case.

Now let us suppose that (5.69) holds true. Then there exist $z \in B_{5r}(p) \setminus E$ and $\eta \in \{x, y\}$ such that $u(\eta) > 0$. In particular $\eta \in E$ and so there exists ζ on the segment joining η and z such that $\zeta \in \partial E$. Notice that, since the ball is convex, we have that $\zeta \in B_{5r}(p)$.

Hence, we have the following picture: $\zeta \in \partial E$, x and y lie in $B_{3r}(p)$ and $B_1(\zeta) \subseteq B_2$ (where the minimization property holds, recall (5.64) and (5.66)).

Thus, with a slight abuse of notation, we suppose, up to a translation, that $\zeta = 0$. So our picture becomes that $0 \in \partial E$, x and y lie in B_{10r}, with our minimizing property in B_1.

So we consider the minimizing pair (u_r, E_r) as in (5.63), which is minimizing in $B_{1/r} \supseteq B_{50}$ (recall (5.66)). In this way, we apply formula (5.54), thus obtaining

$$\|u_r\|_{L^\infty(B_{25})} \leqslant K.$$

Notice that $r^{-1}x, r^{-1}y \in B_{10} \subset B_{25}$, hence

$$|u_r(r^{-1}x)| + |u_r(r^{-1}y)| \leqslant 2K.$$

So we obtain

$$\begin{aligned}|u(x) - u(y)| &= r^{s-\frac{\sigma}{2}}|u_r(r^{-1}x) - u_r(r^{-1}y)| \\ &\leqslant 2K r^{s-\frac{\sigma}{2}} = 2K|x-y|^{s-\frac{\sigma}{2}}.\end{aligned}$$

This proves (5.65), which in turn implies (1.5), thus completing the proof of Theorem 1.1. □

We complete this paper with a brief comment about the two-phase case (i.e. when the function u in Theorem 1.1 is not assumed to be nonnegative to start with). The additional difficulties in this setting arise since the fractional harmonic replacements do not behave nicely with respect to the operation of taking the positive part, namely the positive part of the harmonic replacement is not necessarily harmonic in its positive set. As an example, considering the fractional harmonic replacement introduced in [16], one can consider the fractional harmonic function $u(x) := x_+^s - 1$ in $(0, +\infty)$, with fixed boundary data in $(-\infty, 0) \cup (1, +\infty)$; similarly, in the case of the fractional harmonic replacement introduced here in Section 3, one can consider the case $s = 1/2$ and the harmonic function on $\mathbb{R}^2$ given by $u(x, y) = xy$.

This difficulty arising at the level of the fractional replacements in the two-phase problem reflects also into the proof of the density estimates here (precisely in the computations below (5.14) and (5.46)).

For these reasons, we believe that the investigation of density estimates and continuity properties for two-phase fractional minimizers is an interesting open problem.

Conflict of interest statement

There is no conflict of interest.

References

[1] H.W. Alt, L.A. Caffarelli, Existence and regularity for a minimum problem with free boundary, J. Reine Angew. Math. 325 (1981) 105–144.
[2] H.W. Alt, L.A. Caffarelli, A. Friedman, Variational problems with two phases and their free boundary, Trans. Am. Math. Soc. 282 (2) (1984) 431–461.

[3] I. Athanasopoulos, L.A. Caffarelli, C. Kenig, S. Salsa, An area-Dirichlet integral minimization problem, Commun. Pure Appl. Math. 54 (4) (2001) 479–499.
[4] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Universitext, Springer, New York, 2011, xiv, 599 pp.
[5] L. Caffarelli, J.-M. Roquejoffre, O. Savin, Nonlocal minimal surfaces, Commun. Pure Appl. Math. 63 (9) (2010) 1111–1144.
[6] L. Caffarelli, O. Savin, E. Valdinoci, Minimization of a fractional perimeter-Dirichlet integral functional, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 32 (4) (2015) 901–924.
[7] L. Caffarelli, L. Silvestre, An extension problem related to the fractional Laplacian, Commun. Partial Differ. Equ. 32 (7–9) (2007) 1245–1260.
[8] L. Caffarelli, L. Silvestre, Regularity results for nonlocal equations by approximation, Arch. Ration. Mech. Anal. 200 (1) (2011) 59–88.
[9] G.M. Capriani, The Steiner rearrangement in any codimension, Calc. Var. Partial Differ. Equ. 49 (1–2) (2014) 517–548.
[10] E. Di Nezza, G. Palatucci, E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, Bull. Sci. Math. 136 (5) (2012) 521–573.
[11] S. Dipierro, A. Figalli, E. Valdinoci, Strongly nonlocal dislocation dynamics in crystals, Commun. Partial Differ. Equ. 39 (12) (2014) 2351–2387.
[12] S. Dipierro, M. Medina, E. Valdinoci, Fractional elliptic problems with critical growth in the whole of $\mathbb{R}^n$, preprint, http://arxiv.org/pdf/1506.01748.pdf.
[13] S. Dipierro, O. Savin, E. Valdinoci, All functions are locally s-harmonic up to a small error, J. Eur. Math. Soc. (JEMS) (2016), in press.
[14] S. Dipierro, O. Savin, E. Valdinoci, A nonlocal free boundary problem, SIAM J. Math. Anal. 47 (6) (2015) 4559–4605.
[15] S. Dipierro, O. Savin, E. Valdinoci, Definition of fractional Laplacian for functions with polynomial growth, preprint, http://arxiv.org/pdf/1610.04663v1.pdf.
[16] S. Dipierro, E. Valdinoci, On a fractional harmonic replacement, Discrete Contin. Dyn. Syst. 35 (8) (2015) 3377–3392.
[17] D. dos Prazeres, E.V. Teixeira, Cavity problems in discontinuous media, Calc. Var. Partial Differ. Equ. 55 (1) (2016) 10.
[18] E.B. Fabes, C.E. Kenig, R.P. Serapioni, The local regularity of solutions of degenerate elliptic equations, Commun. Partial Differ. Equ. 7 (1982) 77–116.
[19] J. Heinonen, T. Kilpeläinen, O. Martio, Nonlinear Potential Theory of Degenerate Elliptic Equations, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1993, v, 363 pp.
[20] M. Kassmann, A new formulation of Harnack's inequality for nonlocal operators, C. R. Math. Acad. Sci. Paris 349 (11–12) (2011) 637–640.
[21] E.H. Lieb, M. Loss, Analysis, 2nd ed., Graduate Studies in Mathematics, vol. 14, American Mathematical Society (AMS), Providence, RI, 2001, xxii, 346 pp.
[22] E.V. Teixeira, L. Zhang, An elliptic variational problem involving level surface area on Riemannian manifolds, Rev. Mat. Iberoam. 28 (3) (2012) 759–772.
[23] A.I. Tyulenev, Boundary values of functions in a Sobolev space with Muckenhoupt weight on some non-Lipschitz domains, Sb. Math. 205 (8) (2014) 1133–1159.