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Ann. I. H. Poincaré – AN 34 (2017) 1273–1297

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

Normal form approach to global well-posedness of the quadratic derivative nonlinear Schrödinger equation on the circle

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Received 21 October 2015; received in revised form 11 October 2016; accepted 13 October 2016

Available online 18 October 2016

Abstract

We consider the quadratic derivative nonlinear Schrödinger equation (dNLS) on the circle. In particular, we develop an infinite iteration scheme of normal form reductions for dNLS. By combining this normal form procedure with the Cole–Hopf transformation, we prove unconditional global well-posedness in $L^2(\mathbb{T})$, and more generally in certain Fourier–Lebesgue spaces $\mathcal{F}L^{s,p}(\mathbb{T})$, under the mean-zero and smallness assumptions. As a byproduct, we construct an infinite sequence of quantities that are invariant under the dynamics. We also show the necessity of the smallness assumption by explicitly constructing a finite time blowup solution with non-small mean-zero initial data.

MSC: 35Q55

Keywords: Quadratic derivative nonlinear Schrödinger equation; Normal form; Cole–Hopf transform; Fourier–Lebesgue space; Well-posedness; Finite time blowup

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http://dx.doi.org/10.1016/j.anihpc.2016.10.003

1. Introduction

In this paper, we consider the Cauchy problem for the following quadratic derivative nonlinear Schrödinger equation (dNLS) posed on the circle $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$:

$$\begin{cases} i\partial_t u + \partial_x^2 u = \lambda u \partial_x u \\ u|_{t=0} = \phi, \end{cases} \qquad (t,x) \in \mathbb{R} \times \mathbb{T}, \tag{1.1}$$

where $u(t,x)$ is a complex-valued function and $\lambda \in \mathbb{C} \setminus \{0\}$. Our main goal is to study the global-in-time behaviour of solutions to (1.1). By the transformation $u \mapsto \frac{\lambda}{i}u$, we may assume $\lambda = i$ in (1.1). Therefore, in the remaining part of this paper, we consider the following Cauchy problem:

$$\begin{cases} \partial_t u = i\partial_x^2 u + u\partial_x u \\ u|_{t=0} = \phi. \end{cases} \tag{1.2}$$

Let us first compare some basic properties of (1.2) against the usual NLS with a power-type nonlinearity:

$$\partial_t u = i\partial_x^2 u \pm i|u|^{p-1}u \tag{1.3}$$

and the cubic derivative NLS (DNLS):

$$\partial_t u = i\partial_x^2 u + \partial_x(|u|^2 u). \tag{1.4}$$

It is well known that (1.3) and (1.4) are invariant under modulations: $u \mapsto e^{i\theta}u$, which leads to the conservation of mass (= the L^2-norm). On the other hand, the equation (1.2) is not invariant under such modulations. Another difference from (1.3) and (1.4) is the lack of conservation laws. While (1.3) and (1.4) enjoy the conservation of mass, momentum, and energy, the equation (1.2) does not possess any 'natural' conservation law, except for the conservation of the spatial mean:

$$\int_{\mathbb{T}} u(t,x)dx = \int_{\mathbb{T}} \phi(x)dx. \tag{1.5}$$

The Cauchy problems for the usual NLS (1.3) and the cubic DNLS (1.4) have been studied extensively by many mathematicians [9,26,4,3,23,13,14,28] and they are known to be locally/globally well-posed in Sobolev spaces H^s for some range of s. Moreover, the solution map: $u(0) \in H^s \mapsto u(t) \in H^s$ is known to be uniformly continuous on bounded sets. The situation for dNLS (1.2) is entirely different. Indeed, Christ [6] proved that (1.2) on $M = \mathbb{T}$ or $\mathbb{R}$ is ill-posed in $H^s(M)$ for any $s \in \mathbb{R}$ by exhibiting the following norm inflation phenomenon; given any $\varepsilon > 0$, there exist a solution u to (1.2) on M and $t_\varepsilon \in (0,\varepsilon)$ such that

$$\|u(0)\|_{H^s(M)} < \varepsilon \qquad \text{and} \qquad \|u(t_\varepsilon)\|_{H^s(M)} > \varepsilon^{-1}. \tag{1.6}$$

This in particular implies the failure of continuity of the solution map (at the trivial function) and hence the ill-posedness of (1.2). See also [25] (and [17,27] in the context of the related Benjamin–Ono equation) for the failure of local uniform continuity of the solution map on $\mathbb{R}$.

In view of the above ill-posedness result [6], we can not expect to have well-posedness for (1.4) in the usual Sobolev spaces. In the non-periodic setting, however, there are several local well-posedness results for (1.2), either in weighted Sobolev spaces or by adding an extra condition on initial data. Kenig–Ponce–Vega [16] and Chihara [5] established local well-posedness of (1.2) on $\mathbb{R}$ in weighted Sobolev spaces with sufficiently high regularity. In [22], Stefanov proved local well-posedness of (1.2) in $H^1(\mathbb{R})$ with the small 'disturbance' condition: $\sup_{x\in\mathbb{R}} \left|\int_{-\infty}^x \phi(y)\,dy\right| < \varepsilon$ for some $\varepsilon > 0$. Note that these results make use of the strong dispersive effect on $\mathbb{R}$ such as local smoothing estimates that are not available on the circle. In fact, there seems to be no known well-posedness result of (1.2) on $\mathbb{T}$.

Before we state our main results, let us define the homogeneous Fourier–Lebesgue space $\mathcal{F}L_0^{s,p}(\mathbb{T})$ for *mean-zero* functions on $\mathbb{T}$ via the norm:

$$\|\phi\|_{\mathcal{F}L_0^{s,p}(\mathbb{T})} = \big\| |k|^s \widehat{\phi}(k) \big\|_{\ell^p(\mathbb{Z}_0)},$$

where $\mathbb{Z}_0 := \mathbb{Z} \setminus \{0\}$. Note that, if we assume that an initial condition ϕ has mean 0, then the corresponding solution $u(t)$ to (1.2) defined on a time interval I also has mean 0 for each $t \in I$ thanks to the conservation of the spatial mean (1.5).

We now state our main results.

Theorem 1.1 *(Small data global well-posedness for mean-zero initial data). Suppose that* (s, p) *satisfy* (i) $s > \frac{1}{2} - \frac{1}{p}$, $p > 2$ *or* (ii) $s \geq 0$, $p = 2$. *Then, there exists* $\delta_0 = \delta_0(s, p) > 0$ *such that dNLS (1.2) is globally well-posed in* $\mathcal{F}L_0^{s,p}(\mathbb{T}) \cap \{\|\phi\|_{\mathcal{F}L_0^{s,p}(\mathbb{T})} \leq \delta_0\}$. *Moreover, the uniqueness holds unconditionally. Namely, the uniqueness holds in the entire* $C(\mathbb{R}; \mathcal{F}L_0^{s,p}(\mathbb{T}))$.

Given a solution u to (1.2) constructed above, there exists an infinite sequence $\{\mathcal{Q}_k(t)\}_{k\in\mathbb{Z}_0}$ *such that* $\mathcal{Q}_k = \mathcal{Q}_k[u] : \mathbb{R} \to \mathbb{C}$ *is invariant under the dynamics of (1.2) for each* $k \in \mathbb{Z}_0$.

Note that we have $\mathcal{F}L_0^{s,p}(\mathbb{T}) \subset L_0^2(\mathbb{T}) := \mathcal{F}L_0^{0,2}(\mathbb{T})$ for (s, p) satisfying the condition in Theorem 1.1. Namely, $L_0^2(\mathbb{T})$ is the largest space where Theorem 1.1 holds. We point out that the quantity $\mathcal{Q}_k(t) = \mathcal{Q}_k[u](t)$ constructed in Theorem 1.1 is not a conservation law in the usual sense. Given a global solution u constructed in Theorem 1.1, we have $\mathcal{Q}_k[u](t) = \mathcal{Q}_k[u](0)$ for all $t \in \mathbb{R}$ and $k \in \mathbb{Z}_0$. However, the definition of $\mathcal{Q}_k(t)$ depends on the information of the solution u on the entire interval $[0, t]$, whereas a conservation law in the usual sense depends only on the information of the solution u at this specific time t. See (2.26) for the definition of $\mathcal{Q}_k(t)$.

The smallness condition in Theorem 1.1 is sharp. In fact, we construct an explicit finite time blowup solution for a non-small mean-zero initial condition.

Theorem 1.2 *(Finite time blowup solution). There exists a mean-zero function* $\phi \in L_0^2(\mathbb{T})$ *and* $t_* > 0$ *such that the corresponding solution u to (1.2) on the time interval* $[0, t_*)$ *with* $u|_{t=0} = \phi$ *satisfies*

$$\lim_{t\to t_*-} \|u(t)\|_{L^p(\mathbb{T})} = \infty$$

for any $1 \leq p \leq \infty$. *In particular, we have*

$$\lim_{t\to t_*-} \|u(t)\|_{\mathcal{F}L_0^{s,p}(\mathbb{T})} = \infty$$

for (s, p) *satisfying the condition in Theorem 1.1.*

Recall the following Galilean invariance for (1.2) on $\mathbb{T}$; if u is a solution to (1.2) on $\mathbb{T}$, then

$$u_c(t, x) := u(t, x + ct) + c \tag{1.7}$$

is also a solution to (1.2) for any $c \in \mathbb{R}$. This Galilean invariance allows us to convert a function of a real-valued spatial mean to a mean-zero function. Namely, given an initial condition $\phi \in L^2(\mathbb{T})$ with $\widehat{\phi}(0) \in \mathbb{R}$, we can convert it into a mean-zero initial condition $\phi_c = \phi + c$ with $c = \widehat{\phi}(0)$ and construct a global solution $u_c \in C(\mathbb{R}; L_0^2(\mathbb{T}))$ to (1.2) with $u_c|_{t=0} = \phi_c$ as long as the transformed initial condition ϕ_c satisfies the smallness condition stated in Theorem 1.1. Then, by inverting (1.7), we obtain a global solution $u \in C(\mathbb{R}; L^2(\mathbb{T}))$ to (1.2) with $u|_{t=0} = \phi$.

Let us now point out how Theorem 1.1 does not contradict the ill-posedness result in [6] mentioned above. Christ proved the norm inflation (1.6), using initial data of the form $\phi_\varepsilon(x) = iA_\varepsilon + iB_\varepsilon e^{iN_\varepsilon x}$ with $A_\varepsilon, B_\varepsilon > 0$ and $N_\varepsilon \in \mathbb{N}$. In particular, the spatial mean of ϕ_ε is purely imaginary and hence this ill-posedness result is not applicable to our setting. Lastly, note that the Galilean invariance (1.7) does not allow us to convert a function of a purely imaginary spatial mean into a function of a real spatial mean since it would involve non-real $c \in i\mathbb{R}$.

There are two main ingredients in the proof of Theorem 1.1: (i) normal form reductions and (ii) (modified) Cole–Hopf transformation. The normal form approach will be used to handle local-in-time analysis in the low regularity setting, while the modified Cole–Hopf transformation will be used to construct smooth local-in-time solutions as well as obtain a global-in-time a priori estimate.

(i) Normal form reductions. The basic idea of normal form reductions from dynamical systems is to renormalize the flow by removing non-resonant terms in a nonlinearity at the expense of introducing higher order nonlinear terms. This is achieved by introducing a suitable new unknown; see [1,20]. We perform normal form reductions at the level of the interaction representation $v(t) := e^{it\partial_x^2} u(t)$. If u is a smooth solution to (1.2) with spatial mean 0, then the interaction representation v satisfies

$$\partial_t \widehat{v}(k) = \frac{ik}{2} \sum_{\substack{k=k_1+k_2\\ k_1,k_2\neq 0}} e^{i(k^2-k_1^2-k_2^2)t} \widehat{v}(k_1)\widehat{v}(k_2) = \frac{ik}{2} \sum_{\substack{k=k_1+k_2\\ k_1,k_2\neq 0}} e^{2ik_1k_2t} \widehat{v}(k_1)\widehat{v}(k_2) \tag{1.8}$$

for each $k \in \mathbb{Z}_0$. Here, we used the fact that $\Phi(\mathbf{k}) := k^2 - k_1^2 - k_2^2 = 2k_1k_2$ under $k = k_1 + k_2$. Then, by performing differentiation by parts, i.e. integration by parts without an integral sign, we obtain

$$\begin{aligned}\partial_t \widehat{v}(k) &= \partial_t \left[\frac{k}{2} \sum_{\substack{k=k_1+k_2\\ k_1,k_2\neq 0}} e^{i\Phi(\mathbf{k})t} \frac{\widehat{v}(k_1)\widehat{v}(k_2)}{\Phi(\mathbf{k})} \right] - \frac{k}{2} \sum_{\substack{k=k_1+k_2\\ k_1,k_2\neq 0}} e^{i\Phi(\mathbf{k})t} \frac{\partial_t\{\widehat{v}(k_1)\widehat{v}(k_2)\}}{\Phi(\mathbf{k})} \\ &=: \partial_t \widehat{\mathcal{N}^2(v)}(k) + \widehat{\mathcal{B}^3(v)}(k),\end{aligned} \tag{1.9}$$

for each $k \in \mathbb{Z}_0$. By this process, we have the modulation function $\Phi(\mathbf{k}) = 2k_1k_2$ appearing in the denominators, yielding gain of derivatives.[1] The price we have to pay is that $w := v - \mathcal{N}^2(v)$ satisfies $\partial_t w = \mathcal{B}^3(v)$, where $\mathcal{B}^3(v)$ now consists of a cubic nonlinearity in view of (1.8) and (1.9). In order to handle $\mathcal{B}^3(v)$, we need to perform differentiation by parts again. In fact, we will iterate this differentiation by parts process infinitely many times (with suitable adjustments at each step) in Section 2 to renormalize (1.2) into a simpler equation (see the normal form equation (2.24) below), where nonlinear analysis can be carried out with simple tools such as Hölder's and Young's inequalities. In particular, we do not employ any auxiliary function spaces such as Strichartz spaces and the $X^{s,b}$-spaces, allowing us to prove the unconditional uniqueness in Theorem 1.1.

In [2], Babin–Ilyin–Titi introduced this normal form approach via the differentiation by parts for constructing solutions to the KdV equation on $\mathbb{T}$. Subsequently, this idea was applied to other dispersive PDEs [10,11,18], allowing us to construct solutions to dispersive PDEs without relying on the Fourier restriction norm method. In [10], we further developed this idea and successfully implemented an infinite iteration scheme of the so-called Poincaré–Dulac normal form reductions for the cubic NLS on $\mathbb{T}$ in the low regularity setting. This normal form approach has various applications such as exhibiting nonlinear smoothing [8] and establishing a good energy estimate [21].

(ii) (modified) Cole–Hopf transformation. It is well known that the Cole–Hopf transformation [7,15] transforms the viscous Burgers equation on $\mathbb{R}$:

$$\partial_t u = \partial_x^2 u + u\partial_x u \tag{1.10}$$

into the linear heat equation. More precisely, if u is a smooth solution to the viscous Burgers equation (1.10), then

$$w(t,x) := e^{-\frac{1}{2}\int_{-\infty}^x u(t,y)dy}$$

solves the linear heat equation on $\mathbb{R}$: $\partial_t w = \partial_x^2 w$. A similar trick works for the quadratic dNLS (1.2) on $\mathbb{R}$. If $u \in C^\infty_{t,\mathrm{loc}}(\mathbb{R}; \mathcal{S}(\mathbb{R}))$ solves dNLS (1.2) on $\mathbb{R}$, then

$$w(t,x) := e^{-\frac{i}{2}\int_{-\infty}^x u(t,y)dy} \tag{1.11}$$

solves the linear Schrödinger equation: $\partial_t w = i\partial_x^2 w$. See, for example, [25]. Similar gauge transformations played an important role in studying other dispersive PDEs with derivative nonlinearities such as the cubic DNLS (1.4) and the Benjamin–Ono equation [12,24].

In the periodic case, these gauge transformations have been suitably adjusted to study well-posedness of the cubic DNLS (1.4) and the Benjamin–Ono equation. See [13,19]. For our problem, a naive approach would be the following. Given a mean-zero function ϕ on $\mathbb{T}$, we define the Cole–Hopf transformation $\mathcal{G}_0$ by

$$\mathcal{G}_0[\phi] := e^{-\frac{i}{2}\mathcal{J}(\phi)},$$

[1] Compare this with the standard Fourier restriction norm method, where one gains only $\sim \frac{1}{2}$-power of the modulation function $\Phi(\mathbf{k})$.

where $\mathcal{J}(\phi)$ is the mean-zero primitive of ϕ defined by $\mathcal{J}(\phi)_k := \frac{\phi_k}{ik}$ if $k \neq 0$, and $\mathcal{J}(\phi)_k := 0$ if $k = 0$. Note that $\mathcal{G}_0[\phi]$ is a periodic function on $\mathbb{T}$, since $\mathcal{J}(\phi)$ is periodic. Given a smooth mean-zero solution u to dNLS (1.2) on $\mathbb{T}$, let $w(t) := \mathcal{G}_0[u(t)]$. It turns out that w does not quite satisfy the linear Schrödinger equation. Hence, we need to introduce a suitable adjustment to the Cole–Hopf transformation; see (3.3) below.

Once we appropriately define the modified Cole–Hopf transformation $\mathcal{G}$, we can transform any smooth mean-zero solution u to (1.2) to a solution $W = \mathcal{G}[u]$ to the linear Schrödinger equation. Note that, even if u is known to satisfy (1.2) locally in time, the gauged function W exists globally in time as a solution to the linear equation. Then, an important question is to investigate the invertibility of this transformation. It turns out that the inverse transformation is given by

$$u(t,x) = \mathcal{G}^{-1}[W](t,x) := 2i\,\frac{\partial_x W(t,x)}{W(t,x)}$$

for a smooth mean-zero solution u to (1.2). Thus, $W(t,x) \neq 0$ guarantees the invertibility of the modified Cole–Hopf transformation. In Subsection 3.2, we discuss possible sufficient conditions for the invertibility in details and construct smooth global solutions to (1.2).

Interestingly, by interpreting this modified Cole–Hopf transformation from a geometric point of view, we obtain a necessary condition for possible spatial means of initial data ϕ for (1.2). Given a smooth solution u to (1.2) on some time interval I, $u(t)$ is a periodic function for each $t \in I$. In particular, it is a closed loop in the complex plane. Then, the transformed function $W(t) = \mathcal{G}[u](t)$ is also a closed loop in the complex plane. Therefore, W must have a well defined index around the origin. This observation leads to $\int_{\mathbb{T}} \phi(x)dx = 4\pi n$, $n \in \mathbb{Z}$. See Remark 3.1.

We conclude this introduction by mentioning a hidden connection between the normal form reductions and the modified Cole–Hopf transformation. In Section 2, we obtain an infinite series as a result of an infinite iteration of normal form reductions. It turns out that this series is nothing but the Taylor expansion of (the derivative of the interaction representation of) the transformed function $W = \mathcal{G}[u]$ solving the linear Schrödinger equation. Namely, the two seemingly different approaches reduce (1.2) to the same linear Schrödinger equation (up to a differentiation); see Remark 3.2. Such a reducibility to a linear equation by normal form reductions is an important question, closely related to integrability. For example, see Nikolenko [20] in the smooth setting.

This paper is organized as follows. In Section 2, we implement an infinite iteration scheme of normal form reductions and rewrite (1.2) as the normal form equation (2.24). By establishing nonlinear estimates, we prove unconditional local well-posedness of the normal form equation. In Section 3, we introduce a modified Cole–Hopf transformation, converting (1.2) into the linear Schrödinger equation. By investigating a sufficient condition for inverting the modified Cole–Hopf transform, we prove small data global well-posedness of (1.2) in $L^2_0(\mathbb{T})$. In Section 4, we put together the results from Sections 2 and 3 and prove Theorem 1.1. In Appendix A, we present the proof of Theorem 1.2 by constructing an explicit example.

In view of the time-reversibility of the equation (1.2), we only consider positive times in the following.

2. Normal form reductions

In this section, we perform normal form reductions in an iterative manner and rewrite (1.2) as an equation involving infinite series. Then, we establish an a priori estimate on solutions in the Fourier–Lebesgue spaces. As mentioned in Section 1, our main approach is to apply differentiation by parts iteratively and reduce an equation into a *simpler* form at each step. This procedure can be seen as an infinite dimensional version of the Poincaré–Dulac normal form reduction[2] for a finite dimensional system of ODEs [1]. On the one hand, two iterations were sufficient in [2,11,18]. On the other hand, we needed to iterate normal form reductions infinitely many times in [10] to prove unconditional global well-posedness for the cubic NLS on $\mathbb{T}$. In the following, we also set up an infinite iteration scheme for (1.2). As we see below, our argument for (1.2) is somewhat simpler than that in [10]. This is due to miraculous symmetrizations at each step of the normal form reductions, which allows us to write an equation at each step in a very simple form. In particular, there will be no resonant term in the equation at each step (modulo the correction terms $\mathcal{R}^n(v)$ appearing in (2.8), (2.14), and (2.20)). Note that all the analysis in this section is of local-in-time nature.

[2] In fact, our approach for (1.2) is an infinite dimensional analogue of the Poincaré normal form reduction. Namely, there is no resonant term (modulo the correction terms) at each step of the iteration.

2.1. Formal iteration argument

In the following, we first perform normal form reductions at a formal level. Namely, in this subsection, we do not worry about issues such as the convergence of infinite series, switching the time derivative with an infinite series, etc. These issues will be addressed in the next subsection.

Before proceeding further, let us introduce some notations. Given a periodic function $f \in L^2(\mathbb{T})$, we define its Fourier coefficient f_k by

$$f_k := \frac{1}{2\pi} \int_{\mathbb{T}} f(x) e^{-ikx} dx.$$

The Fourier inversion formula states

$$f(x) = \sum_{k\in\mathbb{Z}} f_k e^{ikx}.$$

Then, we can rewrite (1.2) as

$$\partial_t u_k = -ik^2 u_k + \frac{ik}{2} \sum_{k=k_1+k_2} u_{k_1} u_{k_2}. \tag{2.1}$$

In the following, we assume that $\int \phi\, dx = 0$. Hence, it follows from the conservation of mean (1.5) that $u_0(t) = 0$ for all $t \in \mathbb{R}$ and it is understood that the summation is over non-zero frequencies $\mathbb{Z}_0 := \mathbb{Z} \setminus \{0\}$.

We now introduce the interaction representation v by setting

$$v(t) := S(-t)u(t), \tag{2.2}$$

where $S(t) = e^{it\partial_x^2}$. In terms of the Fourier coefficients, we have $v_k(t) = e^{ik^2 t} u_k(t)$. Then, the equation (2.1) becomes

$$\partial_t v_k = \frac{ik}{2} \sum_{k=k_1+k_2} e^{i(k^2-k_1^2-k_2^2)t} v_{k_1} v_{k_2}. \tag{2.3}$$

Given $\mathbf{k} = (k_1, k_2, \cdots) \in \mathbb{Z}_0^\infty$ and $n \in \mathbb{N}$, we define a length $|\mathbf{k}|_n$ and a modulation function $\Phi_n(\mathbf{k})$ by setting

$$|\mathbf{k}|_n := k_1 + k_2 + \cdots + k_n,$$
$$\Phi_n(\mathbf{k}) := \Big(\sum_{i=1}^n k_i\Big)^2 - \sum_{i=1}^n k_i^2 = \sum_{1\le i<j\le n} 2k_i k_j. \tag{2.4}$$

By definition, we set $\Phi_1(\mathbf{k}) = 0$. Note that the modulation function $\Phi_2(\mathbf{k})$ appears as a phase factor in (2.3). As we see below, the modulation function $\Phi_{n+1}(\mathbf{k})$ appears as a phase factor at the nth step of the iteration argument. On the one hand, if $\Phi_n(\mathbf{k}) \neq 0$, corresponding to the so-called *non-resonant* term, and it is large in particular, then we expect some cancellation under a time integration. On the other hand, if $\Phi_n(\mathbf{k}) = 0$, corresponding to the so-called *resonant* term, then there is no cancellation under a time integration. In the following, we exploit symmetrizations at each step and show that all the terms we obtain are indeed non-resonant.

With $\Phi_2(\mathbf{k}) = 2k_1k_2$, we can rewrite (2.3) as

$$\partial_t v_k = \frac{k}{4} \sum_{|\mathbf{k}|_2=k} i\Phi_2(\mathbf{k}) e^{i\Phi_2(\mathbf{k})t} \frac{v_{k_1} v_{k_2}}{k_1 k_2} =: \mathcal{I}^2(v)(k). \tag{2.5}$$

Note that there is no resonant term $\Phi_2(\mathbf{k}) = 0$ in (2.5) because $\mathbf{k} \in \mathbb{Z}_0^\infty$. For conciseness, we use $\mathcal{I}_k^2(v)$ to denote the kth Fourier coefficient $\mathcal{I}^2(v)(k)$ of the multilinear form $\mathcal{I}^2(v)$ in the following. A similar comment applies to other multilinear expressions.

We apply differentiation by parts and obtain

$$\begin{aligned}\mathcal{I}_k^2(v) &= \partial_t\Big[\frac{k}{4} \sum_{|\mathbf{k}|_2=k} e^{i\Phi_2(\mathbf{k})t} \frac{v_{k_1} v_{k_2}}{k_1 k_2}\Big] - \frac{k}{4} \sum_{|\mathbf{k}|_2=k} e^{i\Phi_2(\mathbf{k})t} \frac{\partial_t(v_{k_1} v_{k_2})}{k_1 k_2} \\ &=: \partial_t \mathcal{N}_k^2(v) + \mathcal{B}_k^3(v).\end{aligned} \tag{2.6}$$

By symmetry, we assume that the time derivative in $\partial_t(v_{k_1}v_{k_2})$ falls only on v_{k_2} and we double the contribution. From (2.6) with (2.3), we have

$$\begin{aligned}\mathcal{B}_k^3(v) &= -2\cdot\frac{k}{4}\sum_{|\mathbf{k}|_2=k} e^{i\Phi_2(\mathbf{k})t}\frac{v_{k_1}\partial_t v_{k_2}}{k_1k_2}\\ &= -\frac{ik}{4}\sum_{k=k_1+k_2} e^{i\Phi_2(\mathbf{k})t}\frac{v_{k_1}}{k_1}\sum_{\substack{k_2=m_1+m_2\\ k_2\neq 0}} e^{i(k_2^2-m_1^2-m_2^2)t}v_{m_1}v_{m_2}.\end{aligned}\tag{2.7}$$

In order to apply a symmetrization, we need to add and subtract the contribution from $m_1+m_2=0$. For this purpose, define $\mathcal{R}_k^2(v)$ and $\mathcal{I}_k^3(v)$ by

$$\mathcal{R}_k^2(v) = \frac{i}{4}v_k M(u) = \frac{i}{4}v_k\sum_{m\in\mathbb{Z}_0} e^{-2im^2t}v_m v_{-m} \quad\text{and}\quad \mathcal{I}_k^3(v) := \mathcal{B}_k^3(v) - \mathcal{R}_k^2(v),\tag{2.8}$$

where $M(u)$ is given by

$$M(u) := \mathbf{P}_0[u^2] = \frac{1}{2\pi}\int_{\mathbb{T}} u^2\,dx.\tag{2.9}$$

Here, $\mathbf{P}_0$ denotes the Dirichlet projection onto the zeroth frequency. We point out that $M(u)\neq\|u\|_{L^2}^2$. From (2.6) and (2.8), we have

$$\mathcal{I}_k^2(v) = \partial_t\mathcal{N}_k^2(v) + \mathcal{R}_k^2(v) + \mathcal{I}_k^3(v).\tag{2.10}$$

By symmetrization in $\{k_1,k_2,k_3\}$, we have

$$\begin{aligned}\mathcal{I}_k^3(v) := \mathcal{B}_k^3(v) - \mathcal{R}_k^2(v) &= -\frac{ik}{4}\sum_{k=k_1+m_1+m_2} e^{i(k^2-k_1^2-m_1^2-m_2^2)t}\frac{v_{k_1}}{k_1}v_{m_1}v_{m_2}\\ &= -\frac{ik}{8}\sum_{|\mathbf{k}|_3=k} 2k_2k_3\, e^{i\Phi_3(\mathbf{k})t}\frac{v_{k_1}v_{k_2}v_{k_3}}{k_1k_2k_3}\\ &= -\frac{k}{4\cdot\binom{3}{2}}\sum_{|\mathbf{k}|_3=k} i\Phi_3(\mathbf{k})e^{i\Phi_3(\mathbf{k})t}\frac{v_{k_1}v_{k_2}v_{k_3}}{k_1k_2k_3}\\ &= -\frac{k}{2^2\cdot 3!}\sum_{|\mathbf{k}|_3=k} i\Phi_3(\mathbf{k})e^{i\Phi_3(\mathbf{k})t}\frac{v_{k_1}v_{k_2}v_{k_3}}{k_1k_2k_3}.\end{aligned}\tag{2.11}$$

Note that the symmetrization cancelled out the resonant terms ($\Phi_3(\mathbf{k})=0$) and there is no resonant term in the final expression in (2.11). Therefore, we can perform differentiation by parts on $\mathcal{I}_k^3(v)$

Remark 2.1. Due to the presence of $e^{i\Phi_n(\mathbf{k})t}$ in their definitions, the multilinear expressions $\mathcal{I}^n(v)$ and $\mathcal{N}^n(v)$ are non-autonomous and in fact depend on t. Thus, strictly speaking, we should denote them by $\mathcal{I}_k^n(t)(v(t))$ and $\mathcal{N}_k^n(t)(v(t))$. For simplicity of notations, however, we suppress such t-dependence when there is no confusion. The same comment applies to $\mathcal{B}^n(v)$ and $\mathcal{R}^n(v)$.

Differentiating $\mathcal{I}_k^3(v)$ by parts yields

$$\begin{aligned}\mathcal{I}_k^3(v) &= \partial_t\left[\frac{-k}{2^2\cdot 3!}\sum_{|\mathbf{k}|_3=k} e^{i\Phi_3(\mathbf{k})t}\frac{v_{k_1}v_{k_2}v_{k_3}}{k_1k_2k_3}\right] + \frac{k}{2^2\cdot 3!}\sum_{|\mathbf{k}|_3=k} e^{i\Phi_3(\mathbf{k})t}\frac{\partial_t(v_{k_1}v_{k_2}v_{k_3})}{k_1k_2k_3}\\ &=: \partial_t\mathcal{N}_k^3(v) + \mathcal{B}_k^4(v).\end{aligned}\tag{2.12}$$

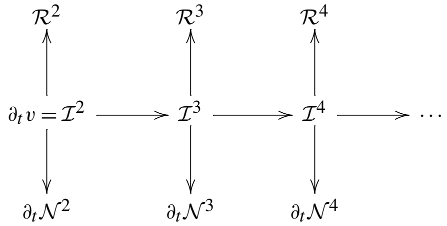

Fig. 1. Schematic diagram of the iteration process.

Then, by symmetry as above, we have

$$\begin{aligned}\mathcal{B}_k^4(v) &= 3 \cdot \frac{k}{2^2 \cdot 3!} \sum_{|\mathbf{k}|_3 = k} e^{i\Phi_3(\mathbf{k})t} \frac{v_{k_1} v_{k_2} \partial_t v_{k_3}}{k_1 k_2 k_3} \\ &= \frac{ik}{2^4} \sum_{|\mathbf{k}|_3 = k} e^{i\Phi_3(\mathbf{k})t} \frac{v_{k_1} v_{k_2}}{k_1 k_2} \sum_{\substack{k_3 = m_1 + m_2 \\ k_3 \neq 0}} e^{i(k_3^2 - m_1^2 - m_2^2)t} v_{m_1} v_{m_2}.\end{aligned} \tag{2.13}$$

Define $\mathcal{R}_k^3(v)$ and $\mathcal{I}_k^4(v)$ by

$$\mathcal{R}_k^3(v) = \frac{-ik}{2^4} M(u) \sum_{|\mathbf{k}|_2 = k} e^{i\Phi_2(\mathbf{k})t} \frac{v_{k_1} v_{k_2}}{k_1 k_2} \quad \text{and} \quad \mathcal{I}_k^4(v) = \mathcal{B}_k^4(v) - \mathcal{R}_k^3(v). \tag{2.14}$$

From (2.12) and (2.14), we have

$$\mathcal{I}_k^3(v) = \partial_t \mathcal{N}_k^3(v) + \mathcal{R}_k^3(v) + \mathcal{I}_k^4(v). \tag{2.15}$$

From (2.13), (2.14), and symmetrization, we have

$$\begin{aligned}\mathcal{I}_k^4(v) &= \frac{ik}{2^5} \sum_{|\mathbf{k}|_4 = k} 2k_3 k_4 e^{i\Phi_4(\mathbf{k})t} \frac{v_{k_1} v_{k_2} v_{k_3} v_{k_4}}{k_1 k_2 k_3 k_4} \\ &= \frac{k}{2^5 \cdot \binom{4}{2}} \sum_{|\mathbf{k}|_4 = k} i\Phi_4(\mathbf{k}) e^{i\Phi_4(\mathbf{k})t} \frac{v_{k_1} v_{k_2} v_{k_3} v_{k_4}}{k_1 k_2 k_3 k_4} \\ &= \frac{k}{2^3 \cdot 4!} \sum_{|\mathbf{k}|_4 = k} i\Phi_4(\mathbf{k}) e^{i\Phi_4(\mathbf{k})t} \frac{v_{k_1} v_{k_2} v_{k_3} v_{k_4}}{k_1 k_2 k_3 k_4}.\end{aligned}$$

Fig. 1 shows this iteration procedure schematically. We iterate this process of differentiation by parts and symmetrization indefinitely.

The following lemma describes the equation we obtain after the nth step.

Lemma 2.2. *Let $n \in \mathbb{N}$. Then, the equation (2.3) can be rewritten as follows:*

$$\begin{aligned}\partial_t v_k &= \mathcal{I}_k^2(v) = \partial_t \mathcal{N}_k^2(v) + \mathcal{R}_k^2(v) + \mathcal{I}_k^3(v) \\ &= \cdots = \partial_t \left(\sum_{j=2}^n \mathcal{N}_k^j(v) \right) + \sum_{j=2}^n \mathcal{R}_k^j(v) + \mathcal{I}_k^{n+1}(v),\end{aligned} \tag{2.16}$$

where $\mathcal{I}^n(v)$, $\mathcal{N}^n(v)$, and $\mathcal{R}^n(v)$ satisfy the following recursive relation:

$$\mathcal{I}_k^n(v) = \partial_t \mathcal{N}_k^n(v) + \mathcal{R}_k^n(v) + \mathcal{I}_k^{n+1}(v), \quad n \geq 2. \tag{2.17}$$

Moreover, the Fourier coefficients of $\mathcal{I}^n(v)$, $\mathcal{N}^n(v)$, and $\mathcal{R}^n(v)$, $n \geq 2$, are given by

$$\mathcal{I}_k^n(v) = \frac{(-1)^n k}{2^{n-1} \cdot n!} \sum_{|\mathbf{k}|_n = k} i\Phi_n(\mathbf{k}) e^{i\Phi_n(\mathbf{k})t} \prod_{j=1}^n \frac{v_{k_j}}{k_j}, \tag{2.18}$$

$$\mathcal{N}_k^n(v) = \frac{(-1)^n k}{2^{n-1}\cdot n!} \sum_{|\mathbf{k}|_n = k} e^{i\Phi_n(\mathbf{k})t} \prod_{j=1}^{n} \frac{v_{k_j}}{k_j}, \tag{2.19}$$

$$\mathcal{R}_k^n(v) = \frac{-i}{4} M(u)\cdot \mathcal{N}_k^{n-1}(v), \tag{2.20}$$

with the convention that

$$\mathcal{N}^1(v) = -v. \tag{2.21}$$

Remark 2.3. We point out that the factor $n!$ appears in the denominator of $\mathcal{N}^n(v)$. This corresponds to a functional version of the Taylor expansion of an exponential function. See Section 3.

Proof. We argue by induction. The case $n = 1, 2, 3$ follows from (2.5), (2.10), and (2.15). Now, suppose that (2.16) – (2.20) hold for some $n \in \mathbb{N}$. Differentiating $\mathcal{I}_k^{n+1}(v)$ by parts, we have

$$\begin{aligned}\mathcal{I}_k^{n+1}(v) &= \partial_t\Bigg[\frac{(-1)^{n+1}k}{2^n\cdot(n+1)!}\sum_{|\mathbf{k}|_{n+1}=k} e^{i\Phi_{n+1}(\mathbf{k})t}\prod_{j=1}^{n+1}\frac{v_{k_j}}{k_j}\Bigg]\\ &\quad + \frac{(-1)^{n+2}k}{2^n\cdot(n+1)!}\sum_{|\mathbf{k}|_{n+1}=k} e^{i\Phi_{n+1}(\mathbf{k})t}\partial_t\Bigg(\prod_{j=1}^{n+1}\frac{v_{k_j}}{k_j}\Bigg) =: \partial_t\mathcal{N}_k^{n+1}(v) + \mathcal{B}_k^{n+2}(v).\end{aligned} \tag{2.22}$$

The first term $\mathcal{N}_k^{n+1}(v)$ readily satisfies (2.19). Let $\mathcal{R}_k^{n+1}(v)$ be as in (2.20). By symmetrization as before, we have

$$\begin{aligned}\mathcal{I}_k^{n+2}(v) &:= \mathcal{B}_k^{n+2}(v) - \mathcal{R}_k^{n+1}(v)\\ &= (n+1)\cdot\frac{(-1)^{n+2}ik}{2^{n+1}\cdot(n+1)!}\sum_{|\mathbf{k}|_{n+1}=k} e^{i\Phi_{n+1}(\mathbf{k})t}\prod_{j=1}^{n}\frac{v_{k_j}}{k_j}\\ &\quad\times\sum_{k_{n+1}=m_1+m_2} e^{i(k_{n+1}^2-m_1^2-m_2^2)t} v_{m_1}v_{m_2}\\ &= \frac{(-1)^{n+2}k}{2^n\cdot n!\cdot 4\cdot\binom{n+2}{2}}\sum_{|\mathbf{k}|_{n+2}=k} i\Phi_{n+2}(\mathbf{k})e^{i\Phi_{n+2}(\mathbf{k})t}\prod_{j=1}^{n+2}\frac{v_{k_j}}{k_j},\end{aligned}$$

which agrees with (2.18). Lastly, (2.16) for the $(n+1)$-st step follows from (2.16) for the nth step and (2.22). □

The proof of Lemma 2.2 shows that, at each step of the normal form reductions, the symmetrization completely eliminates the resonant terms in $\mathcal{I}^n(v)$. Now, suppose that $\mathcal{I}_k^{n+1}(v) \to 0$ as $n \to \infty$. Then, by taking $n \to \infty$ in (2.16), we *formally* arrive at the limit equation:

$$\partial_t v_k = \partial_t\Bigg(\sum_{n=2}^{\infty}\mathcal{N}_k^n(v)\Bigg) + \sum_{n=2}^{\infty}\mathcal{R}_k^n(v). \tag{2.23}$$

Integrating (2.23) in time and applying the Fourier inversion formula, we obtain the following *normal form equation*:

$$v(t) = \phi + \sum_{n=2}^{\infty}\mathcal{N}^n(v(t)) - \sum_{n=2}^{\infty}\mathcal{N}^n(\phi) + \int_0^t \sum_{n=2}^{\infty}\mathcal{R}^n(v(t'))dt'. \tag{2.24}$$

In the following, we refer to (2.24) as the *normal form equation*.

Remark 2.4. (i) With (2.20) and (2.21), we can rewrite (2.23) as

$$\partial_t\Bigg(\sum_{n=1}^{\infty}\mathcal{N}_k^n(v)\Bigg) = \frac{i}{4}M(u)\Bigg(\sum_{n=1}^{\infty}\mathcal{N}_k^n(v)\Bigg), \tag{2.25}$$

where $u(t) = e^{it\partial_x^2} v(t)$. Given a smooth function v on $\mathbb{R} \times \mathbb{T}$, define

$$\mathcal{Q}_k(t) = \mathcal{Q}_k([0,t]) := e^{-\frac{i}{4}\int_0^t M(u)(t')dt'} \sum_{n=1}^{\infty} \mathcal{N}_k^n(v) \tag{2.26}$$

for $k \in \mathbb{Z}_0$. Then, given a global solution v to (2.24), the equation (2.25) formally yields

$$\mathcal{Q}_k(t) = \mathcal{Q}_k(0) \tag{2.27}$$

for any $t \in \mathbb{R}$. Hence, we found an infinite sequence $\{\mathcal{Q}_k(t)\}_{k\in\mathbb{Z}_0}$ of the quantities that are formally invariant under the dynamics of the normal form equation (2.24). We point out, however, that $\mathcal{Q}_k$ is *not* a conservation law in the usual sense. Namely, the definition (2.26) of $\mathcal{Q}_k(t)$ depends on the information of the solution v on the entire interval $[0,t]$.

(ii) Define $q(t,x)$ by setting

$$q_k(t) = e^{-ik^2t}\mathcal{Q}_k(t), \quad k \in \mathbb{Z}_0.$$

Then, it follows from (2.27) that $q_k(t) = e^{-ik^2t}\mathcal{Q}_k(0)$. Namely, q is a solution to the linear Schrödinger equation: $\partial_t q = i\partial_x^2 q$. Therefore, via the infinite iteration of normal form reductions, we have found, at least at a formal level, a transformation $u \mapsto q$, mapping a solution to (1.2) to a solution to the linear Schrödinger equation. See Section 3 for more discussion on this issue.

2.2. Analysis of the normal form equation

In the previous subsection, we reduced the original dNLS (1.2) to the normal form equation (2.24) through an infinite sequence of normal form reductions. In this subsection, we first justify the formal computations performed in the previous section, at least for smooth solutions to (1.2). Then, we prove local well-posedness of (2.24) by establishing certain multilinear estimates. In Section 3, we transfer this well-posedness of (2.24) to the original problem (1.2).

We first introduce a constant $Z_{s,p}$ which we will frequently use.

Definition 2.5. For $s, p \in \mathbb{R}$ satisfying $s > -\frac{1}{p}$ and $p \geq 1$, we define a constant $Z_{s,p}$ by

$$Z_{s,p} := \big\| |k|^{-(s+1)} \big\|_{\ell^{p'}(\mathbb{Z}_0)} = \Big[2\,\zeta\big((s+1)\,p'\big)\Big]^{\frac{1}{p'}} < \infty,$$

where $\zeta(\tau) = \sum_{k=1}^{\infty} k^{-\tau}$ is the Riemann zeta function and p' is the Hölder conjugate of p: $\frac{1}{p} + \frac{1}{p'} = 1$. If $p = 1$, we can also define $Z_{s,p}$ for all $s \geq -1$ by

$$Z_{s,1} := \big\| |k|^{-(s+1)} \big\|_{\ell^{\infty}(\mathbb{Z}_0)} = 1 \quad \text{for all } s \geq -1.$$

For $s, p \in \mathbb{R}$ satisfying $s > \frac{1}{2} - \frac{1}{p}$ and $p > 2$, we also define a constant $z_{s,p}$ by

$$z_{s,p} := Z_{s-1,\frac{2p}{p+2}} < \infty$$

such that we have

$$\|f\|_{L^2} \leq z_{s,p}\|f\|_{\mathcal{F}L_0^{s,p}} \tag{2.28}$$

for any mean-zero function f on $\mathbb{T}$.

The following proposition shows that a smooth solution to (1.2) indeed satisfies the normal form equation (2.24).

Proposition 2.6. *Given $T > 0$, let u be a smooth solution to dNLS (1.2) on $[0,T]$ with a smooth initial data $u|_{t=0} = \phi$ satisfying $\int_{\mathbb{T}} \phi\, dx = 0$. Then, its interaction representation $v(t) = e^{-it\partial_x^2}u(t)$ satisfies the normal form equation (2.24) on $[0,T]$.*

Proof. We break the proof into two parts. We first prove that v satisfies (2.23) for each $k \in \mathbb{Z}_0$, assuming that $v \in C([0,T]; L^2(\mathbb{T}))$. Then, by imposing a higher regularity, we show that v satisfies the normal form equation (2.24).

Part 1: In view of the time reversibility of (1.2) and (2.24), we only consider positive times. Fix $T > 0$ and define constants $C_0 = C_0(T)$ by

$$C_0 = C_0(T) := \sup_{t\in[0,T]} \|u(t)\|_{L^2_x}.$$

Then, from $|v_k(t)| = |u_k(t)|$ and Cauchy–Schwarz inequality, we have

$$\sup_{t\in[0,T]} \|v_k(t)\|_{\ell^\infty} \le C_0 \quad \text{and} \quad \sup_{t\in[0,T]} \|k^{-1} v_k(t)\|_{\ell^1} \le Z_{0,2} C_0.$$

Hence for given $k \in \mathbb{Z}_0$, it follows from (2.18) with (2.4) and Young's inequality that

$$\begin{aligned}
|\mathcal{I}_k^n(v(t))| &\le \frac{|k|}{2^{n-1}n!} \sum_{|\mathbf{k}|_n = k} \big|\Phi_n(\mathbf{k})\big| \prod_{\ell=1}^n \frac{|v_{k_\ell}(t)|}{|k_\ell|} \\
&\le \frac{|k|}{2^{n-2}n!} \sum_{1\le i<j\le n} \sum_{|\mathbf{k}|_n = k} |v_{k_i}(t)|\,|v_{k_j}(t)| \prod_{\substack{1\le \ell\le n\\ \ell\ne i,j}} \frac{|v_{k_\ell}(t)|}{|k_\ell|} \\
&\le \frac{|k|}{2^{n-2}n!} \sum_{1\le i<j\le n} \|v_k(t)\|_{\ell^2}^2 \|k^{-1} v_k(t)\|_{\ell^1}^{n-2} \\
&\le |k| \frac{Z_{0,2}^{n-2} C_0^n}{2^{n-1}(n-2)!}
\end{aligned} \tag{2.29}$$

uniformly in $t \in [0,T]$. Therefore, we have

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \mathcal{I}_k^n(v(t)) = 0 \tag{2.30}$$

for each $k \in \mathbb{Z}_0$. By a similar computation with (2.19) and the triangle inequality $|k| \le \sum_{j=1}^n |k_j|$, we have

$$\sup_{t\in[0,T]} |\mathcal{N}_k^n(v(t))| \le \sup_{t\in[0,T]} \frac{1}{2^{n-1}(n-1)!} \|v_k(t)\|_{\ell^\infty} \|k^{-1} v_k(t)\|_{\ell^1}^{n-1} \le \frac{Z_{0,2}^{n-1} C_0^n}{2^{n-1}(n-1)!}. \tag{2.31}$$

In particular, $\sum_{n=2}^\infty \mathcal{N}_k^n(v(t))$ converges (absolutely and uniformly in $t \in [0,T]$).

From (2.20) and (2.31), we obtain

$$\sup_{t\in[0,T]} |\mathcal{R}_k^n(v(t))| \le \frac{Z_{0,2}^{n-2} C_0^{n-1}}{2^n (n-2)!} \sup_{t\in[0,T]} M(u(t)) \le \frac{Z_{0,2}^{n-2} C_0^{n+1}}{2^n (n-2)!} \tag{2.32}$$

for $n \ge 3$. When $n = 2$, we have

$$\sup_{t\in[0,T]} |\mathcal{R}_k^2(v(t))| \le \frac{1}{4} C_0 \sup_{t\in[0,T]} |v_k(t)| \le \frac{1}{4} C_0^3,$$

which is precisely (2.32) with $n = 2$. Hence, $\sum_{n=2}^\infty \mathcal{R}_k^n(v(t))$ converges absolutely and uniformly in $t \in [0,T]$.

Lastly, it follows from (2.17), (2.29), and (2.32) that $\sum_{n=2}^\infty \partial_t \mathcal{N}_k^n(v(t))$ converges absolutely and uniformly in $t \in [0,T]$. Indeed, we have

$$\begin{aligned}
\sup_{t\in[0,T]} |\partial_t \mathcal{N}_k^n(v(t))| &\le \sup_{t\in[0,T]} |\mathcal{I}_k^n(v(t))| + \sup_{t\in[0,T]} |\mathcal{I}_k^{n+1}(v(t))| + \sup_{t\in[0,T]} |\mathcal{R}_k^n(v(t))| \\
&\le |k| \left[\frac{Z_{0,2}^{n-2} C_0^n}{2^{n-1}(n-2)!} + \frac{Z_{0,2}^{n-1} C_0^{n+1}}{2^n (n-1)!} \right] + \frac{Z_{0,2}^{n-2} C_0^{n+1}}{2^n (n-2)!}.
\end{aligned}$$

Hence, we have $\partial_t\big(\sum_{n=2}^{\infty}\mathcal{N}_k^n(v(t))\big)=\sum_{n=2}^{\infty}\partial_t\mathcal{N}_k^n(v(t))$. Therefore, with (2.16), we have

$$\begin{aligned}
&\left|\partial_t v_k(t)-\partial_t\left(\sum_{n=2}^{\infty}\mathcal{N}_k^n(v(t))\right)-\sum_{n=2}^{\infty}\mathcal{R}_k^j(v)\right| \\
&=\left|\partial_t v_k(t)-\sum_{n=2}^{\infty}\partial_t\mathcal{N}_k^n(v(t))-\sum_{n=2}^{\infty}\mathcal{R}_k^j(v)\right| \\
&\le \lim_{N\to\infty}\sum_{n=N+1}^{\infty}\big(|\partial_t\mathcal{N}_k^n(v(t))|+|\mathcal{R}_k^n(v)|\big)+\lim_{N\to\infty}|\mathcal{I}_k^{N+1}(v(t))|=0
\end{aligned}$$

for each $k\in\mathbb{Z}_0$ and $t\in[0,T]$. This proves (2.23) on $[0,T]$.

Part 2: In order to obtain the normal form equation (2.24), we need to invert the Fourier transform and thus need to use a control on a higher regularity. By a computation similar to (2.31), we have

$$\sup_{k\in\mathbb{Z}_0}\sup_{t\in[0,T]}|k|^2|\mathcal{N}_k^n(v(t))|\le C_2\frac{Z_{0,2}^{n-1}C_0^{n-1}}{2^{n-1}(n-1)!},$$

where $C_2=C_2(T):=\sup_{t\in[0,T]}\|u(t)\|_{H_x^2}$. In particular, $\sum_{k\in\mathbb{Z}_0}\sum_{n=2}^{\infty}\mathcal{N}_k^n(v(t))$ converges absolutely (and uniformly in $[0,T]$). Hence, we have

$$\begin{aligned}
\sum_{n=2}^{\infty}\mathcal{N}^n(v(t))-\sum_{n=2}^{\infty}\mathcal{N}^n(\phi)&=\sum_{n=2}^{\infty}\sum_{k\in\mathbb{Z}_0}\big(\mathcal{N}_k^n(v(t))-\mathcal{N}_k^n(\phi)\big)e^{ikx} \\
&=\sum_{k\in\mathbb{Z}_0}\left(\sum_{n=2}^{\infty}\mathcal{N}_k^n(v(t))-\sum_{n=2}^{\infty}\mathcal{N}_k^n(\phi)\right)e^{ikx}. \qquad (2.33)
\end{aligned}$$

Similarly, $\sum_{k\in\mathbb{Z}_0}\sum_{n=2}^{\infty}\mathcal{R}_k^n(v(t))$ converges absolutely (and uniformly in $[0,T]$) with a uniform bound:

$$\sup_{k\in\mathbb{Z}_0}\sup_{t\in[0,T]}\sum_{k\in\mathbb{Z}_0}\sum_{n=2}^{\infty}|\mathcal{R}_k^n(v(t))|\lesssim\sum_{k\in\mathbb{Z}_0}\frac{1}{|k|^2}\sum_{n=2}^{\infty}C_2\frac{Z_{0,2}^{n-2}C_0^n}{2^n(n-2)!}<\infty.$$

Hence, by Dominated Convergence Theorem, we have

$$\begin{aligned}
\int_0^t\sum_{n=2}^{\infty}\mathcal{R}^n(v(t'))dt'&=\sum_{n=2}^{\infty}\int_0^t\sum_{k\in\mathbb{Z}_0}\mathcal{R}_k^n(v(t'))e^{ikx}dt' \\
&=\sum_{k\in\mathbb{Z}_0}\int_0^t\sum_{n=2}^{\infty}\mathcal{R}_k^n(v(t'))e^{ikx}dt' \qquad (2.34)
\end{aligned}$$

for $t\in[0,T]$.

Therefore, from (2.23), (2.33), and (2.34), we have

$$\begin{aligned}
v(t)&=\sum_{k\in\mathbb{Z}_0}v_k(t)e^{ikx} \\
&=\sum_{k\in\mathbb{Z}_0}\left(\phi_k+\sum_{n=2}^{\infty}\mathcal{N}_k^n(v(t))-\sum_{n=2}^{\infty}\mathcal{N}_k^n(\phi)+\int_0^t\sum_{n=2}^{\infty}\mathcal{R}_k^n(v(t'))dt'\right)e^{ikx} \\
&=\phi+\sum_{n=2}^{\infty}\mathcal{N}^n(v(t))-\sum_{n=2}^{\infty}\mathcal{N}^n(\phi)+\int_0^t\sum_{n=2}^{\infty}\mathcal{R}^n(v(t'))dt'
\end{aligned}$$

for $t\in[0,T]$. This completes the proof. □

In the remaining part of this section, we consider the normal form equation (2.24). In particular, we prove local well-posedness of (2.24) in the Fourier–Lebesgue spaces $\mathcal{F}L_0^{s,p}$ for small initial data. See Proposition 2.8 below. This will be achieved by the contraction mapping principle. In the following lemma, we establish the key multilinear estimates of $\mathcal{N}^n(v)$ in $\mathcal{F}L_0^{s,p}$.

Lemma 2.7. *Assume that* (s,p) *satisfy* (i) $s > -\frac{1}{p},\ p > 1$ *or* (ii) $s \geq -1,\ p = 1$. *Then, for each integer* $n \geq 2$, *there exists* $C_s(n) > 0$ *such that*

$$\|\mathcal{N}^n(v)\|_{\mathcal{F}L_0^{s,p}} \leq C_s(n)\frac{Z_{s,p}^{n-1}}{2^{n-1}(n-1)!}\|v\|^n_{\mathcal{F}L_0^{s,p}} \tag{2.35}$$

and

$$\|\mathcal{N}^n(v) - \mathcal{N}^n(\widetilde{v})\|_{\mathcal{F}L_0^{s,p}} \leq C_s(n)\frac{nZ_{s,p}^{n-1}}{2^{n-1}(n-1)!}\big(\|v\|_{\mathcal{F}L_0^{s,p}} + \|\widetilde{v}\|_{\mathcal{F}L_0^{s,p}}\big)^{n-1}\|v-\widetilde{v}\|_{\mathcal{F}L_0^{s,p}}, \tag{2.36}$$

where $C_s(n)$ *satisfies*

$$C_s(n) = \begin{cases} 1, & \text{if } s \leq 0,\\ n^s, & \text{if } s > 0.\end{cases}$$

Proof. Recall the following estimate:

$$\Big(\sum_{j=1}^n a_j\Big)^\theta \leq \begin{cases}\sum_{j=1}^n a_j^\theta, & \text{if } 0 \leq \theta \leq 1,\\ n^{\theta-1}\sum_{j=1}^n a_j^\theta, & \text{if } 1 \leq \theta < \infty,\end{cases} \tag{2.37}$$

where the first estimate follows from the concavity of $x \mapsto x^\theta$ and the second from Hölder's inequality. Let $w_k := |k|^s v_k$ so that $\|v\|_{\mathcal{F}L_0^{s,p}} = \|w_k\|_{\ell^p}$. Then, by (2.19) and (2.37) followed by Young's and Hölder's inequalities, we have

$$\begin{aligned}\|\mathcal{N}^n(v)\|_{\mathcal{F}L_0^{s,p}} &\lesssim_n \Big\||k|^s k \sum_{|\mathbf{k}|_n = k} e^{i\Phi_n(\mathbf{k})t}\prod_{j=1}^n \frac{v_{k_j}}{k_j}\Big\|_{\ell^p} \leq \Big\||k|^{s+1}\sum_{|\mathbf{k}|_n=k}\prod_{j=1}^n \frac{|w_{k_j}|}{|k_j|^{s+1}}\Big\|_{\ell^p}\\ &\leq C_s(n)\sum_{\ell=1}^n\Big\|\sum_{|\mathbf{k}|_n=k}|w_{k_\ell}|\prod_{j\neq\ell}\frac{|w_{k_j}|}{|k_j|^{s+1}}\Big\|_{\ell^p} \leq C_s(n)n\|w_k\|_{\ell^p}\Big\|\frac{w_k}{|k|^{s+1}}\Big\|^{n-1}_{\ell^1}\\ &\leq C_s(n)nZ_{s,p}^{n-1}\|w_k\|^n_{\ell^p}.\end{aligned} \tag{2.38}$$

Noting that the omitted constant in the first inequality is $1/(2^{n-1}n!)$, we obtain (2.35). The second estimate (2.36) follows from a similar argument with the telescoping sum:

$$\prod_{j=1}^n w_{k_j} - \prod_{j=1}^n \widetilde{w}_{k_j} = \sum_{i=1}^n w_{k_1}\cdots w_{k_{j-1}}\cdot(w_{k_j}-\widetilde{w}_{k_j})\cdot\widetilde{w}_{k_{j+1}}\cdots\widetilde{w}_{k_n}.$$

This completes the proof. □

We conclude this section by proving small data local well-posedness of the normal form equation (2.24).

Proposition 2.8. *Suppose that* (s,p) *satisfy* (i) $s > \frac{1}{2} - \frac{1}{p},\ p > 2$ *or* (ii) $s \geq 0,\ p = 2$. *Then, there exists a constant* $\delta_1 = \delta_1(s,p) > 0$ *such that if* $\phi \in \mathcal{F}L_0^{s,p}(\mathbb{T})$ *satisfies*

$$\|\phi\|_{\mathcal{F}L_0^{s,p}} \leq \delta_1, \tag{2.39}$$

then there exist $T = T(\|\phi\|_{\mathcal{F}L_0^{s,p}}) > 0$ *and a unique solution* v *to the normal form equation* (2.24) *in* $C([0,T];\mathcal{F}L_0^{s,p}(\mathbb{T}))$ *with* $v|_{t=0} = \phi$.

Proof. Part 1: We first construct a local weak solution in $L^\infty([0,T];\mathcal{F}L_0^{s,p}(\mathbb{T}))$.[3] Given $\phi \in \mathcal{F}L_0^{s,p}(\mathbb{T})$, define a map $\Gamma = \Gamma_\phi$ on $L^\infty([0,T];\mathcal{F}L_0^{s,p}(\mathbb{T}))$ by

$$\Gamma[v](t) := \phi + \sum_{n=2}^{\infty}\big(\mathcal{N}^n(t)(v(t)) - \mathcal{N}^n(t)(\phi)\big) + \int_0^t \sum_{n=2}^{\infty} \mathcal{R}^n(t')(v(t'))dt'. \tag{2.40}$$

Define $A_s > 0$ to be a constant such that

$$\frac{C_s(n)n}{2^{n-1}} \le A_s \tag{2.41}$$

for all $n \in \mathbb{N}$. Let $B_R^s = B_R^s(T)$ be the closed ball in $X^s(T) := L^\infty([0,T];\mathcal{F}L_0^{s,p}(\mathbb{T}))$ of radius $R = 2\|\phi\|_{\mathcal{F}L_0^{s,p}}$ centred at the origin. In addition, we assume that $\|\phi\|_{\mathcal{F}L_0^{s,p}}$ is sufficiently small such that

$$A_s(e^{2Z_{s,p}R} - 1) \le \frac{1}{4} \tag{2.42}$$

Lastly, we choose $T = T(R) = T(\|\phi\|_{\mathcal{F}L_0^{s,p}}) > 0$ such that

$$T = \min\left(\frac{1}{2A_s z_{s,p}^2 R^2 e^{Z_{s,p}R}}, \frac{1}{A_s z_{s,p}^2 R^2 e^{Z_{s,p}R}\big(2 + e^{Z_{s,p}R}\big)}\right). \tag{2.43}$$

Let $v, \widetilde{v} \in B_R^s$. Then, from Lemma 2.7 with (2.20), (2.28), (2.42), and (2.43), we have

$$\begin{aligned}
\|\Gamma[v]\|_{X^s(T)} &\le \|\phi\|_{\mathcal{F}L_0^{s,p}} + \sum_{n=2}^{\infty}\big(\|\mathcal{N}^n(v(t))\|_{X^s(T)} + \|\mathcal{N}^n(\phi)\|_{X^s(T)}\big)\\
&\quad + \frac{1}{4}Tz_{s,p}^2\|v\|_{X^s(T)}^2 \sum_{n=1}^{\infty}\|\mathcal{N}^n(v)\|_{X^s(T)}\\
&\le \|\phi\|_{\mathcal{F}L_0^{s,p}} + \sum_{n=2}^{\infty}\frac{C_s(n)}{2^{n-1}}\frac{Z_{s,p}^{n-1}}{(n-1)!}\big(\|v\|_{X^s(T)}^n + \|\phi\|_{\mathcal{F}L_0^{s,p}}^n\big)\\
&\quad + \frac{1}{4}Tz_{s,p}^2\|v\|_{X^s(T)}^2\sum_{n=1}^{\infty}\frac{C_s(n)}{2^{n-1}}\frac{Z_{s,p}^{n-1}}{(n-1)!}\|v\|_{X^s(T)}^n\\
&= \|\phi\|_{\mathcal{F}L_0^{s,p}}\Big\{A_s\Big(e^{Z_{s,p}\|\phi\|_{\mathcal{F}L_0^{s,p}}} - 1\Big) + 1\Big\} + A_s\|v\|_{X^s(T)}\Big(e^{Z_{s,p}\|v\|_{X^s(T)}} - 1\Big)\\
&\quad + \frac{1}{4}TA_s z_{s,p}^2\|v\|_{X^s(T)}^3 e^{Z_{s,p}\|v\|_{X^s(T)}}\\
&\le R.
\end{aligned} \tag{2.44}$$

Similarly, we have

$$\begin{aligned}
\big\|\Gamma[v] - \Gamma[\widetilde{v}]\big\|_{X^s(T)} &\le \sum_{n=2}^{\infty}\big\|\mathcal{N}^n(v) - \mathcal{N}^n(\widetilde{v})\big\|_{X^s(T)} + T\sum_{n=2}^{\infty}\|\mathcal{R}^n(v) - \mathcal{R}^n(\widetilde{v})\|_{X^s(T)}\\
&\le A_s\sum_{n=2}^{\infty}\frac{Z_{s,p}^{n-1}}{(n-1)!}\big(\|v\|_{X^s(T)} + \|\widetilde{v}\|_{X^s(T)}\big)^{n-1}\|v - \widetilde{v}\|_{X^s(T)}\\
&\quad + \frac{1}{4}TA_s z_{s,p}^2\sum_{n=1}^{\infty}\frac{Z_{s,p}^{n-1}}{(n-1)!}\|v\|_{X^s(T)}^n\big(\|v\|_{X^s(T)} + \|\widetilde{v}\|_{X^s(T)}\big)\|v - \widetilde{v}\|_{X^s(T)}
\end{aligned}$$

[3] Here, by a "weak" solution, we mean a solution without continuity in time.

$$
\begin{aligned}
&+\frac{1}{4}TA_s z_{s,p}^2\|\widetilde{v}\|_{X^s(T)}^2\sum_{n=1}^{\infty}\frac{Z_{s,p}^{n-1}}{(n-1)!}\big(\|v\|_{X^s(T)}+\|\widetilde{v}\|_{X^s(T)}\big)^{n-1}\|v-\widetilde{v}\|_{X^s(T)}\\
&\le A_s\big(e^{2Z_{s,p}R}-1\big)\|v-\widetilde{v}\|_{X^s(T)}+\frac{1}{4}TA_s z_{s,p}^2\big(2R^2e^{Z_{s,p}R}+R^2e^{2Z_{s,p}R}\big)\|v-\widetilde{v}\|_{X^s(T)}\\
&\le\frac{1}{2}\|v-\widetilde{v}\|_{X^s(T)}. \qquad (2.45)
\end{aligned}
$$

Hence, it follows from (2.44) and (2.45) that Γ is a contraction on B_R^s. Therefore, there exists a unique weak solution $v\in L^\infty([0,T];\mathcal{F}L_0^{s,p}(\mathbb{T}))$ to (2.24) as long as $\|\phi\|_{\mathcal{F}L_0^{s,p}}\le\delta_1\ll 1$, satisfying (2.42).

Note that the argument above does not yield continuity in time of the solution $v\in L^\infty([0,T];\mathcal{F}L_0^{s,p}(\mathbb{T}))$. This is due to the presence of the boundary terms $\mathcal{N}^n(t)(v(t))$ without time integration in the definition (2.40) of $\Gamma[v]$. In the following, we will present an additional argument to show the fixed point v indeed lies in $C([0,T];\mathcal{F}L_0^{s,p}(\mathbb{T}))$.

Part 2: Next, we prove the persistence of regularity. We now suppose that $\phi\in\mathcal{F}L_0^{s+\sigma,p}(\mathbb{T})$ for some $\sigma>0$. Let $v\in B_R^s\subset L^\infty([0,T];\mathcal{F}L_0^{s,p})$ be the solution to (2.24) with $v|_{t=0}=\phi$ constructed in Part 1, where $T=T(\|\phi\|_{\mathcal{F}L_0^{s,p}})$ is chosen as in (2.43).

Proceeding as in (2.38) with $w_k=|k|^s v_k$, we have

$$
\begin{aligned}
\|\mathcal{N}^n(v)\|_{\mathcal{F}L_0^{s+\sigma,p}}&\le\frac{1}{2^{n-1}n!}\bigg\||k|^{s+\sigma+1}\sum_{|\mathbf{k}|_n=k}\prod_{j=1}^{n}\frac{|w_{k_j}|}{|k_j|^{s+1}}\bigg\|_{\ell^p}\\
&\le\frac{C_{s+\sigma}(n)nZ_{s,p}^{n-1}}{2^{n-1}n!}\|v\|_{\mathcal{F}L_0^{s,p}}^{n-1}\|v\|_{\mathcal{F}L_0^{s+\sigma,p}}. \qquad (2.46)
\end{aligned}
$$

Then, proceeding as in (2.44) with (2.46), we have

$$
\begin{aligned}
\|v\|_{X^{s+\sigma}(T)}&\le\|\phi\|_{\mathcal{F}L_0^{s+\sigma,p}}\\
&\quad+\sum_{n=2}^{\infty}\frac{C_{s+\sigma}(n)}{2^{n-1}}\frac{Z_{s,p}^{n-1}}{(n-1)!}\big(\|v\|_{X^s(T)}^{n-1}\|v\|_{X^{s+\sigma}(T)}+\|\phi\|_{\mathcal{F}L_0^{s,p}}^{n-1}\|\phi\|_{\mathcal{F}L_0^{s+\sigma,p}}\big)\\
&\quad+\frac{1}{4}Tz_{s,p}^2\sum_{n=1}^{\infty}\frac{C_{s+\sigma}(n)}{2^{n-1}}\frac{Z_{s,p}^{n-1}}{(n-1)!}\|v\|_{X^s(T)}^{n+1}\|v\|_{X^{s+\sigma}(T)}\\
&=\|\phi\|_{\mathcal{F}L_0^{s+\sigma,p}}\Big\{A_{s+\sigma}\Big(e^{Z_{s,p}\|\phi\|_{\mathcal{F}L_0^{s,p}}}-1\Big)+1\Big\}+A_{s+\sigma}\|v\|_{X^{s+\sigma}(T)}\Big(e^{Z_{s,p}\|v\|_{X^s(T)}}-1\Big)\\
&\quad+\frac{1}{4}TA_{s+\sigma}z_{s,p}^2\|v\|_{X^s(T)}^2e^{Z_{s,p}\|v\|_{X^s(T)}}\|v\|_{X^{s+\sigma}(T)}, \qquad (2.47)
\end{aligned}
$$

where $A_{s+\sigma}$ is as in (2.41). Given $\sigma>0$, we choose $\delta(s,\sigma)>0$ sufficiently small such that $R=2\|\phi\|_{\mathcal{F}L_0^{s,p}}$ with $\|\phi\|_{\mathcal{F}L_0^{s,p}}\le\delta(s,\sigma)$ satisfies

$$
A_{s+\sigma}(e^{Z_{s,p}R}-1)\le\frac{1}{4}. \qquad (2.48)
$$

If necessary, we make T smaller such that

$$
T\le\frac{1}{A_{s+\sigma}z_{s,p}^2R^2e^{Z_{s,p}R}}. \qquad (2.49)
$$

Hence, from (2.47), (2.48), and (2.49), we obtain

$$
\|v\|_{X^{s+\sigma}(T)}\le\frac{5}{2}\|\phi\|_{\mathcal{F}L_0^{s+\sigma,p}}.
$$

Therefore, we conclude that $v\in L^\infty([0,T];\mathcal{F}L_0^{s+\sigma,p})$. The important point is that, while $T=T(s,\sigma)$ now depends on $\|\phi\|_{\mathcal{F}L_0^{s,p}}$ and σ, it is independent of $\|\phi\|_{\mathcal{F}L_0^{s+\sigma,p}}$.

Part 3: Lastly, we prove that the solution constructed in Part 1 is indeed in $C([0,T];\mathcal{F}L_0^{s,p}(\mathbb{T}))$. We first prove continuity in $\mathcal{F}L_0^{s,p}(\mathbb{T})$ for smoother solutions. Given small $\phi\in\mathcal{F}L_0^{s+1,p}(\mathbb{T})$, let $v\in B_R^s\subset L^\infty([0,T];\mathcal{F}L_0^{s,p}(\mathbb{T}))$ be the weak solution with $v|_{t=0}=\phi$ constructed in Part 2, where $T=T(s,\sigma)$ with $\sigma=1$. Note that from Part 2, we have

$$\|v\|_{L^\infty([0,T];\mathcal{F}L_0^{s+1,p})}\le R_{s+1}:=R_{s+1}(\|\phi\|_{\mathcal{F}L_0^{s+1,p}}). \tag{2.50}$$

Write (2.24) as

$$v(t)=\phi+\mathcal{N}(t)+\mathcal{R}(t), \tag{2.51}$$

where

$$\mathcal{N}(t)=\sum_{n=2}^\infty\mathcal{N}^n(v(t))-\sum_{n=2}^\infty\mathcal{N}^n(\phi)\quad\text{and}\quad\mathcal{R}(t)=\int_0^t\sum_{n=2}^\infty\mathcal{R}^n(v(t'))dt'.$$

Thanks to the time integral, using Lemma 2.7 with (2.20), one can easily show that $\mathcal{R}(t)\in C([0,T];\mathcal{F}L_0^{s,p}(\mathbb{T}))$. Indeed, given $t_1,t_2\in\mathbb{R}$, we have

$$\|\mathcal{R}(t_1)-\mathcal{R}(t_2)\|_{\mathcal{F}L_0^{s,p}}\le\frac{1}{4}A_sz_{s,p}^2R^3e^{Z_{s,p}R}|t_1-t_2|. \tag{2.52}$$

By the triangle inequality, we have

$$\begin{aligned}\|\mathcal{N}^n(t_1)(v(t_1))-\mathcal{N}^n(t_2)(v(t_2))\|_{\mathcal{F}L_0^{s,p}}&\le\|\mathcal{N}^n(t_1)(v(t_1))-\mathcal{N}^n(t_2)(v(t_1))\|_{\mathcal{F}L_0^{s,p}}\\&\quad+\|\mathcal{N}^n(t_2)(v(t_1))-\mathcal{N}^n(t_2)(v(t_2))\|_{\mathcal{F}L_0^{s,p}}\\&=:\mathrm{I}_n+\mathrm{II}_n.\end{aligned} \tag{2.53}$$

Arguing as in (2.45) with (2.42), we have

$$\sum_{n=2}^\infty\mathrm{II}_n\le A_s\sum_{n=2}^\infty\frac{(2Z_{s,p}R)^{n-1}}{(n-1)!}\|v(t_1)-v(t_2)\|_{\mathcal{F}L_0^{s,p}}\le\frac{1}{4}\|v(t_1)-v(t_2)\|_{\mathcal{F}L_0^{s,p}}. \tag{2.54}$$

By Mean Value Theorem with $w_k:=|k|^sv_k$, (2.4), and (2.37), we have

$$\begin{aligned}\mathrm{I}_n&\le\frac{1}{2^{n-1}n!}\Bigg\||k|^{s+1}\sum_{|\mathbf{k}|_n=k}|\Phi(\mathbf{k})(t_1-t_2)|\prod_{\ell=1}^n\frac{|w_{k_\ell}(t_1)|}{|k_\ell|^{s+1}}\Bigg\|_{\ell^p}\\&\le|t_1-t_2|\frac{1}{2^{n-2}n!}\Bigg\||k|^{s+1}\sum_{1\le i<j\le n}\sum_{|\mathbf{k}|_n=k}|k_i||k_j|\prod_{\ell=1}^n\frac{|w_{k_\ell}(t_1)|}{|k_\ell|^{s+1}}\Bigg\|_{\ell^p}\\&\le|t_1-t_2|\frac{C_s(n)n}{2^{n-1}(n-2)!}\big\||k|w_k(t_1)\big\|_{\ell^p}\bigg\|\frac{w_k(t_1)}{|k|^s}\bigg\|_{\ell^1}\bigg\|\frac{w_k(t_1)}{|k|^{s+1}}\bigg\|_{\ell^1}^{n-2}\\&\le|t_1-t_2|\frac{C_s(n)nZ_{s,p}^{n-1}}{2^{n-1}(n-2)!}\|v(t_1)\|^2_{\mathcal{F}L_0^{s+1,p}}\|v(t_1)\|^{n-2}_{\mathcal{F}L_0^{s,p}}.\end{aligned}$$

Choosing $B_s>0$ such that $2^{-(n-1)}C_s(n)nZ_{s,p}\le B_s$ for all $n\in\mathbb{N}$, and recalling (2.50), we have

$$\sum_{n=2}^\infty\mathrm{I}_n\le B_sR_{s+1}^2e^{Z_{s,p}R}|t_1-t_2| \tag{2.55}$$

Hence, from (2.51) – (2.55) we have

$$\|v(t_1)-v(t_2)\|_{\mathcal{F}L_0^{s,p}}\lesssim B_sR_{s+1}^2e^{Z_{s,p}R}|t_1-t_2|+A_sz_{s,p}^2R^3e^{Z_{s,p}R}|t_1-t_2|.$$

Therefore, we conclude that $v\in C(\mathbb{R};\mathcal{F}L_0^{s,p}(\mathbb{T}))$.

Now, given small $\phi \in \mathcal{F}L_0^{s,p}(\mathbb{T})$, let $v \in L^\infty([0,T];\mathcal{F}L_0^{s,p}(\mathbb{T}))$ be the weak solution with $v|_{t=0} = \phi$ constructed in Part 2, where $T = T(s,\sigma)$ with $\sigma = 1$. Let $\{\phi^{(j)}\}_{j\in\mathbb{N}} \subset \mathcal{F}L_0^{s+1,p}(\mathbb{T})$ such that $\phi^{(j)}$ converges to ϕ in $\mathcal{F}L_0^{s,p}(\mathbb{T})$, as $j \to \infty$. Denote by $v^{(j)}$ the corresponding global solutions in $L^\infty([0,T];\mathcal{F}L_0^{s+1,p}(\mathbb{T})) \cap C([0,T];\mathcal{F}L_0^{s,p}(\mathbb{T}))$. Recalling that $T = T(s,\sigma)$ depends only on $\|\phi^{(j)}\|_{\mathcal{F}L_0^{s,p}}$ and $\sigma > 0$ but is independent of $\|\phi^{(j)}\|_{\mathcal{F}L_0^{s+1,p}}$, we can choose $T = T(s,\sigma)$ independent of $j \in \mathbb{N}$. Then, arguing as in (2.45), we have

$$\|v - v^{(j)}\|_{X^s(T)} \le 2\|\phi - \phi^{(j)}\|_{\mathcal{F}L_0^{s,p}}$$

Therefore, $v^{(j)}(t)$ converges to $v(t)$ in the $\mathcal{F}L_0^{s,p}$-topology, uniformly in $t \in [0,T]$. Hence, as a uniform limit of continuous functions, we conclude that $v \in C([0,T];\mathcal{F}L_0^{s,p}(\mathbb{T}))$. □

Remark 2.9 *(Unconditional uniqueness).* Given $\phi \in \mathcal{F}L_0^{s,p}$ satisfying (2.39), suppose that $v, \widetilde{v} \in C([0,T];\mathcal{F}L_0^{s,p})$ are two solutions to (2.24) with $v|_{t=0} = \widetilde{v}|_{t=0} = \phi$. Without loss of generality, assume that v is the solution constructed in Proposition 2.8. In particular, we have $v \in B_R^s(T)$ with $R = 2\|\phi\|_{\mathcal{F}L_0^{s,p}}$. In the following, redefine the local existence time T in (2.43) and (2.49) by replacing R with $4R$.

In general, we may have $\widetilde{v} \notin B_R^s(T)$. By the continuity in time with $\widetilde{v}|_{t=0} = \phi$, there exists small $\tau_1 > 0$ such that $\sup_{t\in[0,\tau_1]} \|\widetilde{v}(t)\|_{\mathcal{F}L_0^{s,p}} \le 2R$. Then, it follows from (2.45) that $v = \widetilde{v}$ in $C([0,\tau_1];\mathcal{F}L_0^{s,p})$. Since $\|\widetilde{v}(\tau_1)\|_{\mathcal{F}L_0^{s,p}} = \|v(\tau_1)\|_{\mathcal{F}L_0^{s,p}} \le R$, there exists small $\tau_2 > 0$ such that $\sup_{t\in[\tau_1,\tau_2]} \|\widetilde{v}(t)\|_{\mathcal{F}L_0^{s,p}} \le 2R$. Then, it follows from (2.45) that $v = \widetilde{v}$ in $C([\tau_1,\tau_2];\mathcal{F}L_0^{s,p})$ and conclude that $\|\widetilde{v}(\tau_2)\|_{\mathcal{F}L_0^{s,p}} \le R$. In this way, we can cover the entire interval $[0,T]$ and conclude that $v = \widetilde{v}$ in $C([0,T];\mathcal{F}L_0^{s,p})$. Namely, we have unconditional uniqueness.

Remark 2.10. In Part 2 of the proof of Proposition 2.8, we proved the persistence of regularity only for a fixed value of $p \ge 2$. In general, the persistence of regularity also holds for different values of p as long as a proper embedding holds. Let $B_R^{s,p} = B_R^{s,p}(T)$ be the closed ball in $X^{s,p}(T) := L^\infty([0,T];\mathcal{F}L_0^{s,p}(\mathbb{T}))$ in the following.

Suppose that $\phi \in \mathcal{F}L_0^{s,p}(\mathbb{T})$ with $s > \frac{1}{2} - \frac{1}{p}$ such that $\mathcal{F}L_0^{s,p}(\mathbb{T}) \subset L^2(\mathbb{T})$. Let $v \in B_R^{0,2} \subset X^{0,2}(T)$ be the solution to (2.24) with $v|_{t=0} = \phi$ constructed in Part 1, where $T = T(\|\phi\|_{L^2})$ is chosen as in (2.43). Proceeding as in (2.38) and (2.46) with $w_k = |k|^s v_k$, we have

$$\|\mathcal{N}^n(v)\|_{\mathcal{F}L_0^{s,p}} \le \frac{C_s(n) n Z_{0,2}^{n-1}}{2^{n-1} n!} \|v\|_{L^2}^{n-1} \|v\|_{\mathcal{F}L_0^{s,p}}. \tag{2.56}$$

Then, a computation analogous to (2.47) with (2.56) yields

$$\begin{aligned} \|v\|_{X^{s,p}(T)} \le{}& \|\phi\|_{\mathcal{F}L_0^{s,p}} \Big\{ A_s\Big(e^{Z_{0,2}\|\phi\|_{L^2}} - 1\Big) + 1\Big\} + A_s \|v\|_{X^{s,p}(T)} \Big(e^{Z_{0,2}\|v\|_{X^{0,2}(T)}} - 1\Big) \\ &+ \frac{1}{4} T A_s \|v\|_{X^{0,2}(T)}^2 e^{Z_{0,2}\|v\|_{X^{0,2}(T)}} \|v\|_{X^{s,p}(T)}. \end{aligned} \tag{2.57}$$

Then, by choosing R and T sufficiently small such that

$$A_s(e^{Z_{0,2}R} - 1) \le \frac{1}{4} \qquad \text{and} \qquad T \le \frac{1}{A_s R^2 e^{Z_{0,2}R}},$$

it follows from (2.57) that

$$\|v\|_{X^{s,p}(T)} \le \frac{5}{2} \|\phi\|_{\mathcal{F}L_0^{s,p}}.$$

Therefore, we conclude that v also lies in $L^\infty([0,T];\mathcal{F}L_0^{s,p})$.

3. Small data global existence of smooth solutions

3.1. Cole–Hopf transformation

In the non-periodic setting, we can use the Cole–Hopf transformation (1.11) to transform smooth solutions to (1.2) on $\mathbb{R}$ into solutions of the linear Schrödinger equation. In the periodic case, however, we need to make a suitable

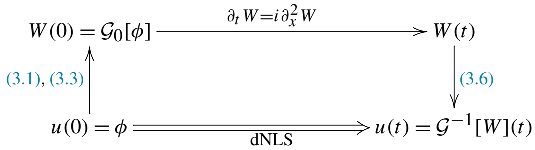

Fig. 2. Relation between dNLS and the linear Schrödinger equation.

adjustment so that a modified Cole–Hopf transformation converts smooth mean-zero solutions to (1.2) on $\mathbb{T}$ into solutions of the linear Schrödinger equation.

Given a mean-zero function ϕ on $\mathbb{T}$, we follow the Cole–Hopf transformation (1.11) on $\mathbb{R}$ and define a gauge transformation $\mathcal{G}_0$ by setting

$$\mathcal{G}_0[\phi] := e^{-\frac{i}{2}\mathcal{J}(\phi)}, \tag{3.1}$$

where $\mathcal{J}(\phi)$ is the mean-zero primitive of ϕ given by

$$\mathcal{J}(\phi)_k = \begin{cases} \frac{\phi_k}{ik}, & k \neq 0, \\ 0, & k = 0. \end{cases}$$

Note that $\mathcal{G}_0[\phi]$ is a periodic function on $\mathbb{T}$, since $\mathcal{J}(\phi)$ is periodic. Suppose that u is a smooth mean-zero solution to dNLS (1.2) on $\mathbb{T}$ and let $w(t) := \mathcal{G}_0[u(t)]$. Unfortunately, such w does not satisfies the linear Schrödinger equation. Indeed, we have

$$i\partial_t w + \partial_x^2 w = -\frac{1}{4}\mathbf{P}_0[u^2] \cdot w, \tag{3.2}$$

where $\mathbf{P}_0[u^2]$ is defined in (2.9).

In view of (3.2), we define a new "gauge" transformation $\mathcal{G}$ on functions depending on both x and t. Given a smooth function $u(t,x)$ on $[0,T] \times \mathbb{T}$ such that $\int_{\mathbb{T}} u(t)dx = 0$ for all $t \in [0,T]$, define a gauge transformation $\mathcal{G}$ by

$$W(t) = \mathcal{G}[u](t) := e^{-\frac{i}{4}\int_0^t \mathbf{P}_0[u^2(t')]dt'} e^{-\frac{i}{2}\mathcal{J}(u(t))}. \tag{3.3}$$

In particular, we have $\mathcal{G}[u](0) = \mathcal{G}_0(u(0))$. Note that

$$\|\partial_x W(0)\|_{L^2} \le \frac{1}{2}\left\|e^{\frac{1}{2}\operatorname{Im}\mathcal{J}(u(0))}\right\|_{L^\infty}\|u(0)\|_{L^2} \le \frac{1}{2}e^{\frac{1}{2}Z_{0,2}\|u(0)\|_{L^2}}\|u(0)\|_{L^2}, \tag{3.4}$$

where $Z_{0,2}$ is as in Definition 2.5.

Suppose that $u(t,x)$ is a smooth mean-zero solution to (1.2) on an interval $[0,T]$. Define W by (3.3). Then, it is easy to see that W satisfies the linear Schrödinger equation on $(0,T) \times \mathbb{T}$:

$$\partial_t W = i\partial_x^2 W. \tag{3.5}$$

Given a smooth mean-zero initial condition ϕ, we can use the gauge transformation $\mathcal{G}$ to construct a smooth solution u to (1.2). Indeed, set $W(0) = \mathcal{G}_0[\phi]$ and let $W(t)$ be the global solution to the linear Schrödinger equation (3.5). Then, applying the inverse gauge transformation:

$$u(t,x) = \mathcal{G}^{-1}[W](t,x) := 2i\frac{\partial_x W(t,x)}{W(t,x)}, \tag{3.6}$$

we see that u is a solution to (1.2) as long as (3.6) makes sense. See Fig. 2. Note that the smoothness of ϕ (and hence of $W(t)$) was needed to consider the pointwise division in (3.6). Obviously, (3.6) makes sense as long as $W(t,x) \neq 0$.

Now, let us introduce a geometric view point. The image of a complex-valued periodic function is a closed loop in $\mathbb{C}$. Thus, the inverse transformation (3.6) makes sense at time t if the loop $W(t)$ stays away from the origin in the complex plane. Note that the trivial solution $u(t) \equiv 0$ to (1.2) is transformed into $W(t) \equiv 1$. Thus, a small initial condition $u(0)$ to (1.2) are transformed into a small loop $W(0)$ around $1 \in \mathbb{C}$. In particular, it is away from the origin in the complex plane. Then, we expect that the solution $W(t)$ to (3.5) remains as a small loop around $1 \in \mathbb{C}$ for all $t \in \mathbb{R}$, allowing us to apply the inverse transformation (3.6). In the following, we make this intuition precise. In particular, by assuming that a smooth mean-zero initial condition $u(0) = \phi$ is sufficiently small, we show that $W(t)$ stays away from the origin for all time, giving rise to a smooth global solution u to (1.2).

Remark 3.1. Given a smooth periodic function $u(t,x)$ on $[0,T] \times \mathbb{T}$ such that $\fint_{\mathbb{T}} u(t)dx = \mu \in \mathbb{C}$ for all $t \in [0,T]$, let $W(t)$ be the gauge transformation of $u(t)$ defined in (3.3). We need to make sure that $W(t)$ is a periodic function for each $t \in [0,T]$. From the geometric point of view, $W(t)$ must be a closed loop in $\mathbb{C}$ for each $t \in [0,T]$. In particular, the *index* (= winding number) of the loop $\gamma = W(t)$ at the origin must be well defined. By a direct computation, we have

$$\operatorname{Ind}_\gamma(0) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z} = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\partial_x W(t)}{W(t)}dx = -\frac{1}{4\pi}\int_{\mathbb{T}} u(t)dx = -\frac{\mu}{4\pi},$$

where the third equality follows from (3.6). Hence, we must have $\mu \in 4\pi\mathbb{Z}$.

In this paper, we only consider the mean-zero functions u, corresponding to the loop W of index 0 at the origin. It may be of interest to study well-posedness of (1.2) with $\fint_{\mathbb{T}} u(0)dx = -4\pi m$, $m \in \mathbb{Z}$, corresponding to the loop $W(0)$ of index m at the origin.

Remark 3.2. The normal form reduction performed in Section 2 corresponds to the Taylor expansion of (the derivative of the interaction representation of) the gauge transformation W defined in (3.3). Indeed, we have

$$\begin{aligned}\partial_x S(-t)W(t) &= e^{-\frac{i}{4}\fint_0^t \mathbf{P}_0[u^2(t')]dt'}\partial_x S(-t)e^{-\frac{i}{2}\mathcal{J}(u(t))}\\ &= e^{-\frac{i}{4}\fint_0^t \mathbf{P}_0[u^2(t')]dt'}\partial_x S(-t)\Bigg[\sum_{n=0}^\infty \frac{1}{n!}\Big(-\frac{i}{2}\mathcal{J}(u(t))\Big)^n\Bigg]\\ &= e^{-\frac{i}{4}\fint_0^t \mathbf{P}_0[u^2(t')]dt'}\partial_x \sum_{n=1}^\infty \mathcal{M}^n(u(t)).\end{aligned} \tag{3.7}$$

Here, $\mathcal{M}^n(u(t))$, $n \geq 1$, is given by

$$\big(\mathcal{M}^n(u(t))\big)_k = \frac{1}{n!}\Big(-\frac{1}{2}\Big)^n \sum_{|\mathbf{k}|_n = k} e^{i\Phi_n(\mathbf{k})t}\prod_{j=1}^n \frac{v_{k_j}(t)}{k_j} = \frac{1}{2k}\mathcal{N}_k^n(v(t)), \tag{3.8}$$

where v denotes the interaction representation of u defined in (2.2) and the last equality follows from (2.19) and (2.21). Then, from (3.7), (3.8) and (2.26), indeed we have

$$\big(\partial_x S(-t)W(t)\big)_k = \frac{i}{2}e^{-\frac{i}{4}\fint_0^t \mathbf{P}_0[u^2(t')]dt'}\sum_{n=1}^\infty \mathcal{N}_k^n(v(t)) = \frac{i}{2}\mathcal{Q}_k(t) \tag{3.9}$$

for $k \in \mathbb{Z}_0$. Since W is a solution to the linear Schrödinger equation, $\big(S(-t)W(t)\big)_k$ is conserved under the dynamics. This fact can be also seen from (3.9) and Remark 2.4:

$$\big(S(-t)W(t)\big)_k = \frac{1}{2k}\mathcal{Q}_k(t) = \frac{1}{2k}\mathcal{Q}_k(0) = W_k(0).$$

3.2. Global existence of smooth solutions

In this subsection, we prove global existence of smooth solutions to (1.2) with small mean-zero initial data. As mentioned in the previous subsection, the main goal is to make sure that the loop $W(t)$ in the complex plane does not intersect the origin for any $t \in \mathbb{R}$. The following simple lemma provides a sufficient condition.

Lemma 3.3. *Suppose that $W^0 \in H^{\frac{1}{2}+\varepsilon}(\mathbb{T})$, $\varepsilon > 0$, satisfies*

$$|W_0^0| - \sum_{k\neq 0}|W_k^0| \geq \delta > 0, \tag{3.10}$$

for some $\delta > 0$*. Here,* W_k^0 *is the* k*-th Fourier coefficient of* W^0*. Then, when viewed as a loop in the complex plane, the solution* $W(t)$ *to the linear Schrödinger equation* (3.5) *with* $W|_{t=0} = W^0$ *never intersect the origin for any* $t \in \mathbb{R}$*. Furthermore, we have*

$$|W(t,x)| \ge \delta > 0.$$

Proof. By Sobolev embedding and the unitarity of the linear Schrödinger flow, we have $W(t,x) \in C(\mathbb{R} \times \mathbb{T})$. Note that the solution $W(t)$ to the linear Schrödinger equation (3.5) is given by

$$W(t,x) = \sum_{k\in\mathbb{Z}} W_k^0 e^{ikx} e^{-ik^2 t}.$$

Therefore, from (3.10), we have

$$|W(t,x)| = \left| W_0^0 + \sum_{k\neq 0} W_k^0 e^{ikx} e^{-ik^2 t} \right| \ge |W_0^0| - \sum_{k\neq 0} |W_k^0| \ge \delta > 0,$$

for all $(t,x) \in \mathbb{R} \times \mathbb{T}$. □

Remark 3.4. The condition (3.10) is sharp. Consider

$$W^0(x) = -2\zeta(2) + \sum_{k\neq 0} \frac{1}{k^2} e^{ikx},$$

where $\zeta(\tau) = \sum_{k=1}^{\infty} k^{-\tau}$ is the Riemann zeta function. Then, clearly (3.10) is violated. Moreover, we have $W^0(0) = 0$.

As a corollary of Lemma 3.3, we obtain the following a priori bound on the L^2-norm of smooth solutions to (1.2).

Lemma 3.5. *Suppose that* u *is a smooth global solution to* (1.2) *with a mean-zero initial condition:* $\int_{\mathbb{T}} u(0)\,dx = 0$*. Let* $W(t) = \mathcal{G}[u](t)$ *be the gauge transformation defined in* (3.3)*. If* $W^0 = W(0)$ *satisfies* (3.10) *for some* $\delta > 0$*, then we have the following a priori bound:*

$$\|u(t)\|_{L^2(\mathbb{T})} \le \frac{1}{\delta} e^{\frac{1}{2} Z_{0,2} \|u(0)\|_{L^2}} \|u(0)\|_{L^2}$$

for all $t \in \mathbb{R}$*.*

Proof. By Lemma 3.3, we have $|W(t,x)| \ge \delta > 0$ for all $(t,x) \in \mathbb{R} \times \mathbb{T}$. Then, from (3.6), the unitary of the linear Schrödinger flow on $\dot{H}^1$, and (3.4), we have

$$\|u(t)\|_{L^2(\mathbb{T})} \le \frac{2}{\delta} \|\partial_x W(t)\|_{L^2(\mathbb{T})} = \frac{2}{\delta} \|\partial_x W(0)\|_{L^2(\mathbb{T})} \le \frac{1}{\delta} e^{\frac{1}{2} Z_{0,2} \|u(0)\|_{L^2}} \|u(0)\|_{L^2}. \qquad \square$$

Given a smooth mean-zero function ϕ on $\mathbb{T}$, let W be the solution to (3.5) with $W(0) = W^0 := \mathcal{G}_0[\phi]$. In the following proposition, we transfer the condition (3.10) on W^0 to a condition on ϕ guaranteeing that the loop $W(t)$ does not intersect the origin for any $t \in \mathbb{R}$. This allows us to apply the inverse transformation (3.6) and construct a smooth global solution $u(t) = \mathcal{G}^{-1}[W](t)$ to (1.2).

Proposition 3.6. *Suppose that* (s,p) *satisfy* (i) $s > \frac{1}{2} - \frac{1}{p}$, $p > 2$ *or* (ii) $s \ge 0$, $p = 2$*. Let* ϕ *be a smooth function on* $\mathbb{T}$*, satisfying* $\int_{\mathbb{T}} \phi\,dx = 0$*. Define* $M = M(\phi)$ *by*

$$M := \sup_{x\in\mathbb{T}} |\mathcal{J}(\phi)(x)|. \tag{3.11}$$

If $M < \pi$ *and*

$$e^{\frac{1}{2} Z_{s,p} \|\phi\|_{\mathcal{F}L_0^{s,p}}} < 2e^{-\frac{M}{2}} \cos\left(\frac{M}{2}\right), \tag{3.12}$$

then there exists a smooth global solution u to dNLS (1.2) with $u|_{t=0} = \phi$. Moreover, there exists $C(\phi) > 0$ such that we have

$$\|u(t)\|_{L^2(\mathbb{T})} \le C(M, \|\phi\|_{\mathcal{F}L_0^{s,p}}) < \infty, \tag{3.13}$$

for all $t \in \mathbb{R}$.

Remark 3.7. This type of small "disturbance" condition also appears in the corresponding problem on $\mathbb{R}$. In [22], Stefanov proved local well-posedness of the quadratic dNLS (1.2) in $H^1(\mathbb{R})$, assuming that $\sup_{x\in\mathbb{R}} \left| \int_{-\infty}^{x} \phi(y)\,dy \right|$ is sufficiently small. This is analogous to the condition $M < \pi$ in Proposition 3.6.

Proof. Let $W^0 = \mathcal{G}_0[\phi]$. Then, we have

$$\begin{aligned} W^0(x) = e^{-\frac{i}{2}\mathcal{J}(\phi)(x)} &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{-i}{2}\right)^n [\mathcal{J}(\phi)(x)]^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{-i}{2}\right)^n \sum_{k\in\mathbb{Z}} \sum_{|\mathbf{k}|_n = k} \prod_{j=1}^{n} \mathcal{J}(\phi)_{k_j} e^{ikx} =: \sum_{k\in\mathbb{Z}} W_k^0 e^{ikx}, \end{aligned} \tag{3.14}$$

where

$$W_k^0 = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{-i}{2}\right)^n \sum_{|\mathbf{k}|_n = k} \prod_{j=1}^{n} \mathcal{J}(\phi)_{k_j}.$$

Then, by Young's inequality and

$$\|\mathcal{J}(\phi)_k\|_{\ell^1} = \sum_{k\ne 0} \frac{|\phi_k|}{|k|} \le Z_{s,p}\|\phi\|_{\mathcal{F}L_0^{s,p}}, \tag{3.15}$$

we have

$$\begin{aligned} \sum_{k\in\mathbb{Z}} |W_k^0| &\le \sum_{n=0}^{\infty} \frac{1}{2^n n!} \sum_{k\in\mathbb{Z}} \sum_{|\mathbf{k}|_n = k} \prod_{j=1}^{n} |\mathcal{J}(\phi)_{k_j}| \le \sum_{n=0}^{\infty} \frac{1}{2^n n!} \|\mathcal{J}(\phi)_k\|_{\ell^1}^n \\ &\le \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{Z_{s,p}\|\phi\|_{\mathcal{F}L_0^{s,p}}}{2}\right)^n = e^{\frac{1}{2}Z_{s,p}\|\phi\|_{\mathcal{F}L_0^{s,p}}}. \end{aligned} \tag{3.16}$$

Note that the absolute convergence of the summations in n and k of (3.16) justifies the interchange of the summations in (3.14).

On the other hand, let $r(x) := \exp\left\{\mathrm{Re}(-\frac{i}{2}\mathcal{J}(\phi)(x))\right\}$ and $\theta(x) := \mathrm{Im}(-\frac{i}{2}\mathcal{J}(\phi)(x))$. Then, by (3.11), we have

$$W^0(x) = r(x)e^{i\theta(x)}, \qquad r(x) \ge e^{-\frac{M}{2}}, \qquad \text{and} \quad |\theta(x)| \le \frac{M}{2}$$

for all $x \in \mathbb{T}$. Hence, by Mean Value Theorem, there exists $x_0 \in \mathbb{T}$ such that

$$\begin{aligned} |W_0^0| &= \frac{1}{2\pi}\left|\int_{\mathbb{T}} W^0 dx\right| = \frac{1}{2\pi}\left|\int_{\mathbb{T}} r(x)e^{i\theta(x)}dx\right| \ge \frac{1}{2\pi}\left|\int_{\mathbb{T}} r(x)\cos\theta(x)dx\right| \\ &= \left|r(x_0)\cos\theta(x_0)\right| \\ &\ge e^{-\frac{M}{2}}\cos\left(\frac{M}{2}\right). \end{aligned} \tag{3.17}$$

The condition $0 \le M < \pi$ guarantees that the right-hand side of (3.17) is positive.

In view of (3.12), (3.16) and (3.17), we see that the condition (3.10) in Lemma 3.3 is satisfied for some $\delta > 0$. Hence, letting W denote the smooth global solution to (3.5) with $W|_{t=0} = W^0$, Lemma 3.3 allows us to apply the inverse transformation (3.6) and construct a smooth global solution u to (1.2) with $u|_{t=0} = \phi$. Lastly, the global L^2-bound (3.13) follows from Lemma 3.5. □

We now transfer the conditions in Proposition 3.6 to a smallness condition on the $\mathcal{F}L_0^{s,p}$-norm of smooth mean-zero initial data.

Corollary 3.8. *Suppose that* (s, p) *satisfy* (i) $s > \frac{1}{2} - \frac{1}{p}$, $p > 2$ *or* (ii) $s \geq 0$, $p = 2$. *Then, there exists* $\delta_2 > 0$ *such that if a smooth mean-zero function* ϕ *on* $\mathbb{T}$ *satisfies*

$$\|\phi\|_{\mathcal{F}L_0^{s,p}(\mathbb{T})} \leq \delta_2,$$

then there exists a smooth global solution u *to dNLS* (1.2) *with* $u|_{t=0} = \phi$*, satisfying* (3.13).

Proof. From (3.15), we have

$$M = \sup_{x\in\mathbb{T}} |\mathcal{J}(\phi)(x)| \leq Z_{s,p}\|\phi\|_{\mathcal{F}L_0^{s,p}} \leq Z_{s,p}\,\delta_2. \tag{3.18}$$

Hence, the first condition $M < \pi$ in Proposition 3.6 is satisfied if $Z_{s,p}\,\delta_2 < \pi$.

Let $f(x) = 2e^{-2x}\cos x$. Then, there exists unique $\alpha \in \left(0, \frac{\pi}{2}\right)$ such that $f(\alpha) = 1$ since f is strictly decreasing on $[0, \frac{\pi}{2}]$, $f(0) = 2$, and $f\left(\frac{\pi}{2}\right) = 0$. Now, choose $\delta_2 > 0$ sufficiently small such that $Z_{s,p}\delta_2 < 2\alpha$. Then, we have

$$e^{\frac{1}{2}Z_{s,p}\|\phi\|_{\mathcal{F}L_0^{s,p}}} \leq e^{\frac{1}{2}Z_{s,p}\delta_2} < e^{\alpha} = 2e^{-\alpha}\cos\alpha < 2e^{-\frac{M}{2}}\cos\left(\tfrac{M}{2}\right),$$

where the last inequality follows from (3.18) and the fact that $g(x) = e^{-x}\cos x$ is strictly decreasing on $x \in \left[0, \frac{\pi}{2}\right]$. This shows that the second condition (3.12) in Proposition 3.6 also holds. □

4. Proof of Theorem 1.1

We are now ready to present the proof of Theorem 1.1. The main ingredients are the normal form reductions and small data global existence of smooth solutions via the gauge transformation discussed in Sections 2 and 3.

Given small $\delta_0 > 0$ (to be chosen later), fix $\phi \in \mathcal{F}L_0^{s,p}$ such that $\|\phi\|_{\mathcal{F}L_0^{s,p}} \leq \delta_0$. For $j \in \mathbb{N}$, let $\phi^{(j)} = \mathbf{P}_{\leq j}\phi$, where $\mathbf{P}_{\leq j}$ is the Dirichlet projection onto the frequencies $\{|k| \leq j\}$. Note that we have $\{\phi^{(j)}\}_{j\in\mathbb{N}} \subset C^\infty(\mathbb{T})$, $\int_{\mathbb{T}} \phi^{(j)} dx = 0$, and $\|\phi^{(j)}\|_{\mathcal{F}L_0^{s,p}} \leq \delta_0$ for all $j \in \mathbb{N}$. Then, by Corollary 3.8, there exist global smooth solutions $u^{(j)}$ to (1.2) with $u^{(j)}|_{t=0} = \phi^{(j)}$, as long as $\delta_0 \leq \delta_2$.

We first consider the case $(s, p) = (0, 2)$. Letting $W^{(j),0} = \mathcal{G}_0[\phi^{(j)}]$, it follows from (3.16) and (3.17) with (3.18) that

$$\begin{aligned}|W_0^{(j),0}| - \sum_{k\neq 0} |W_k^{(j),0}| &= 2|W_0^{(j),0}| - \sum_{k\in\mathbb{Z}} |W_k^{(j),0}| \\ &\geq 2e^{-\frac{1}{2}Z_{0,2}\delta_0}\cos\left(\frac{Z_{0,2}\delta_0}{2}\right) - e^{\frac{1}{2}Z_{0,2}\delta_0} =: A(\delta_0) > 0\end{aligned}$$

for all $j \in \mathbb{N}$. From the continuity of $A(\delta_0)$ and $A(0) = 1$, we have $A(\delta_0) \geq \frac{1}{2}$ for all sufficiently small $\delta_0 > 0$. Hence, it follows from Lemma 3.5 that by choosing $\delta_0 > 0$ sufficiently small, we have

$$\|u^{(j)}(t)\|_{L^2} \leq \|u(t)\|_{L^2(\mathbb{T})} \leq 2e^{\frac{1}{2}Z_{0,2}\delta_0}\delta_0 < \delta_1, \tag{4.1}$$

for all $t \in \mathbb{R}$ and all $j \in \mathbb{N}$, where δ_1 is as in Proposition 2.8.

Fix $T = T(\delta_1)$, where T is the local existence time in Proposition 2.8. Then, from a slight modification of (2.45) applied to $v^{(j)}(t) = S(-t)u^{(j)}(t)$, we obtain

$$\|u^{(j)} - u^{(\ell)}\|_{C([0,T];L^2)} \leq 2\|\phi^{(j)} - \phi^{(\ell)}\|_{L^2} \tag{4.2}$$

for all $j, \ell \in \mathbb{N}$. In view of (4.1), we can iterate the argument on intervals $[iT, (i+1)T]$, $i = 1, 2, \ldots$, and obtain

$$\|u^{(j)} - u^{(\ell)}\|_{C([0,\tau];L^2)} \leq 2^{[\frac{\tau}{T}]+1}\|\phi^{(j)} - \phi^{(\ell)}\|_{L^2} \tag{4.3}$$

for any $\tau > 0$. Since $\phi^{(j)}$ converges in L^2, it follows from (4.3) that $u^{(j)}$ converges to some u in $C(\mathbb{R}; L^2)$ endowed with the compact-open topology, i.e. $u^{(j)}$ converges to some $u(t)$ in L^2, uniformly on each compact time interval. In particular, we have

$$\partial_t u - i\partial_x^2 u - u\partial_x u = \lim_{j\to\infty} \Big\{\partial_t u^{(j)} - i\partial_x^2 u^{(j)} - \tfrac{1}{2}\partial_x\big[(u^{(j)})^2\big]\Big\} = 0$$

in the sense of distributions. Therefore, u is a distributional solution to (1.2) with $u|_{t=0} = \phi$. The continuous dependence follows from a slight modification of (2.45). See (4.2) above.

Next, we discuss the issue of uniqueness. Given $\phi \in L^2$ such that $\|\phi\|_{L^2} < \delta_0$, let $u \in C(\mathbb{R}; L^2)$ be solutions to (1.2). Then, it follows from (2.1) that $u_k \in C^1(\mathbb{R}; \mathbb{C})$ for each $k \in \mathbb{Z}_0$. Letting $v(t) = S(-t)u(t)$, we also have $v_k \in C^1(\mathbb{R}; \mathbb{C})$ for each $k \in \mathbb{Z}_0$. Moreover, $\{v_k\}_{k\in\mathbb{Z}_0}$ satisfies (2.3) for each $k \in \mathbb{Z}_0$. We now need to verify the differentiation by parts step (2.6). Recall the following lemma from [10].

Lemma 4.1. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}'_t$. Suppose that $\sum_n f_n$ converges (absolutely) in $\mathcal{D}'_t$. Then, $\sum_n \partial_t f_n$ converges (absolutely) in $\mathcal{D}'_t$ and $\partial_t(\sum_n f_n) = \sum_n \partial_t f_n$.*

By Young's inequality, we have

$$\sum_{|\mathbf{k}|_n = k} \bigg| e^{i\Phi_n(\mathbf{k})t} \prod_{j=1}^n \frac{v_{k_j}(t)}{k_j} \bigg| \le \bigg\| \frac{v_k(t)}{k} \bigg\|_{\ell^2}^2 \bigg\| \frac{v_k(t)}{k} \bigg\|_{\ell^1}^{n-2} \lesssim \|v_k(t)\|_{\ell^2}^n.$$

Hence, Lemma 4.1 justifies the computation in (2.6). Note that in transition from (2.6) to (2.7), we used the product rule: $\partial_t(v_{k_1}v_{k_2}) = \partial_t v_{k_1} v_{k_2} + v_{k_1}\partial_t v_{k_2}$. This is justified by the fact that $v_k \in C^1(\mathbb{R}; \mathbb{C})$ for each $k \in \mathbb{Z}_0$. Proceeding in a similar manner, we can justify all the subsequent steps in the normal form reductions.

Noting that Part 1 of Proposition 2.6 relies only on the L^2-regularity of $v(t)$, we see that (2.30) holds. Moreover, the normal form equation (2.24) holds for each frequency $k \in \mathbb{Z}_0$ on the Fourier side. Namely, we have

$$v_k(t) = \phi_k + \sum_{n=2}^\infty \mathcal{N}_k^n(v(t)) - \sum_{n=2}^\infty \mathcal{N}_k^n(\phi) + \int_0^t \sum_{n=2}^\infty \mathcal{R}_k^n(v(t'))dt' \tag{4.4}$$

for each $k \in \mathbb{Z}_0$. Then, the uniqueness part of Theorem 1.1 follows from the corresponding uniqueness statement for (4.4). See Remark 2.9.

The general case (s, p) with (i) $s > \frac{1}{2} - \frac{1}{p},\ p > 2$ or (ii) $s \ge 0,\ p = 2$ follows from the persistence of regularity for the normal form equation (2.24) and (2.28); see Remark 2.10. This completes the proof of Theorem 1.1.

We conclude this section by stating the following corollary.

Corollary 4.2. *Assume the hypotheses of Theorem 1.1. Let u be a global solution to (1.2) with $u|_{t=0} = \phi \in \mathcal{F}L_0^{s,p}$. Then, $\mathcal{Q}_k = \mathcal{Q}_k[u]$, $k \in \mathbb{Z}_0$, defined in (2.26) is invariant under the dynamics of (1.2).*

Proof. In view of Remark 2.4 and Proposition 2.6, this proposition follows from a standard approximation argument, using a computation similar to (2.45), and thus we omit details. □

Conflict of interest statement

The authors have no conflict of interest to declare.

Acknowledgements

J.C. was supported by NIMS grant funded by Korean Government (No. A23100000). S.K. was partially supported by NRF of Korea (No. 2015R1D1A1A01058832) and Posco Science Fellowship. T.O. was supported by the European Research Council (grant no. 637995 "ProbDynDispEq"). The authors would like to thank Dario Bambusi for the reference [20].

Appendix A. Large data finite time blowup solution

In this appendix, we present the proof of Theorem 1.2. Recall that in establishing global well-posedness of (1.2) through the gauge transformation (3.3), it was essential to guarantee that the gauged function W stays away from the origin. This was achieved by imposing smallness assumption, since if $u(0) = \phi$ is small, the gauge transformation $W(0) = \mathcal{G}_0[\phi]$ is a loop close to $1 \in \mathbb{C}$. In the following, we construct an example of a finite time blowup solution to (1.2) by first constructing an example of $W(t)$ which approaches the origin such that the inverse gauge transformation (3.6) ceases to make sense. As $W(t)$ evolves linearly, it suffices to find a linear solution that touches the origin in finite time. By choosing

$$W(t,x) := 1 - ie^{-it}\cos x,$$

we see that $W\big(\frac{\pi}{2}, 0\big) = 0$. Indeed, the loop $W(t)$ touches the origin for the first time at $t = \frac{\pi}{2}$.

On the other hand, the inverse gauge transformation (3.6) gives a solution u to (1.2):

$$u(x,t) = \frac{-2e^{-it}\sin x}{1 - ie^{-it}\cos x}.$$

Note that $\int_{\mathbb{T}} u(0)dx = 0$. We claim that the L^1-norm of $u(t)$ diverges as $t \to \frac{\pi}{2}-$. The modulus of u is

$$|u(t,x)| = \frac{2|\sin x|}{\sqrt{1 - 2\sin t\cos x + \cos^2 x}}.$$

In particular, when $t = \frac{\pi}{2}$,

$$\big|u\big(\tfrac{\pi}{2}, x\big)\big| = \frac{2|\sin x|}{1 - \cos x}.$$

For $|x| \ll 1$, we have

$$2|\sin x| \geq 2\left|x - \frac{x^3}{3!}\right| \geq |x| \qquad \text{and} \qquad 1 - \cos x \leq \frac{x^2}{2!}.$$

Hence

$$\int_{\mathbb{T}} \big|u\big(\tfrac{\pi}{2}, x\big)\big| dx \geq \int_{|x| \ll 1} \frac{2}{|x|} dx = \infty.$$

Therefore, every L^p-norm of $u(t)$, $1 \leq p \leq \infty$, diverges as $t \to \frac{\pi}{2}$. Also, given (s,p) as in Theorem 1.1, the $\mathcal{F}L_0^{s,p}$-norm of $u(t)$ diverges as $t \to \frac{\pi}{2}$.

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