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Nonhomogeneous boundary conditions for the spectral fractional Laplacian

Nicola Abatangelo, Louis Dupaigne *

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Abstract

We present a construction of harmonic functions on bounded domains for the *spectral fractional Laplacian* operator and we classify them in terms of their divergent profile at the boundary. This is used to establish and solve boundary value problems associated with nonhomogeneous boundary conditions. We provide a weak-L^1 theory to show how problems with measure data at the boundary and inside the domain are well-posed. We study linear and semilinear problems, performing a sub- and supersolution method. We finally show the existence of *large solutions* for some power-like nonlinearities.

MSC: primary 35B40; secondary 35B30, 45P05, 35C15

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1. Introduction

Given a bounded domain Ω of the Euclidean space $\mathbb{R}^N$, the *spectral fractional Laplacian* operator $(-\triangle|_\Omega)^s$, $s \in (0,1)$, is classically defined as a fractional power of the Laplacian with homogeneous Dirichlet boundary conditions, seen as a self-adjoint operator in the Lebesgue space $L^2(\Omega)$, see (2) below. This provides a nonlocal operator of elliptic type with *homogeneous* boundary conditions. Recent bibliography on this operator can be found e.g. in Bonforte, Sire and Vázquez [5], Grubb [16], Caffarelli and Stinga [7], Servadei and Valdinoci [21].

One aspect of the theory is however left unanswered: the formulation of natural *nonhomogeneous* boundary conditions. A first attempt can be found in the work of Dhifli, Mâagli and Zribi [11]. The investigations that have resulted in the present paper turn out, we hope, to shed some further light on this question. We provide a weak formulation, which is well-posed in the sense of Hadamard, for linear problems of the form

$$\begin{cases} (-\triangle|_\Omega)^s u = \mu & \text{in } \Omega, \\ \dfrac{u}{h_1} = \zeta & \text{on } \partial\Omega \end{cases} \tag{1}$$

* Corresponding author.
E-mail address: louis.dupaigne@math.cnrs.fr (L. Dupaigne).

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where h_1 is a reference function, see (8) below, with prescribed singular behavior at the boundary. Namely, h_1 is bounded above and below by constant multiples of $\delta^{-(2-2s)}$, where

$$\delta(x) := \operatorname{dist}(x, \partial\Omega)$$

is the distance to the boundary and the left-hand side of the boundary condition must be understood as a limit as δ converges to zero. In other words, unlike the classical Dirichlet problem for the Laplace operator, nonhomogeneous boundary conditions must be singular. In addition, if the data μ, ζ are smooth, the solution blows up at the fixed rate $\delta^{-(2-2s)}$. This is very similar indeed to the theory of nonhomogeneous boundary conditions for the classical (sometimes called "restricted") fractional Laplacian – although in that case the blow-up rate is of order $\delta^{-(1-s)}$ – as analyzed from different perspectives by Grubb [15] and the first author [1] of the present note. In fact, for the special case of positive s-harmonic functions, that is when $\mu = 0$, the singular boundary condition was already identified in previous works emphasizing the probabilistic and potential theoretic aspects of the problem: see e.g. Bogdan, Byczkowski, Kulczycki, Ryznar, Song and Vondraček [4] for an explicit example in the framework of the classical fractional Laplacian, as well as Song and Vondraček [24], Glover, Pop-Stojanovic, Rao, Šikić, Song and Vondraček [13] and Song [23] for the spectral fractional Laplacian. Let us also mention the work of Guan [17], who considered similar issues in the context of the regional fractional Laplacian and generalizations thereof.

Turning to nonlinear problems, even more singular boundary conditions arise: in the above system, if $\mu = -u^p$ for suitable values of p, one may choose $\zeta = +\infty$, in the sense that the solution u will blow up at a higher rate with respect to $\delta^{-(2-2s)}$ and controlled by the (scale-invariant) one $\delta^{-2s/(p-1)}$. Note that the value $\zeta = +\infty$ is not admissible for linear problems. This was already observed by the first author in the context of the fractional Laplacian, see [2], clarifying earlier work by Felmer and Quaas [12] and this is what we prove here for the spectral fractional Laplacian. Interestingly, the range of admissible exponents p is different according to which operator one works with.

1.1. Main results

For clarity, we list here the definitions and statements that we use, with reference to the sections of the paper where the proofs can be found. First recall the definition of the spectral fractional Laplacian:

Definition 1. Let $\Omega \subset \mathbb{R}^N$ a bounded domain and let $\{\varphi_j\}_{j\in\mathbb{N}}$ be a Hilbert basis of $L^2(\Omega)$ consisting of eigenfunctions of the Dirichlet Laplacian $-\triangle|_\Omega$, associated to the eigenvalues λ_j, $j \in \mathbb{N}$, i.e.[1] $\varphi_j \in H^1_0(\Omega) \cap C^\infty(\Omega)$ and $-\triangle\varphi_j = \lambda_j\varphi_j$ in Ω. Given $s \in (0, 1)$, consider the Hilbert space[2]

$$H(2s) := \left\{ v = \sum_{j=1}^{\infty} \widehat{v}_j \varphi_j \in L^2(\Omega) : \|v\|^2_{H(2s)} = \sum_{j=0}^{\infty} \lambda_j^{2s} |\widehat{v}_j|^2 < \infty \right\}.$$

The spectral fractional Laplacian of $u \in H(2s)$ is the function belonging to $L^2(\Omega)$ given by the formula

$$(-\triangle|_\Omega)^s u = \sum_{j=1}^{\infty} \lambda_j^s \widehat{u}_j \varphi_j. \tag{2}$$

Note that $C_c^\infty(\Omega) \subset H(2s) \hookrightarrow L^2(\Omega)$. So, the operator $(-\triangle|_\Omega)^s$ is unbounded, densely defined and with bounded inverse $(-\triangle|_\Omega)^{-s}$ in $L^2(\Omega)$. Alternatively, for almost every $x \in \Omega$,

$$(-\triangle|_\Omega)^s u(x) = PV \int_\Omega [u(x) - u(y)] J(x, y)\, dy + \kappa(x) u(x), \tag{3}$$

where, letting $p_\Omega(t, x, y)$ denote the heat kernel of $-\triangle|_\Omega$,

[1] See Brezis [6, Theorem 9.31].

[2] When Ω is smooth, $H(2s)$ coincides with the Sobolev space $H^{2s}(\Omega)$ if $0 < s < 1/4$, $H^s_{00}(\Omega)$ if $s \in \{1/4, 3/4\}$, $H^{2s}_0(\Omega)$ otherwise; see Lions and Magenes [18, Theorems 11.6 and 11.7 pp. 70–72].

$$J(x,y)=\frac{s}{\Gamma(1-s)}\int_0^\infty \frac{p_\Omega(t,x,y)}{t^{1+s}}\,dt \qquad \text{and} \qquad \kappa(x)=\frac{s}{\Gamma(1-s)}\int_\Omega \left(1-\int_\Omega p_\Omega(t,x,y)\,dy\right)\frac{dt}{t^{1+s}} \tag{4}$$

are respectively the *jumping kernel* and the *killing measure*,[3] see Song, Vondraček [24, formulas (3.3) and (3.4)]. For the reader's convenience, we provide a proof of (3) in the Appendix. We assume from now on that

$$\Omega \text{ is of class } C^{1,1}.$$

In particular, sharp bounds are known for the heat kernel p_Ω, see (27) below, and provide in turn sharp estimates for $J(x,y)$, see (33) below, so that the right-hand side of (3) remains well-defined for every $x\in\Omega$ under the assumption that $u\in C^{2s+\varepsilon}_{loc}(\Omega)\cap L^1(\Omega,\delta(x)dx)$ for some $\varepsilon>0$. This allows us to *define* the spectral fractional Laplacian of functions which *do not* vanish on the boundary of Ω. As a simple example, observe that the function $u=1$ does not belong to $H(2s)$ if $s\geq 1/4$, yet it solves (1) for $\mu=\kappa$ and $\zeta=0$.

Definition 2. The Green function and the Poisson kernel of the spectral fractional Laplacian are defined respectively by

$$G^s_\Omega(x,y) \;=\; \frac{1}{\Gamma(s)}\int_0^\infty p_\Omega(t,x,y)\,t^{s-1}\,dt, \qquad x,\,y\in\Omega, x\neq y,\; s\in(0,1], \tag{5}$$

and by

$$P^s_\Omega(x,y) \;:=\; -\,\frac{\partial}{\partial\nu_y}\,G^s_\Omega(x,y), \qquad x\in\Omega, y\in\partial\Omega, \tag{6}$$

where ν is the outward unit normal to $\partial\Omega$.

In Section 2, we shall prove that P^s_Ω is well-defined (see Lemma 14) and review some useful identities involving the Green function G^s_Ω and the Poisson kernel P^s_Ω. Now, let us define weak solutions of (1).

Definition 3. Consider the test function space

$$\mathcal{T}(\Omega) \;:=\; (-\triangle|_\Omega)^{-s}C^\infty_c(\Omega) \tag{7}$$

and the weight

$$h_1(x)=\int_{\partial\Omega} P^s_\Omega(x,y)\,d\sigma(y), \qquad x\in\Omega. \tag{8}$$

Given two Radon measures $\mu\in\mathcal{M}(\Omega)$ and $\zeta\in\mathcal{M}(\partial\Omega)$ with

$$\int_\Omega \delta(x)\,d|\mu|(x) \;<\; \infty, \qquad |\zeta|(\partial\Omega)\;<\;\infty, \tag{9}$$

a function $u\in L^1_{loc}(\Omega)$ is a weak solution to

$$\begin{cases} (-\triangle|_\Omega)^s u = \mu & \text{in } \Omega \\ \dfrac{u}{h_1} = \zeta & \text{on } \partial\Omega \end{cases} \tag{10}$$

if, for any $\psi\in\mathcal{T}(\Omega)$,

$$\int_\Omega u\,(-\triangle|_\Omega)^s\psi \;=\; \int_\Omega \psi\,d\mu - \int_{\partial\Omega}\frac{\partial\psi}{\partial\nu}\,d\zeta. \tag{11}$$

[3] In the language of potential theory of killed stochastic processes. Note that the integral in (3) must be understood in the sense of principal values. To see this, look at (33).

Remark 4. We emphasize that the boundary condition is encoded both in formula (11) and in the choice of test functions $\mathcal{T}(\Omega)$. We shall prove on the one hand that[4] $\mathcal{T}(\Omega) \subseteq C^1_0(\overline{\Omega})$, see Lemma 28, so that all integrals above are well-defined indeed and on the other hand that all weak solutions enjoy the extra integrability condition $u \in L^1(\Omega, \delta(x)dx)$, see Lemma 5 below. In particular, equation (11) is a weak formulation of (10), as we state now.

Lemma 5.

1. *(Weak solutions are distributional solutions) Assume that $u \in L^1_{loc}(\Omega)$ is a weak solution of (10). Then in fact, $u \in L^1(\Omega, \delta(x)dx)$ and $(-\triangle|_\Omega)^s u = \mu$ in the sense of distributions i.e. for any $\psi \in C^\infty_c(\Omega)$, $\delta^{-1}(-\triangle|_\Omega)^s \psi$ is bounded and*

$$\int_\Omega u(-\triangle|_\Omega)^s \psi = \int_\Omega \psi \, d\mu,$$

moreover the boundary condition holds in the sense that for every $\phi \in C(\overline{\Omega})$

$$\lim_{t\downarrow 0} \frac{1}{t} \int_{\{\delta(x)<t\}} \frac{u(x)}{h_1(x)} \phi(x)\, dx = \int_{\partial\Omega} \phi(x)\, d\zeta(x)$$

whenever $\mu \in \mathcal{M}(\Omega)$ satisfies (9) and $\zeta \in L^1(\partial\Omega)$.

2. *(For smooth data, weak solutions are classical) Assume that $u \in L^1_{loc}(\Omega)$ is a weak solution of (10), where $\mu \in C^\alpha(\overline{\Omega})$ for some α such that $\alpha + 2s \notin \mathbb{N}$ and $\zeta \in C(\partial\Omega)$. Then, $(-\triangle|_\Omega)^s u$ is well-defined by (3) for every $x \in \Omega$, $(-\triangle|_\Omega)^s u(x) = \mu(x)$ for all $x \in \Omega$ and for all $x_0 \in \partial\Omega$,*

$$\lim_{\substack{x \to x_0 \\ x\in\Omega}} \frac{u(x)}{h_1(x)} = \zeta(x_0).$$

3. *(Classical solutions are weak solutions) Assume that $u \in C^{2s+\varepsilon}_{loc}(\Omega)$ is such that $u/h_1 \in C(\overline{\Omega})$. Let $\mu = (-\triangle|_\Omega)^s u$ be given by (3) and $\zeta = u/h_1|_{\partial\Omega}$. Then, u is a weak solution of (10).*

We present some facts about harmonic functions in Section 3 with an eye kept on their singular boundary trace in Section 4. We prove the well-posedness of (10) in Section 5, namely

Theorem 6. *Given two Radon measures $\mu \in \mathcal{M}(\Omega)$ and $\zeta \in \mathcal{M}(\partial\Omega)$ such that (9) holds, there exists a unique function $u \in L^1_{loc}(\Omega)$ which is a weak solution to (10). Moreover, for a.e. $x \in \Omega$,*

$$u(x) = \int_\Omega G^s_\Omega(x,y)\, d\mu(y) + \int_{\partial\Omega} P^s_\Omega(x,y)\, d\zeta(y) \tag{12}$$

and

$$\|u\|_{L^1(\Omega,\delta(x)dx)} \le C(\Omega, N, s)\left(\|\delta\mu\|_{\mathcal{M}(\Omega)} + \|\zeta\|_{\mathcal{M}(\partial\Omega)}\right). \tag{13}$$

In addition, the following estimates hold:

$$\|u\|_{L^p(\Omega,\delta(x)dx)} \le C_1 \|\delta\mu\|_{\mathcal{M}(\Omega)} \qquad \text{if } \zeta = 0 \text{ and } p \in \left[1, \frac{N+1}{N+1-2s}\right) \tag{14}$$

$$\|u\|_{C^\alpha(\overline{\omega})} \le C_2\left(\|\mu\|_{L^\infty(\omega)} + \|\zeta\|_{\mathcal{M}(\partial\Omega)}\right) \qquad \text{if } \omega \subset\subset \Omega \text{ and } \alpha \in (0, 2s) \tag{15}$$

$$\|u\|_{C^{2s+\alpha}(\overline{\omega})} \le C_3\left(\|\mu\|_{C^\alpha(\overline{\omega})} + \|\zeta\|_{\mathcal{M}(\partial\Omega)}\right) \qquad \text{if } \omega \subset\subset \Omega \text{ and } 2s+\alpha \notin \mathbb{N}. \tag{16}$$

In the above $C_1 = C_1(\Omega, N, s, p)$, $C_2 = C_2(\Omega, \omega, N, s, \alpha)$, $C_3 = C_3(\Omega, \omega, N, s, \alpha)$.

[4] But it is not true that $C^\infty_c(\Omega) \subseteq \mathcal{T}(\Omega)$.

Remark 7. Note that the spectral fractional Laplacian shares the same interior regularity (15), (16) with the fractional Laplacian (see Silvestre [22]) and the regional fractional Laplacian (see Mou and Yi [20]), since this is due to the singularity rate of the kernel defining the operators, which is the same for the three of them. In the context of the regional fractional Laplacian, a representation formula similar to (12) can be derived from Guan [17, Theorem 1.3]. Note that the asymptotic behavior of solutions near the boundary is yet again different in that context.

In Section 6 we solve nonlinear Dirichlet problems, by proving

Theorem 8. *Let $g(x,t):\Omega\times\mathbb{R}^+\longrightarrow\mathbb{R}^+$ be a Carathéodory function and $h:\mathbb{R}^+\to\mathbb{R}^+$ a nondecreasing function such that $g(x,0)=0$ and for a.e. $x\in\Omega$ and all $t>0$,*

$$0\le g(x,t)\le h(t)\qquad \textit{where}\qquad h(\delta^{-(2-2s)})\delta\in L^1(\Omega)$$

Then, problem

$$\begin{cases}(-\triangle|_\Omega)^s u=-g(x,u) & \textit{in } \Omega\\ \dfrac{u}{h_1}=\zeta & \textit{on } \partial\Omega\end{cases}\tag{17}$$

has a solution $u\in L^1(\Omega,\delta(x)dx)$ for any $\zeta\in C(\partial\Omega)$, $\zeta\ge 0$. In addition, if $t\mapsto g(x,t)$ is nondecreasing then the solution is unique.

The above theorem and the linear theory that we developed are insufficient to encompass the large solutions that can appear and which have a higher rate of explosion. Precisely, with Section 7 we prove

Theorem 9. *Let*

$$p\in\left(1+s,\frac{1}{1-s}\right).$$

Then, there exists a function $u\in L^1(\Omega,\delta(x)dx)\cap C^\infty(\Omega)$ solving

$$\begin{cases}(-\triangle|_\Omega)^s u=-u^p & \textit{in } \Omega,\\ \dfrac{u}{h_1}=+\infty & \textit{on } \partial\Omega\end{cases}\tag{18}$$

in the following sense: the first equality holds pointwise and in the sense of distributions, the boundary condition is understood as a pointwise limit. In addition, there exists a constant $C=C(\Omega,N,s,p)$ such that

$$0\le u\le C\delta^{-\frac{2s}{p-1}}.$$

Remark 10. If one replaces the spectral fractional Laplacian by the classical fractional Laplacian $(-\triangle)^s$, then an analogous of Theorem 9 holds for

$$p\in\left(1+2s,\frac{1+s}{1-s}\right),$$

see [2], as well as the pioneering works of Felmer and Quaas and Chen, Felmer and Quaas [12,8]. We suspect that both exponent ranges are optimal.

2. Green function and Poisson kernel

In the following three lemmas,[5] we establish some useful identities for the Green function defined by (5).

[5] Which hold even if the domain Ω is not $C^{1,1}$.

Lemma 11. *(See [14, formula (17)].) Let $f \in L^2(\Omega)$. For almost every $x \in \Omega$, $G^s_\Omega(x,\cdot)f \in L^1(\Omega)$ and*

$$(-\triangle|_\Omega)^{-s} f(x) = \int_\Omega G^s_\Omega(x,y) f(y)\,dy \qquad \textit{for a.e. } x \in \Omega.$$

Proof. If φ_j is an eigenfunction of $-\triangle|_\Omega$, then

$$\begin{aligned}\int_\Omega G^s_\Omega(x,y)\varphi_j(y)\,dy \;&=\; \int_0^\infty \frac{t^{s-1}}{\Gamma(s)} \int_\Omega p_\Omega(t,x,y)\varphi_j(y)\,dy\,dt \\ &= \int_0^\infty \frac{t^{s-1}}{\Gamma(s)} e^{-\lambda_j t}\varphi_j(x)\,dt \;=\; \frac{\lambda_j^{-s}}{\Gamma(s)}\varphi_j(x)\int_0^\infty t^{s-1}e^{-t}\,dt \;=\; \lambda_j^{-s}\varphi_j(x) = (-\triangle|_\Omega)^{-s}\varphi_j(x)\end{aligned}$$

By linearity, if $f \in L^2(\Omega)$ is a linear combination of eigenvectors $f = \sum_{j=1}^M \widehat{f}_j\varphi_j$, then

$$\int_\Omega G^s_\Omega(x,y)\sum_{j=1}^M \widehat{f}_j\varphi_j(y)\,dy \;=\; \sum_{j=1}^M \widehat{f}_j\lambda_j^{-s}\varphi_j(x),$$

and so, letting

$$\mathbb{G}^s_\Omega f \;:=\; \int_\Omega G^s_\Omega(\cdot,y)\,f(y)\,dy, \tag{19}$$

we have

$$\|\mathbb{G}^s_\Omega f\|^2_{H(2s)} = \sum_{j=1}^M \lambda_j^{2s}\cdot|\widehat{f}_j|^2\lambda_j^{-2s} = \|f\|^2_{L^2(\Omega)}. \tag{20}$$

Thus the map $\mathbb{G}^s_\Omega : f \longmapsto \mathbb{G}^s_\Omega f$ uniquely extends to a linear isometry from $L^2(\Omega)$ to $H(2s)$, which coincides with $(-\triangle|_\Omega)^{-s}$. It remains to prove that the identity (19) remains valid a.e. for $f \in L^2(\Omega)$. By standard parabolic theory, the function $(t,x) \mapsto \int_\Omega p_\Omega(t,x,y)dy$ is bounded (by 1) and smooth in $[0,T]\times\omega$ for every $T>0$, $\omega \subset\subset \Omega$. Hence, for every $x\in\Omega$, $G^s_\Omega(x,\cdot)\in L^1(\Omega)$. Assume first that $f=\psi\in C^\infty_c(\Omega)$ and take a sequence $\{\psi_k\}_{k\in\mathbb{N}}$ in the linear span of the eigenvectors $\{\varphi_j\}_{j\in\mathbb{N}}$ such that $\{\psi_k\}_{k\in\mathbb{N}}$ converges to ψ in $L^2(\Omega)$. The convergence is in fact uniform and so (19) holds for $f=\psi$. Indeed, by standard elliptic regularity, there exist constants $C=C(N,\Omega)$, $k=k(N)$ such that any eigenfunction satisfies

$$\|\nabla\varphi_j\|_{L^\infty(\Omega)} \le (C\lambda_j)^k\|\varphi_j\|_{L^2(\Omega)} = (C\lambda_j)^k.$$

In particular, taking C larger if needed,

$$\left\|\frac{\varphi_j}{\delta}\right\|_{L^\infty(\Omega)} \le (C\lambda_j)^k. \tag{21}$$

Now write $\psi = \sum_{j=1}^\infty \widehat{\psi}_j\varphi_j$ and fix $m\in\mathbb{N}$. Integrating by parts m times yields

$$\widehat{\psi}_j = \int_\Omega \psi\varphi_j = -\frac{1}{\lambda_j}\int_\Omega \psi\,\triangle\varphi_j = -\frac{1}{\lambda_j}\int_\Omega \triangle\psi\,\varphi_j = \ldots = \frac{(-1)^m}{\lambda_j^m}\int_\Omega \triangle^m\psi\,\varphi_j$$

which implies that

$$|\widehat{\psi}_j| \le \frac{\|\triangle^m\psi\|_{L^2(\Omega)}}{\lambda_j^m}, \tag{22}$$

i.e. the spectral coefficients of ψ converge to 0 faster than any polynomial. This and (21) imply that $\{\psi_k\}_{k\in\mathbb{N}}$ converges to ψ uniformly, as claimed.

Take at last $f \in L^2$ and a sequence $\{f_k\}_{k\in\mathbb{N}}$ in $C_c^\infty(\Omega)$ of nonnegative functions such that $\{f_k\}_{k\in\mathbb{N}}$ converges to $|f|$ a.e. and in $L^2(\Omega)$. By (20), $\|\mathbb{G}_\Omega^s f_k\|_{L^2} \le \|f_k\|_{L^2}$ and by Fatou's lemma, we deduce that $G_\Omega^s(x,\cdot)f \in L^1(\Omega)$ for a.e. $x \in \Omega$ and the desired identity follows. $\square$

Lemma 12. *(See [14, formula (8)].) For a.e. $x, y \in \Omega$,*

$$\int_\Omega G_\Omega^{1-s}(x,\xi)G_\Omega^s(\xi,y)\,d\xi = G_\Omega^1(x,y) \tag{23}$$

Proof. Clearly, given an eigenfunction φ_j,

$$(-\triangle|_\Omega)^{-s}(-\triangle|_\Omega)^{s-1}\varphi_j = \lambda_j^{-s}\lambda_j^{s-1}\varphi_j = (-\triangle|_\Omega)^{-1}\varphi_j$$

so $(-\triangle|_\Omega)^{-s} \circ (-\triangle|_\Omega)^{s-1} = (-\triangle|_\Omega)^{-1}$ in $L^2(\Omega)$. By the previous lemma and Fubini's theorem, we deduce that for $\varphi \in L^2(\Omega)$ and a.e. $x \in \Omega$,

$$\int_{\Omega^2} G_\Omega^{1-s}(x,\xi)G_\Omega^s(\xi,y)\varphi(y)\,d\xi\,dy = \int_\Omega G_\Omega^1(x,y)\varphi(y)\,dy$$

and so (23) holds almost everywhere. $\square$

Lemma 13. *For any $\psi \in C_c^\infty(\Omega)$,*

$$(-\triangle|_\Omega)^s\psi \;=\; (-\triangle)\circ(-\triangle|_\Omega)^{s-1}\psi \;=\; (-\triangle|_\Omega)^{s-1}\circ(-\triangle)\psi \tag{24}$$

Proof. The identity clearly holds if ψ is an eigenfunction. If $\psi \in C_c^\infty(\Omega)$, its spectral coefficients have fast decay and the result follows by writing the spectral decomposition of ψ. Indeed, thanks to (22) and (21), we may easily work by density to establish (24). $\square$

Let us turn to the definition and properties of the Poisson kernel. Recall that, for $x \in \Omega$, $y \in \partial\Omega$, the Poisson kernel of the Dirichlet Laplacian is given by

$$P_\Omega^1(x,y) \;=\; -\,\frac{\partial}{\partial\nu_y}\,G_\Omega^1(x,y).$$

Lemma 14. *The function*

$$P_\Omega^s(x,y) \;:=\; -\frac{\partial}{\partial\nu_y}G_\Omega^s(x,y)$$

is well-defined for $x \in \Omega$, $y \in \partial\Omega$ and $P_\Omega^s(x,\cdot) \in C(\partial\Omega)$ for any $x \in \Omega$. Furthermore, there exists a constant $C > 0$ depending on N, s, Ω only such that

$$\frac{1}{C}\frac{\delta(x)}{|x-y|^{N+2-2s}} \le P_\Omega^s(x,y) \le C\frac{\delta(x)}{|x-y|^{N+2-2s}} \tag{25}$$

and

$$\int_\Omega G_\Omega^{1-s}(x,\xi)P_\Omega^s(\xi,y)\,d\xi \;=\; P_\Omega^1(x,y). \tag{26}$$

Remark 15. When Ω is merely Lipschitz, one must work with the Martin kernel in place of the Poisson kernel, see [13].

Proof of Lemma 14. Take $x, z \in \Omega$, $y \in \partial\Omega$. Then,

$$\frac{G^s_\Omega(x,z)}{\delta(z)} = \frac{1}{\Gamma(s)\,\delta(z)} \int_0^\infty p_\Omega(t,x,z)t^{s-1}\,dt = \frac{|z-x|^{2s}}{\Gamma(s)\,\delta(z)} \int_0^\infty p_\Omega(|z-x|^2\tau,x,z)\tau^{s-1}\,d\tau.$$

Since Ω has $C^{1,1}$ boundary, given $x \in \Omega$, $p_\Omega(\cdot,x,\cdot) \in C^1((0,+\infty)\times\overline{\Omega})$ and the following heat kernel bound holds (cf. Davies, Simon and Zhang [9,10,25]):

$$\left[\frac{\delta(x)\delta(y)}{t} \wedge 1\right] \frac{1}{c_1 t^{N/2}} e^{-|x-y|^2/(c_2 t)} \le p_\Omega(t,x,y) \le \left[\frac{\delta(x)\delta(y)}{t} \wedge 1\right] \frac{c_1}{t^{N/2}} e^{-c_2|x-y|^2/t}, \tag{27}$$

where c_1, c_2 are constants depending on Ω, N only. So,

$$\frac{|z-x|^{2s}}{\delta(z)} p_\Omega(|z-x|^2\tau,x,z)\tau^{s-1} \le C|z-x|^{2s-N-2}\delta(x)\tau^{s-2-N/2}e^{-c_2/\tau} \tag{28}$$

and the reverse inequality

$$\frac{1}{C}|z-x|^{2s-N-2}\delta(x)\tau^{s-2-N/2}e^{-1/(c_2\tau)} \le \frac{|z-x|^{2s}}{\delta(z)} p_\Omega(|z-x|^2\tau,x,z)\tau^{s-1}$$

also holds for $\tau \ge \delta(x)\delta(z)|z-x|^{-2}$. As $z \to y \in \partial\Omega$, the right-hand-side of (28) obviously converges in $L^1(0,+\infty,d\tau)$ so we may apply the generalized dominated convergence theorem to deduce that $P^s_\Omega(x,y)$ is well-defined, satisfies (25) and

$$P^s_\Omega(x,y) = -\frac{\partial}{\partial\nu_y}G^s_\Omega(x,y) = \lim_{z\to y}\frac{G^s_\Omega(x,z)}{\delta(z)} = -\frac{1}{\Gamma(s)}\int_0^\infty \frac{\partial}{\partial\nu_y}p_\Omega(t,x,y)\,t^{s-1}\,dt.$$

From this last formula we deduce also that, for any fixed $x \in \Omega$, the function $P^s_\Omega(x,\cdot) \in C(\partial\Omega)$: indeed, having chosen a sequence $\{y_k\}_{k\in\mathbb{N}} \subset \partial\Omega$ converging to some $y \in \partial\Omega$, we have

$$\left|P^s_\Omega(x,y_k) - P^s_\Omega(x,y)\right| \le \frac{1}{\Gamma(s)}\int_0^\infty \left|\frac{\partial}{\partial\nu_y}p_\Omega(t,x,y_k) - \frac{\partial}{\partial\nu_y}p_\Omega(t,x,y)\right| t^{s-1}\,dt$$

where, by (27)

$$\left|\frac{\partial}{\partial\nu_y}p_\Omega(t,x,y)\right| \le \frac{c_1\,\delta(x)}{t^{N/2+1}}\,e^{-c_2|x-y|^2/t} \le \frac{c_1\,\delta(x)}{t^{N/2+1}}\,e^{-c_2\delta(x)^2/t} \qquad \text{for any } y\in\partial\Omega,$$

so that $\left|P^s_\Omega(x,y_k) - P^s_\Omega(x,y)\right| \to 0$ as $k\uparrow\infty$ by dominated convergence.

By similar arguments, G^s_Ω is a continuous function on $\overline{\Omega}^2 \setminus \{(x,y) : x = y\}$. And so, by (23), we have

$$-\frac{\partial}{\partial\nu_y}\int_\Omega G^{1-s}_\Omega(x,\xi)G^s_\Omega(\xi,y)\,d\xi = P^1_\Omega(x,y).$$

Let us compute the derivative of the left-hand side alternatively. We have

$$\int_\Omega G^{1-s}_\Omega(x,\xi)\frac{G^s_\Omega(\xi,z)}{\delta(z)}\,d\xi = \int_{\mathbb{R}^+\times\Omega} f(t,\xi,z)\,dtd\xi,$$

where, having fixed $x \in \Omega$,

$$f(t,\xi,z) = \frac{G^{1-s}_\Omega(x,\xi)}{\Gamma(s)\,\delta(z)}p_\Omega(t,\xi,z)t^{s-1} \le C|x-\xi|^{2s-N}\left[\frac{\delta(x)\delta(\xi)}{|x-\xi|^2}\wedge 1\right]t^{s-2-N/2}\delta(\xi)e^{-c_2|z-\xi|^2/t}$$

For fixed $\varepsilon > 0$, and $\xi \in \Omega\setminus B(y,\varepsilon)$, $z \in B(y,\varepsilon/2)$, we deduce that

$$f(t,\xi,z) \le C|x-\xi|^{2s-N} t^{s-2-N/2} e^{-c_\varepsilon/t} \in L^1((0,+\infty)\times\Omega).$$

Similarly, if $t > \varepsilon$,

$$f(t,\xi,z) \le C|x-\xi|^{2s-N} t^{s-2-N/2} \in L^1((\varepsilon,+\infty)\times\Omega).$$

Now,

$$\int_0^\varepsilon t^{s-1-N-2} e^{-c_2\frac{|\xi-z|^2}{t}} dt \le |\xi-z|^{2s-N} \int_0^{+\infty} \tau^{s-1-N/2} e^{-c_2/\tau} d\tau.$$

Hence, there exists a constant $C > 0$ independent of ε such that

$$\int_{(0,\varepsilon)\times B(y,\varepsilon)} f(t,\xi,z)dt\,d\xi \le C\varepsilon^{2+2s}.$$

It follows from the above estimates and dominated convergence that

$$P_\Omega^1(x,y) = \lim_{z\to y} \int_{(0,+\infty)\times\Omega} f(t,\xi,z)\,dtd\xi = \int_\Omega G_\Omega^{1-s}(x,\xi) P_\Omega^s(\xi,y)d\xi \tag{29}$$

i.e. (26) holds. □

Remark 16. Thanks to the heat kernel bound (27), the following estimate also holds:

$$\frac{1}{C}\frac{1}{|x-y|^{N-2s}}\left(1\wedge\frac{\delta(x)\delta(y)}{|x-y|^2}\right) \le G_\Omega^s(x,y) \le \frac{C}{|x-y|^{N-2s}}\left(1\wedge\frac{\delta(x)\delta(y)}{|x-y|^2}\right) \tag{30}$$

for some constant $C = C(\Omega, N, s)$. Also observe for computational convenience that

$$\frac{1}{2}\left(1\wedge\frac{\delta(x)\delta(y)}{|x-y|^2}\right) \le \frac{\delta(x)\delta(y)}{\delta(x)\delta(y)+|x-y|^2} \le \left(1\wedge\frac{\delta(x)\delta(y)}{|x-y|^2}\right).$$

3. Harmonic functions and interior regularity

Definition 17. A function $h \in L^1(\Omega, \delta(x)dx)$ is s-harmonic in Ω if for any $\psi \in C_c^\infty(\Omega)$ there holds

$$\int_\Omega h\,(-\triangle|_\Omega)^s\psi \;=\; 0.$$

The above definition makes sense thanks to the following lemma.

Lemma 18. *For any $\psi \in C_c^\infty(\Omega)$, $(-\triangle|_\Omega)^s\psi \in C_0^1(\overline{\Omega})$ and there exists a constant $C = C(s, N, \Omega, \psi) > 0$ such that*

$$|(-\triangle|_\Omega)^s\psi| \;\le\; C\delta \qquad in\ \Omega. \tag{31}$$

In addition, if $\psi \ge 0$, $\psi \not\equiv 0$, then

$$(-\triangle|_\Omega)^s\psi \;\le\; -C\delta \qquad in\ \Omega\setminus\operatorname{supp}\psi. \tag{32}$$

Proof. Thanks to (21) and (22), $(-\triangle|_\Omega)^s\psi \in C_0^1(\overline{\Omega})$ and

$$\left|\frac{(-\triangle|_\Omega)^s\psi}{\delta}\right| \le \sum_{j=1}^\infty \lambda_j^s|\widehat{\psi}_j|\left\|\frac{\varphi_j}{\delta}\right\|_{L^\infty(\Omega)} < \infty$$

and (31) follows. Let us turn to the case where $\psi \ge 0$, $\psi \not\equiv 0$. By the heat kernel bound (27), there exists $C = C(\Omega, N, s) > 0$ such that

$$\frac{1}{C|x-y|^{N+2s}}\left[\frac{\delta(x)\delta(y)}{|x-y|^2}\wedge 1\right]\le J(x,y)\le \frac{C}{|x-y|^{N+2s}}\left[\frac{\delta(x)\delta(y)}{|x-y|^2}\wedge 1\right]. \tag{33}$$

Now, we apply formula (3) and assume that $x\in\Omega\setminus\operatorname{supp}\psi$. Denote by x^* a point of maximum of ψ and let $2r=\operatorname{dist}(x^*,\operatorname{supp}\psi)$. Then for $y\in B_r(x^*)$, it holds $\psi(y),\delta(y)\ge c_1>0$, $|x-y|\le c_2$ and so

$$\begin{aligned}(-\triangle|_\Omega)^s\psi(x) &= -\int_\Omega \psi(y)\,J(x,y)\,dy\\ &\le -C\int_{B_r(x^*)}\frac{\psi(y)}{|x-y|^{N+2s}}\left[\frac{\delta(x)\delta(y)}{|x-y|^2}\wedge 1\right]dy\\ &\le -C\,\delta(x). \qquad \square\end{aligned}$$

Lemma 19. *The function $P^s_\Omega(\cdot,z)\in L^1(\Omega,\delta(x)dx)$ is s-harmonic in Ω for any fixed $z\in\partial\Omega$.*

Proof. Thanks to (25), $P^s_\Omega(\cdot,z)\in L^1(\Omega,\delta(x)dx)$. Pick $\psi\in C^\infty_c(\Omega)$ and exploit (24):

$$\int_\Omega P^s_\Omega(\cdot,z)\,(-\triangle|_\Omega)^s\psi=\int_\Omega P^s_\Omega(\cdot,z)\,(-\triangle|_\Omega)^{s-1}[-\triangle\psi].$$

Applying Lemma 11, the Fubini's Theorem and (26), the above quantity is equal to

$$\int_\Omega P^1_\Omega(\cdot,z)\,(-\triangle)\psi=0. \qquad \square$$

Lemma 20. *For any finite Radon measure $\zeta\in\mathcal{M}(\partial\Omega)$, let*

$$h(x)\ =\int_{\partial\Omega}P^s_\Omega(x,z)\,d\zeta(z),\qquad x\in\Omega. \tag{34}$$

Then, h is s-harmonic in Ω. In addition, there exists a constant $C=C(N,s,\Omega)>0$ such that

$$\|h\|_{L^1(\Omega,\delta(x)dx)}\le C\|\zeta\|_{\mathcal{M}(\partial\Omega)}. \tag{35}$$

Conversely, for any s-harmonic function $h\in L^1(\Omega,\delta(x)dx)$, $h\ge 0$, there exists a finite Radon measure $\zeta\in\mathcal{M}(\partial\Omega)$, $\zeta\ge 0$, such that (34) holds.

Proof. Since $P(x,\cdot)$ is continuous, h is well-defined. By (25),

$$\delta(x)|h(x)|\le C\int_{\partial\Omega}\frac{d|\zeta|(z)}{|x-z|^{N-2s}}$$

so that $h\in L^1(\Omega,\delta(x)dx)$ and (35) holds. Pick now $\psi\in C^\infty_c(\Omega)$:

$$\int_\Omega h(x)\,(-\triangle|_\Omega)^s\psi(x)\,dx=\int_{\partial\Omega}\left(\int_\Omega P^s_\Omega(x,z)\,(-\triangle|_\Omega)^s\psi(x)\,dx\right)d\zeta(z)=0$$

in view of Lemma 19. Conversely, let h denote a nonnegative s-harmonic function. By Definition 17 and by equation (24), we have for any $\psi\in C^\infty_c(\Omega)$

$$\begin{aligned}0=\int_\Omega h(x)(-\triangle|_\Omega)^s\psi(x)\,dx&=\int_\Omega h(x)\,(-\triangle|_\Omega)^{s-1}\circ(-\triangle)\,\psi(x)\,dx\ =\\ &=\int_\Omega\left(\int_\Omega G^{1-s}_\Omega(x,\xi)h(\xi)\,d\xi\right)(-\triangle\psi(x)\,dx,\end{aligned}$$

so that $\int_\Omega G_\Omega^{1-s}(x,\xi)h(\xi)d\xi$ is a (standard) nonnegative harmonic function. In particular (cf. e.g. [3, Corollary 6.15]), there exists a finite Radon measure $\zeta \in \mathcal{M}(\partial\Omega)$ such that $\int_\Omega G_\Omega^{1-s}(x,\xi)h(\xi)d\xi = \int_{\partial\Omega} P_\Omega^1(x,y)d\zeta(y)$. We now exploit equation (26) to deduce that

$$\int_\Omega G_\Omega^{1-s}(x,\xi)\left[h(\xi) - \int_{\partial\Omega} P_\Omega^s(\xi,y)d\zeta(y)\right]d\xi = 0.$$

Since

$$\int_\Omega \varphi_1(x)\left(\int_\Omega G_\Omega^{1-s}(x,\xi)h(\xi)\,d\xi\right)dx = \int_\Omega h\,(-\triangle|_\Omega)^{s-1}\varphi_1 = \frac{1}{\lambda_1^{1-s}}\int_\Omega h\,\varphi_1 < \infty,$$

it holds $\int_\Omega G_\Omega^{1-s}(x,\xi)h(\xi)d\xi \in C^\infty(\Omega)\cap L^1(\Omega,\delta(x)dx)$. Thanks to (30), we are allowed to let G_Ω^s act on it. By (23), this leads to

$$\int_\Omega G_\Omega^1(x,\xi)\left[h(\xi) - \int_{\partial\Omega} P_\Omega^s(\xi,y)\,d\zeta(y)\right]d\xi = 0.$$

Take at last $\psi \in C_c^\infty(\Omega)$ and $\varphi = (-\triangle|_\Omega)^{-1}\psi$. Then,

$$0 = \int_\Omega \varphi(x)\left[\int_\Omega G_\Omega^1(x,\xi)\left[h(\xi) - \int_{\partial\Omega} P_\Omega^s(\xi,y)d\zeta(y)\right]d\xi\right]dx = \int_\Omega \psi(\xi)\left[h(\xi) - \int_{\partial\Omega} P_\Omega^s(\xi,y)\,d\zeta(y)\right]d\xi$$

and so (34) holds a.e. and in fact everywhere thanks to Lemma 21 below. □

We turn now to the proof of some interior regularity estimates which will be later used to establish Lemma 5 on the relationship between classical and weak solutions. Similar and related results are contained in Mou and Yi [20], but for a different operator—the regional fractional Laplacian. As in [20, Theorems B, C and D], the idea here is to reduce the analysis to the fractional Laplacian case, as treated by Silvestre [22].

Lemma 21. *Take $\alpha > 0$ such that $2s+\alpha \notin \mathbb{N}$ and $f \in C^\alpha_{loc}(\Omega)$. If $u \in L^1(\Omega,\delta(x)dx)$ solves*

$$(-\triangle|_\Omega)^s u = f \qquad \text{in } \mathcal{D}'(\Omega),$$

then $u \in C^{2s+\alpha}_{loc}(\Omega)$, the above equation holds pointwise, and given any compact sets $K \subset\subset K' \subset\subset \Omega$, there exists a constant $C = C(s,N,\alpha,K,K',\Omega)$ such that

$$\|u\|_{C^{2s+\alpha}(K)} \le C\left(\|f\|_{C^\alpha(K')} + \|u\|_{L^1(\Omega,\delta(x)dx)}\right).$$

Similarly, if $f \in L^\infty_{loc}(\Omega)$ and $\alpha \in (0,2s)$,

$$\|u\|_{C^{\alpha}(K)} \le C\left(\|f\|_{L^\infty(K')} + \|u\|_{L^1(\Omega,\delta(x)dx)}\right).$$

In particular, if h is s-harmonic, then $h \in C^\infty(\Omega)$ and the equality $(-\triangle|_\Omega)^s h(x) = 0$ holds at every point $x \in \Omega$.

Proof. We only prove the former inequality, the proof of the latter follows mutatis mutandis. Given $x \in \Omega$, let

$$v(x) = \int_\Omega G_\Omega^{1-s}(x,y)u(y)\,dy.$$

Observe that v is well-defined and

$$\|v\|_{L^1(\Omega,\delta(x)dx)} \le C(\Omega,N,s)\|u\|_{L^1(\Omega,\delta(x)dx)}. \tag{36}$$

Indeed, letting $\varphi_1 > 0$ denote an eigenvector associated to the principal eigenvalue of the Laplace operator, it follows from the Fubini's theorem and Lemma 11 that

$$\int_\Omega \varphi_1(x) \int_\Omega G_\Omega^{1-s}(x,y)\,|u(y)|\,dy\,dx = \lambda_1^{s-1} \int_\Omega |u(y)|\,\varphi_1(y)\,dy.$$

In addition, $-\triangle v = f$ in $\mathcal{D}'(\Omega)$, since for $\varphi \in C_c^\infty(\Omega)$,

$$\int_\Omega v\,(-\triangle)\varphi = \int_\Omega u\,(-\triangle|_\Omega)^s \varphi,$$

thanks to the Fubini's theorem, equation (24), Lemma 11 and Definition 17. Observe now that if $\varphi \in C_c^\infty(\Omega)$, then

$$(-\triangle|_\Omega)^{1-s}\varphi = \frac{s}{\Gamma(1-s)} \int_0^{+\infty} t^{s-1} \left(\frac{\varphi - e^{-t\triangle|_\Omega}\varphi}{t} \right) dt.$$

The above identity is straightforward if φ is an eigenfunction and remains true for $\varphi \in C_c^\infty(\Omega)$ by density, using the fast decay of spectral coefficients, see (22). So,

$$\int_\Omega u\varphi\,dx = \int_\Omega v(-\triangle|_\Omega)^{1-s}\varphi\,dx =$$

$$= \frac{s}{\Gamma(1-s)} \int_\Omega \int_0^\infty v\,t^{s-1} \left(\frac{\varphi - e^{-t\triangle|_\Omega}\varphi}{t} \right) dt\,dx =$$

$$= \frac{s}{\Gamma(1-s)} \int_\Omega \int_0^\infty \varphi t^{s-1} \left(\frac{v - e^{-t\triangle|_\Omega}v}{t} \right) dt\,dx$$

and

$$u = \frac{s}{\Gamma(1-s)} \int_0^{+\infty} t^{s-1} \left(\frac{v - e^{-t\triangle|_\Omega}v}{t} \right) dt.$$

Choose $\overline{f} \in C_c^\alpha(\mathbb{R}^N)$ such that $\overline{f} = f$ in K', $\|\overline{f}\|_{C^\alpha(\mathbb{R}^N)} \le C\|f\|_{C^\alpha(K')}$ and let

$$\overline{u}(x) = c_{N,s} \int_{\mathbb{R}^N} |x-y|^{-(N-2s)}\overline{f}(y)\,dy$$

solve $(-\triangle)^s\overline{u} = \overline{f}$ in $\mathbb{R}^N$. It is well-known (see e.g. [22]) that $\|\overline{u}\|_{C^{2s+\alpha}(\mathbb{R}^N)} \le C\|\overline{f}\|_{C^\alpha(\mathbb{R}^N)}$, for a constant C depending only on s, α, N and the measure of the support of $\overline{f}$. It remains to estimate $u - \overline{u}$. Letting

$$\overline{v}(x) = c_{N,1-s} \int_{\mathbb{R}^N} |x-y|^{-(N-2(1-s))}\overline{u}(y)\,dy,$$

we have as previously that $-\triangle\overline{v} = \overline{f}$ and

$$\overline{u} = \frac{s}{\Gamma(1-s)} \int_0^{+\infty} t^{s-1} \left(\frac{\overline{v} - e^{-t\triangle}\overline{v}}{t} \right) dt,$$

where this time $e^{-t\triangle}\overline{v}(x) = \frac{1}{(4\pi t)^{N/2}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}}\overline{v}(y)\,dy$. Hence,

$$\frac{\Gamma(1-s)}{s}(u - \overline{u}) = \int_0^{+\infty} t^{s-1} \left(\frac{(v-\overline{v}) - e^{-t\triangle}(v-\overline{v})}{t} \right) dt + \int_0^{+\infty} t^{s-1} \left(\frac{e^{-t\triangle|_\Omega}v - e^{-t\triangle}\overline{v}}{t} \right) dt.$$

Fix a compact set K'' such that $K \subset\subset K'' \subset\subset K'$. Since $v - \overline{v}$ is harmonic in K',

$$\|v-\overline{v}\|_{C^{2s+\alpha+2}(K'')} \le C\|v-\overline{v}\|_{L^1(K')} \le C\|u\|_{L^1(\Omega,\delta(x)dx)}.$$

By parabolic regularity,

$$\left\| \int_0^{+\infty} t^{s-1}\left(\frac{(v-\overline{v})-e^{-t\triangle}(v-\overline{v})}{t}\right)dt \right\|_{C^{2s+\alpha}(K)} \le C\|u\|_{L^1(\Omega,\delta(x)dx)}.$$

In addition, the function $w = e^{-t\triangle|_\Omega}v - e^{-t\triangle}\overline{v}$ solves the heat equation inside Ω with initial condition $w(0,\cdot) = v-\overline{v}$. Since $\triangle(v-\overline{v}) = 0$ in K', it follows from parabolic regularity again that $w(t,x)/t$ remains bounded in $C^{2s+\alpha}(K)$ as $t \to 0^+$, so that again

$$\left\| \int_0^{+\infty} t^{s-1}\left(\frac{e^{-t\triangle|_\Omega}v - e^{-t\triangle}\overline{v}}{t}\right)dt \right\|_{C^{2s+\alpha}(K)} \le C\|u\|_{L^1(\Omega,\delta(x)dx)}. \qquad \square$$

Lemma 22. *Take $\alpha > 0$, $\alpha \notin \mathbb{N}$, and $u \in C^{2s+\alpha}_{loc}(\Omega) \cap L^1(\Omega, \delta(x)dx)$. Given any compact set $K \subset\subset K' \subset\subset \Omega$, there exists a constant $C = C(s, N, \alpha, K, K', \Omega)$ such that*

$$\|(-\triangle|_\Omega)^s u\|_{C^\alpha(K)} \le C\left(\|u\|_{C^{2s+\alpha}(K')} + \|u\|_{L^1(\Omega,\delta(x)dx)}\right).$$

Proof. With a slight abuse of notation, we write

$$(-\triangle|_\Omega)^{s-1}u(x) = \int_\Omega G_\Omega^{1-s}(x,y)\,u(y)\,dy.$$

By Lemma 21 we have

$$\begin{aligned}\|(-\triangle|_\Omega)^{s-1}u\|_{C^{2+\alpha}(K)} &\le C\left(\|u\|_{C^{2s+\alpha}(K')} + \|(-\triangle|_\Omega)^{s-1}u\|_{L^1(\Omega,\delta(x)dx)}\right) \le \\ &\le C\left(\|u\|_{C^{2s+\alpha}(K')} + \|u\|_{L^1(\Omega,\delta(x)dx)}\right).\end{aligned}$$

Obviously it holds also

$$\|(-\triangle)\circ(-\triangle|_\Omega)^{s-1}u\|_{C^\alpha(K)} \le \|(-\triangle|_\Omega)^{s-1}u\|_{C^{2+\alpha}(K)}.$$

By (24),

$$(-\triangle)\circ(-\triangle|_\Omega)^{s-1}u = (-\triangle|_\Omega)^s u \qquad \text{in } \mathcal{D}'(\Omega),$$

which concludes the proof. $\square$

Proposition 23. *Let $f \in L^1(\Omega, \delta(x)dx)$ and $u \in L^1_{loc}(\Omega)$. The function*

$$u(x) = \int_\Omega G_\Omega^s(x,y)\,f(y)\,dy$$

belongs to $L^p(\Omega, \delta(x)dx)$ for any $p \in \left[1, \dfrac{N+1}{N+1-2s}\right)$.

Proof. We start by applying the Jensen's Inequality,

$$\left|\int_\Omega G_\Omega^s(x,y)\,f(y)\,dy\right|^p \le \|f\|^{p-1}_{L^1(\Omega,\delta(x)dx)} \int_\Omega \left|\frac{G_\Omega^s(x,y)}{\delta(y)}\right|^p \delta(y)\,f(y)\,dy,$$

so that

$$\int_\Omega |u(x)|^p\,\delta(x)\,dx \le \|f\|^p_{L^1(\Omega,\delta(x)dx)} \sup_{y\in\Omega}\int_\Omega \left|\frac{G^s_\Omega(x,y)}{\delta(y)}\right|^p \delta(x)\,dx$$

and by (30) we have to estimate

$$\sup_{y\in\Omega}\int_\Omega \frac{1}{|x-y|^{(N-2s)p}}\cdot\frac{\delta(x)^{p+1}}{\left[|x-y|^2+\delta(x)\delta(y)\right]^p}\,dx$$

Pick $\varepsilon>0$. Clearly,

$$\sup_{\{y\,:\,\delta(y)\ge\varepsilon\}}\int_\Omega \frac{1}{|x-y|^{(N-2s)p}}\cdot\frac{\delta(x)^{p+1}}{\left[|x-y|^2+\delta(x)\delta(y)\right]^p}\,dx \le C_\varepsilon.$$

Thanks to Lemma 40, we may now reduce to the case where the boundary is flat, i.e. when in a neighborhood A of a given point $y\in\Omega$ such that $\delta(y)<\varepsilon$, there holds $A\cap\partial\Omega\subseteq\{y_N=0\}$ and $A\cap\Omega\subseteq\{y_N>0\}$. Without loss of generality, we assume that $y=(0,y_N)$ and $x=(x',x_N)\in B\times(0,1)\subseteq\mathbb{R}^{N-1}\times\mathbb{R}$. We are left with proving that

$$\int_B dx'\int_0^1 dx_N\,\frac{1}{\left[|x'|^2+|x_N-y_N|^2\right]^{(N-2s)p/2}}\cdot\frac{x_N^{p+1}}{\left[|x'|^2+|x_N-y_N|^2+x_Ny_N\right]^p}$$

is a bounded quantity. Make the change of variables $x_N=y_Nt$ and pass to polar coordinates in x', with $|x'|=y_N\rho$. Then, the above integral becomes

$$y_N^{-(N+1-2s)p+N+1}\int_0^{1/y_N} d\rho\int_0^{1/y_N} dt\,\frac{\rho^{N-2}}{\left[\rho^2+|t-1|^2\right]^{(N-2s)p/2}}\cdot\frac{t^{p+1}}{\left[\rho^2+|t-1|^2+t\right]^p}. \tag{37}$$

Now, we split the integral in the t variable into $\int_0^{1/2}+\int_{1/2}^{3/2}+\int_{3/2}^{1/y_N}$. Note that the exponent $-(N+1-2s)p+N+1$ is positive for $p<(N+1)/(N+1-2s)$. We drop multiplicative constants in the computations that follow. The first integral is bounded above by a constant multiple of

$$\int_0^{1/y_N} d\rho\int_0^{1/2} dt\,\frac{\rho^{N-2}}{\left[\rho^2+1\right]^{(N-2s)p/2}}\cdot\frac{t^{p+1}}{\left[\rho^2+1+t\right]^p}\lesssim\int_0^{1/y_N} d\rho\,\frac{\rho^{N-2}}{\left[\rho^2+1\right]^{(N+2-2s)p/2}}$$

which remains bounded as $y_N\downarrow 0$ since

$$p\ge 1>\frac{N-1}{N+2-2s}\qquad\text{implies}\qquad (N+2-2s)p-N+2>1.$$

The second integral is of the order of

$$\int_0^{1/y_N} d\rho\int_{1/2}^{3/2} dt\,\frac{\rho^{N-2}}{\left[\rho^2+|t-1|^2\right]^{(N-2s)p/2}}\cdot\frac{1}{\left[\rho^2+1\right]^p}=$$

$$=\int_0^{1/y_N} d\rho\int_0^{1/2} dt\,\frac{\rho^{N-2}}{\left[\rho^2+t^2\right]^{(N-2s)p/2}}\cdot\frac{1}{\left[\rho^2+1\right]^p}$$

$$=\int_0^{1/y_N} d\rho\,\frac{\rho^{N-1-(N-2s)p}}{\left[\rho^2+1\right]^p}\int_0^{1/(2\rho)} dt\,\frac{1}{\left[1+t^2\right]^{(N-2s)p/2}}$$

$$\lesssim\int_0^\infty d\rho\,\frac{\rho^{N-1-(N-2s)p}}{\left[\rho^2+1\right]^p}$$

which is finite since $p < (N+1)/(N+1-2s) < N/(N-2s)$ implies $N-1-(N-2s)p > -1$ and $p \geq 1 > N/(N+2-2s)$ implies $2p - N + 1 + (N-2s)p > 1$.

We are left with the third integral which is controlled by

$$\int_0^{1/y_N} d\rho\, \rho^{N-2} \int_{3/2}^{1/y_N} dt\, \frac{t^{p+1}}{\left[\rho^2+t^2\right]^{(N+2-2s)p/2}} \leq$$

$$\leq \int_0^{1/y_N} d\rho\, \rho^{N-(N+1-2s)p} \int_{3/(2\rho)}^{1/(y_N\rho)} dt\, \frac{t^{p+1}}{\left[1+t^2\right]^{(N+2-2s)p/2}}$$

$$\lesssim \int_0^{1/y_N} d\rho\, \rho^{N-(N+1-2s)p}.$$

The exponent $N-(N+1-2s)p > -1$ since $p < (N+1)/(N+1-2s)$, so this third integral is bounded above by a constant multiple of $y_N^{-1-N+(N+1-2s)p}$ which simplifies with the factor in front of (37). □

4. Boundary behavior

We first provide the boundary behavior of the reference function h_1. Afterwards, in Proposition 25 below, we will deal with the weighted trace left on the boundary by harmonic functions induced by continuous boundary data.

Lemma 24. *Let h_1 be given by (8). There exists a constant $C = C(N,\Omega,s) > 0$ such that*

$$\frac{1}{C}\delta^{-(2-2s)} \leq h_1 \leq C\delta^{-(2-2s)}. \tag{38}$$

Proof. Restrict without loss of generality to the case where x lies in a neighborhood of $\partial\Omega$. Take $x^* \in \partial\Omega$ such that $|x-x^*| = \delta(x)$, which exists by compactness of $\partial\Omega$. Take $\Gamma \subset \partial\Omega$ a neighborhood of x^* in the topology of $\partial\Omega$. By Lemma 40 in the Appendix, we can think of $\Gamma \subset \{x_N = 0\}$, $x^* = 0$ and $x = (0,\delta(x)) \in \mathbb{R}^{N-1}\times\mathbb{R}$ without loss of generality. in such a way that it is possible to compute

$$\int_\Gamma \frac{\delta(x)}{|x-z|^{N+2-2s}}\, d\sigma(z) \asymp \int_{B_r} \frac{\delta(x)}{[|z'|^2+\delta(x)^2]^{N/2+1-s}}\, dz'.$$

Recalling (25), we have reduced the estimate to

$$\int_{\partial\Omega} P_\Omega^s(x,z)\, d\sigma(z) \asymp$$

$$\asymp \int_{B_r} \frac{\delta(x)}{[|z'|^2+\delta(x)^2]^{N/2+1-s}}\, dz' \asymp \int_0^r \frac{\delta(x)\, t^{N-2}}{[t^2+\delta(x)^2]^{N/2+1-s}}\, dt$$

$$= \int_0^{r/\delta(x)} \frac{\delta(x)^N t^{N-2}}{[\delta(x)^2 t^2+\delta(x)^2]^{N/2+1-s}}\, dt \asymp \delta(x)^{2s-2} \int_0^{r/\delta(x)} \frac{t^{N-2}\, dt}{[t^2+1]^{N/2+1-s}}$$

and this concludes the proof, since

$$\int_0^{r/\delta(x)} \frac{t^{N-2}\, dt}{[t^2+1]^{N/2+1-s}} \asymp 1. \qquad \square$$

In the following we will use the notation

$$\mathbb{P}^s_\Omega g := \int_{\partial\Omega} P^s_\Omega(\cdot,\theta)\, g(\theta)\, d\sigma(\theta)$$

where σ denotes the Hausdorff measure on $\partial\Omega$, whenever $g \in L^1(\Omega)$.

Proposition 25. *Let $\zeta \in C(\partial\Omega)$. Then, for any $z \in \partial\Omega$,*

$$\frac{\mathbb{P}^s_\Omega \zeta(x)}{h_1(x)} \xrightarrow[x\to z]{x\in\Omega} \zeta(z) \qquad \textit{uniformly on } \partial\Omega. \tag{39}$$

Proof. Let us write

$$\begin{aligned}\left|\frac{\mathbb{P}^s_\Omega \zeta(x)}{h_1(x)} - \zeta(z)\right| &= \left|\frac{1}{h_1(x)}\int_{\partial\Omega} P^s_\Omega(x,\theta)\,\zeta(\theta)\,d\sigma(\theta) - \frac{h_1(x)\,\zeta(z)}{h_1(x)}\right| \leq \\ &\leq \frac{1}{h_1(x)}\int_{\partial\Omega} P^s_\Omega(x,\theta)|\zeta(\theta)-\zeta(z)|\,d\sigma(\theta) \leq C\delta(x)^{3-2s}\int_{\partial\Omega}\frac{|\zeta(\theta)-\zeta(z)|}{|x-\theta|^{N+2-2s}}\,d\sigma(\theta) \leq \\ &\leq C\delta(x)\int_{\partial\Omega}\frac{|\zeta(\theta)-\zeta(z)|}{|x-\theta|^N}\,d\sigma(\theta).\end{aligned}$$

It suffices now to repeat the computations in [1, Lemma 3.1.5] to show that the obtained quantity converges to 0 as $x \to z$. □

With an approximation argument started from the last Proposition, we can deal with a $\zeta \in L^1(\partial\Omega)$ datum.

Theorem 26. *For any $\zeta \in L^1(\partial\Omega)$ and any $\phi \in C^0(\overline{\Omega})$ it holds*

$$\frac{1}{t}\int_{\{\delta(x)\leq t\}} \frac{\mathbb{P}^s_\Omega\zeta(x)}{h_1(x)}\,\phi(x)\,dx \xrightarrow[t\downarrow 0]{} \int_{\partial\Omega}\phi(y)\,\zeta(y)\,d\sigma(y).$$

Proof. For a general $\zeta \in L^1(\partial\Omega)$, consider a sequence $\{\zeta_k\}_{k\in\mathbb{N}} \subset C(\partial\Omega)$ such that

$$\int_{\partial\Omega}|\zeta_k(y)-\zeta(y)|\,d\sigma(y) \xrightarrow[k\uparrow\infty]{} 0. \tag{40}$$

For any fixed $k \in \mathbb{N}$, we have

$$\left|\frac{1}{t}\int_{\{\delta(x)<t\}}\frac{\mathbb{P}^s_\Omega\zeta(x)}{h_1(x)}\,\phi(x)\,dx - \int_{\partial\Omega}\phi(x)\,\zeta(x)\,d\sigma(x)\right| \leq$$

$$\left|\frac{1}{t}\int_{\{\delta(x)<t\}}\frac{\mathbb{P}^s_\Omega\zeta(x)-\mathbb{P}^s_\Omega\zeta_k(x)}{h_1(x)}\,\phi(x)\,dx\right| \tag{41}$$

$$+\left|\frac{1}{t}\int_{\{\delta(x)<t\}}\frac{\mathbb{P}^s_\Omega\zeta_k(x)}{h_1(x)}\,\phi(x)\,dx - \int_{\partial\Omega}\phi(x)\,\zeta_k(x)\,d\sigma(x)\right| \tag{42}$$

$$+\left|\int_{\partial\Omega}\phi(x)\,\zeta_k(x)\,d\sigma(x) - \int_{\partial\Omega}\phi(x)\,\zeta(x)\,d\sigma(x)\right|. \tag{43}$$

Call $\lambda_k := \zeta_k - \zeta$: the term (41) equals

$$\frac{1}{t}\int_{\{\delta(x)<t\}} \frac{\mathbb{P}^s_\Omega \lambda_k(x)}{h_1(x)}\,\phi(x)\,d\sigma(x) = \int_{\partial\Omega}\left(\frac{1}{t}\int_{\{\delta(x)<t\}} \frac{P^s_\Omega(x,y)}{h_1(x)}\,\phi(x)\,dx\right)\lambda_k(y)\,d\sigma(y)$$

Call

$$\Phi(t,y) := \frac{1}{t}\int_{\{\delta(x)<t\}} \frac{P^s_\Omega(x,y)}{h_1(x)}\,\phi(x)\,dx.$$

Combining equations (25), (38) and the boundedness of ϕ, we can prove that Φ is uniformly bounded in t and y. Indeed,

$$|\Phi(t,y)| \le \frac{\|\phi\|_{L^\infty(\Omega)}}{t}\int_{\{\delta(x)<t\}} \frac{\delta(x)^{3-2s}}{|x-y|^{N+2-2s}}\,dx \le \frac{\|\phi\|_{L^\infty(\Omega)}}{t}\int_{\{\delta(x)<t\}} \frac{\delta(x)}{|x-y|^{N}}\,dx$$

and reducing our attention to the flat case (see Lemma 40 in the Appendix for the complete justification) we estimate (the ' superscript denotes an object living in $\mathbb{R}^{N-1}$)

$$\frac{1}{t}\int_0^t\int_{B'} \frac{x_N}{\left[|x'|^2+x_N^2\right]^{N/2}}\,dx'\,dx_N \le \frac{1}{t}\int_0^t\int_{B'_{1/x_N}} \frac{d\xi}{\left[|\xi|^2+1\right]^{N/2}}\,dx_N \le \int_{\mathbb{R}^{N-1}} \frac{d\xi}{\left[|\xi|^2+1\right]^{N/2}}.$$

Thus $\int_{\partial\Omega}\Phi(t,y)\lambda_k(y)d\sigma(y)$ is arbitrarily small in k in view of (40).

The term (42) converges to 0 as $t\downarrow 0$ because the convergence

$$\frac{\mathbb{P}^s_\Omega \zeta_k(x)}{h_1(x)}\,\phi(x) \xrightarrow[x\to z]{x\in\Omega} \zeta_k(z)\,\phi(z)$$

is uniform in $z\in\partial\Omega$ in view of Proposition 25.

Finally, the term (43) is arbitrarily small with $k\uparrow+\infty$, because of (40). This concludes the proof of the theorem, because

$$\lim_{t\downarrow 0}\left|\frac{1}{t}\int_{\{\delta(x)<t\}} \frac{\mathbb{P}^s_\Omega \zeta(x)}{h_1(x)}\,\phi(x)\,dx - \int_{\partial\Omega}\phi(x)\,\zeta(x)\,d\sigma(x)\right| \le$$

$$\le \|\Phi\|_{L^\infty((0,t_0)\times\partial\Omega)}\int_{\partial\Omega}|\zeta_k(y)-\zeta(y)|\,d\sigma(y) + \|\phi\|_{L^\infty(\partial\Omega)}\int_{\partial\Omega}|\zeta_k(y)-\zeta(y)|\,d\sigma(y)$$

and letting $k\uparrow+\infty$ we deduce the thesis as a consequence of (40). □

Moreover we have also

Theorem 27. *For any $\mu\in\mathcal{M}(\Omega)$, such that*

$$\int_\Omega \delta\,d|\mu| < \infty, \tag{44}$$

and any $\phi\in C^0(\overline{\Omega})$ it holds

$$\frac{1}{t}\int_{\{\delta(x)\le t\}} \frac{\mathbb{G}^s_\Omega \mu(x)}{h_1(x)}\,\phi(x)\,dx \xrightarrow[t\downarrow 0]{} 0. \tag{45}$$

Proof. By using the Jordan decomposition of $\mu = \mu^+ - \mu^-$ into its positive and negative part, we can suppose without loss of generality that $\mu \geq 0$. Fix some $s' \in (0, s \wedge 1/2)$. Exchanging the order of integration we claim that

$$\int_{\{\delta(x)\leq t\}} G^s_\Omega(x,y)\,\delta(x)^{2-2s}dx \leq \begin{cases} C\,t^{2-2s'}\,\delta(y)^{2s'} & \delta(y)\geq t \\ C\,t\,\delta(y) & \delta(y)<t \end{cases} \tag{46}$$

where $C = C(N,\Omega,s)$ and does not depend on t, which yields

$$\frac{1}{t}\int_\Omega \left(\int_{\{\delta(x)<t\}} G^s_\Omega(x,y)\,\frac{dx}{h_1(x)} \right) d\mu(y) \;\leq\; C\,t^{1-2s'} \int_{\{\delta(y)\geq t\}} \delta(y)^{2s'}d\mu(y) + C \int_{\{\delta(x)<t\}} \delta(y)\,d\mu(y).$$

The second addend converges to 0 as $t \downarrow 0$ by (44). Since $t^{1-2s'}\delta(y)^{2s'}$ converges pointwisely to 0 in Ω as $t \downarrow 0$ and $t^{1-2s'}\delta(y)^{2s'} \leq \delta(y)$ in $\{\delta(y)\geq t\}$, then the first addend converges to 0 by dominated convergence. This suffices to deduce our thesis (45).

Let us turn now to the proof of the claimed estimate (46). For the first part we refer to [11, Proposition 7] to say

$$\int_{\{\delta(x)\leq t\}} G^s_\Omega(x,y)\,\delta(x)^{2-2s}dx \leq t^{2-2s'} \int_{\{\delta(x)\leq t\}} G^s_\Omega(x,y)\,\delta(x)^{-2s+2s'}dx \leq C\,t^{2-2s'}\,\delta^{2s'}.$$

We focus our attention on the case where $\partial\Omega$ is locally flat, i.e. we suppose that in a neighborhood A of y it holds $A\cap\partial\Omega \subseteq \{x_N = 0\}$ (see Lemma 40 in the Appendix to reduce the general case to this one). So, since $\delta(x) = x_N$ and retrieving estimate (30) on the Green function, we are dealing with (the $'$ superscript denotes objects that live in $\mathbb{R}^{N-1}$)

$$\int_0^t \int_{B'(y')} \frac{y_N\,x_N^{3-2s}}{|x'-y'|^2+(x_N-y_N)^2+x_N\,y_N} \cdot \frac{dx'}{\left[|x'-y'|^2+(x_N-y_N)^2\right]^{(N-2s)/2}}\,dx_N.$$

From now on we drop multiplicative constants depending only on N and s. Suppose without loss of generality $y'=0$. Set $x_N = y_N\eta$ and switch to polar coordinates in the x' variable:

$$\int_0^{t/y_N}\int_0^1 \frac{y_N^{5-2s}\,\eta^{3-2s}}{r^2+y_N^2(\eta-1)^2+\eta\,y_N^2} \cdot \frac{r^{N-2}dr}{\left[r^2+y_N^2(\eta-1)^2\right]^{(N-2s)/2}}\,d\eta$$

and then set $r = y_N\rho$ to get

$$y_N^2 \int_0^{t/y_N} \eta^{3-2s} \int_0^{1/y_N} \frac{\rho^{N-2}}{\left[\rho^2+(\eta-1)^2\right]^{(N-2s)/2}} \cdot \frac{d\rho}{\rho^2+(\eta-1)^2+\eta}\,d\eta \;\leq$$

$$\leq\; y_N^2 \int_0^{t/y_N} \eta^{3-2s} \int_0^{1/y_N} \frac{\rho}{\left[\rho^2+(\eta-1)^2\right]^{(3-2s)/2}} \cdot \frac{d\rho}{\rho^2+(\eta-1)^2+\eta}\,d\eta.$$

Consider now $s\in(1/2,1)$. The integral in the ρ variable is less than

$$\int_0^{1/y_N} \frac{\rho}{\left[\rho^2+(\eta-1)^2\right]^{(5-2s)/2}}\,d\rho \leq |\eta-1|^{-3+2s}$$

so that, integrating in the η variable,

$$y_N^2 \int_0^{t/y_N} \eta^{3-2s}\,|\eta-1|^{-3+2s}\,d\eta \leq t\,y_N$$

and we prove (46) in the case $s \in (1/2, 1)$. Now we study the case $s \in (0, 1/2]$. Split the integration in the η variable into $\int_0^2$ and $\int_2^{1/y_N}$: the latter can be treated in the same way as above. For the other one we exploit the inequality $\rho^2 + (\eta-1)^2 + \eta \geq \frac{3}{4}$ to deduce[6]:

$$y_N^2 \int_0^2 \eta^{3-2s} \left(\int_0^{1/y_N} \frac{\rho}{\left[\rho^2 + (\eta-1)^2 + \eta\right]^{(3-2s)/2}} \cdot \frac{d\rho}{\rho^2 + (\eta-1)^2} \right) d\eta \leq$$

$$\leq y_N^2 \int_0^2 \eta^{3-2s} \left(\int_0^{1/y_N} \frac{\rho}{\left[\rho^2 + (\eta-1)^2\right]^{(3-2s)/2}}\, d\rho \right) d\eta \leq y_N^2 \int_0^2 \frac{\eta^{3-2s}}{|\eta-1|^{1-2s}}\, d\eta \leq y_N^2.$$

Note now that, in our set of assumptions, $y_N = \delta(y) < t$. So $y_N^2 \leq t y_N$ and we get to the desired conclusion (46) also in the case $s \in (0, 1/2]$. $\square$

5. The Dirichlet problem

Recall the definition of test functions (7).

Lemma 28. $\mathcal{T}(\Omega) \subseteq C_0^1(\overline{\Omega}) \cap C^\infty(\Omega)$. *Moreover, for any* $\psi \in \mathcal{T}(\Omega)$ *and* $z \in \partial\Omega$,

$$-\frac{\partial \psi}{\partial \nu}(z) = \int_\Omega P_\Omega^s(y,z)\,(-\triangle|_\Omega)^s \psi(y)\, dy. \tag{47}$$

Proof. Take $\psi \in \mathcal{T}(\Omega)$ and let $f = (-\triangle|_\Omega)^s \psi$. Since $f \in C_c^\infty(\Omega)$, the spectral coefficients of f have fast decay (see (22)) and so the same holds true for ψ. It follows that $\psi \in C_0^1(\overline{\Omega})$ and $\mathcal{T}(\Omega) \subseteq C_0^1(\overline{\Omega})$. By Lemma 11, for all $x \in \overline{\Omega}$,

$$\psi(x) = \int_\Omega G_\Omega^s(x,y)\, f(y)\, dy$$

Using Lemma 14, (28) and the dominated convergence theorem, (47) follows.

Since $(-\triangle|_\Omega)^s$ is self-adjoint in $H(2s)$, we know that the equality $(-\triangle|_\Omega)^s \psi = f$ holds in $\mathcal{D}'(\Omega)$ and the interior regularity follows from Lemma 21. $\square$

Lemma 29 *(Maximum principle for classical solutions). Let* $u \in C_{loc}^{2s+\varepsilon}(\Omega) \cap L^1(\Omega, \delta(x)\, dx)$ *such that*

$$(-\triangle|_\Omega)^s u \geq 0 \ \text{ in } \Omega, \qquad \liminf_{x \to \partial\Omega} u(x) \geq 0.$$

Then $u \geq 0$ *in* Ω. *In particular this holds when* $u \in \mathcal{T}(\Omega)$.

Proof. Suppose $x^* \in \Omega$ such that $u(x^*) = \min_\Omega u < 0$. Then

$$(-\triangle|_\Omega)^s u(x^*) = \int_\Omega [u(x^*) - u(y)]\, J(x,y)\, dy + \kappa(x^*) u(x^*) < 0,$$

a contradiction. $\square$

Lemma 30 *(Maximum principle for weak solutions). Let* $\mu \in \mathcal{M}(\Omega)$, $\zeta \in \mathcal{M}(\partial\Omega)$ *be two Radon measures satisfying* (9) *with* $\mu \geq 0$ *and* $\zeta \geq 0$. *Consider* $u \in L_{loc}^1(\Omega)$ *a weak solution to the Dirichlet problem* (10). *Then* $u \geq 0$ *a.e. in* Ω.

Proof. Take $f \in C_c^\infty(\Omega)$, $f \geq 0$ and $\psi = (-\triangle|_\Omega)^{-s} f \in \mathcal{T}(\Omega)$. By Lemma 29, $\psi \geq 0$ in Ω and by Lemma 28 $-\frac{\partial\psi}{\partial\nu} \geq 0$ on $\partial\Omega$. Thus, by (11), $\int_\Omega uf \geq 0$. Since this is true for every $f \in C_c^\infty(\Omega)$, the result follows. $\square$

[6] In the computation that follows, in the particular case $s = \frac{1}{2}$ the term $|1-\eta|^{2s-1}$ must be replaced by $-\ln|1-\eta|$, but this is harmless.

5.1. *Proof of Theorem 6*

Uniqueness is a direct consequence of the comparison principle, Lemma 30. Let us prove that formula (12) defines the desired weak solution. Observe that if u is given by (12), then $u \in L^1(\Omega, \delta(x)dx)$. Indeed,

$$\int_\Omega \left| \varphi_1(x) \int_\Omega G^s_\Omega(x,y)d\mu(y) \right| dx \le \int_\Omega \int_\Omega G^s_\Omega(x,y)\varphi_1(x)dx\, d|\mu|(y)$$
$$= \frac{1}{\lambda_1^s} \int_\Omega \varphi_1(y)\, d|\mu|(y) \le C\|\delta\mu\|_{\mathcal{M}(\Omega)} \tag{48}$$

This, along with Lemma 20, proves that $u \in L^1(\Omega, \delta(x)dx)$ and (13). Now, pick $\psi \in \mathcal{T}(\Omega)$ and compute, via the Fubini's Theorem, Lemma 11 and Lemma 28,

$$\int_\Omega u(x)(-\triangle|_\Omega)^s\psi(x)\, dx =$$
$$= \int_\Omega \left(\int_\Omega G^s_\Omega(x,y)d\mu(y) \right) (-\triangle|_\Omega)^s\psi(x)\, dx + \int_\Omega \left(\int_{\partial\Omega} P^s_\Omega(x,z)\, d\zeta(z) \right) (-\triangle|_\Omega)^s\psi(x)\, dx =$$
$$= \int_\Omega \left(\int_\Omega G^s_\Omega(x,y)(-\triangle|_\Omega)^s\psi(x)\, dx \right) d\mu(y) + \int_{\partial\Omega} \left(\int_\Omega P^s_\Omega(x,z)(-\triangle|_\Omega)^s\psi(x)\, dx \right) d\zeta(z) =$$
$$= \int_\Omega \psi(y)\, d\mu(y) - \int_{\partial\Omega} \frac{\partial\psi}{\partial\nu}(z)\, d\zeta(z). \qquad \square$$

5.2. *Proof of Lemma 5*

Proof of 1. Consider a sequence $\{\eta_k\}_{k\in\mathbb{N}} \subset C^\infty_c(\Omega)$ of bump functions such that $0 \le \eta_1 \le \ldots \le \eta_k \le \eta_{k+1} \le \ldots \le 1$ and $\eta_k(x) \uparrow \chi_\Omega(x)$ as $k \uparrow \infty$. Consider $\psi \in C^\infty_c(\Omega)$ and define $f_k := \eta_k(-\triangle|_\Omega)^s\psi \in C^\infty_c(\Omega)$, $\psi_k := (-\triangle|_\Omega)^{-s} f_k \in \mathcal{T}(\Omega)$.

Let us first note that the integral

$$\int_\Omega u\, (-\triangle|_\Omega)^s\psi$$

makes sense in view of (31) and (13). The sequence $\{f_k\}_{k\in\mathbb{N}}$ trivially converges a.e. to $(-\triangle|_\Omega)^s\psi$, while

$$|\psi_k(x) - \psi(x)| \le \int_\Omega G^s_\Omega(x,y)|(-\triangle|_\Omega)^s\psi(y)| \cdot (1-\eta_k(y))\, dy$$

converges to 0 for any $x \in \Omega$ by dominated convergence. Since u is a weak solution, it holds

$$\int_\Omega u\, f_k = \int_\Omega \psi_k\, d\mu - \int_{\partial\Omega} \frac{\partial\psi_k}{\partial\nu}\, d\zeta.$$

By dominated convergence $\int_\Omega u\, f_k \to \int_\Omega u(-\triangle|_\Omega)^s\psi$ and $\int_\Omega \psi_k\, d\mu \to \int_\Omega \psi\, d\mu$. Indeed for any $k \in \mathbb{N}$,

$$|f_k| \le |(-\triangle|_\Omega)^s\psi| \qquad \text{and} \qquad |\psi_k| \le \frac{1}{\lambda_1^s} \left\| \frac{(-\triangle|_\Omega)^s\psi}{\varphi_1} \right\|_{L^\infty(\Omega)} \varphi_1,$$

where the latter inequality follows from the maximum principle or the representation formula $\psi_k(x) = \int_\Omega G^s_\Omega(x,y) f_k(y)dy$. Finally, the convergence

$$\int_{\partial\Omega} \frac{\partial \psi_k}{\partial \nu}\, d\zeta \xrightarrow[k\uparrow\infty]{} 0$$

holds by dominated convergence, since

$$\int_{\partial\Omega} \frac{\partial \psi_k}{\partial \nu}\, d\zeta = \int_{\Omega} \mathbb{P}^s_\Omega \zeta\, f_k$$

by the Fubini's Theorem, and $\mathbb{P}^s_\Omega \zeta \in L^1(\Omega, \delta(x)dx)$ while $|f_k| \le |(-\triangle|_\Omega)^s \psi| \le C\delta$ by Lemma 18, so that

$$\int_{\Omega} \mathbb{P}^s_\Omega \zeta\, f_k \xrightarrow[k\uparrow\infty]{} \int_{\Omega} \mathbb{P}^s_\Omega \zeta\, f \;=\; 0$$

because $\mathbb{P}^s_\Omega \zeta$ is s-harmonic and $f \in C^\infty_c(\Omega)$.

The proof of the boundary trace can be found in Theorems 26 and 27, by recalling the representation formula provided by Theorem 6 for the solution to (10). □

Proof of 2. Recall that u is represented by

$$u(x) = \int_{\Omega} G^s_\Omega(x,y)\, \mu(y)\, dy + \int_{\partial\Omega} P^s_\Omega(x,y)\, \zeta(y)\, d\sigma(y).$$

By Point 1. and Lemma 21, $u \in C^{2s+\alpha}_{loc}(\Omega)$. Moreover, $u \in L^1(\Omega, \delta(x)dx)$ thanks to (13). So, we can pointwisely compute $(-\triangle|_\Omega)^s u$ by using (3) and (A.1): this entails by the self-adjointness of the operator in (A.1) that

$$\int_{\Omega} (-\triangle|_\Omega)^s u\, \psi = \int_{\Omega} u (-\triangle|_\Omega)^s \psi = \int_{\Omega} \mu\psi, \qquad \text{for any } \psi \in C^\infty_c(\Omega)$$

and we must conclude that $(-\triangle|_\Omega)^s u = \mu$ a.e. By continuity the equality holds everywhere.

We turn now to the boundary trace. The contribution given by $\mathbb{G}^s_\Omega \mu$ is irrelevant, because it is a bounded function as it follows from

$$|\mathbb{G}^s_\Omega \mu(x)| \le C\|\mu\|_{L^\infty(\Omega)} \int_{\Omega} \frac{dy}{|x-y|^{N-2s}}$$

where we have used (30). Therefore, by Propositions 25, there also holds for all $x_0 \in \partial\Omega$,

$$\lim_{x\to x_0, x\in\Omega} \frac{u(x)}{h_1(x)} = \lim_{x\to x_0, x\in\Omega} \frac{\mathbb{G}^s_\Omega \mu(x) + \mathbb{P}^s_\Omega \zeta(x)}{h_1(x)} = \zeta(x_0). \qquad \square$$

Proof of 3. By Lemma 22 $\mu \in C^\varepsilon_{loc}(\Omega)$. In addition, we have assumed that $\zeta \in C(\partial\Omega)$. Consider

$$v(x) = \int_{\Omega} G^s_\Omega(x,y)\, \mu(y)\, dy + \int_{\partial\Omega} P^s_\Omega(x,z)\, \zeta(z)\, d\sigma(z)$$

the weak solution associated to data μ and ζ. By the previous point of the Lemma, v is a classical solution to the equation, so that in a pointwise sense it holds

$$(-\triangle|_\Omega)^s(u-v) = 0 \quad \text{in } \Omega, \qquad \frac{u-v}{h_1} = 0 \quad \text{on } \partial\Omega.$$

By applying Lemma 29 we conclude that $|u-v| \le \varepsilon h_1$ for any $\varepsilon > 0$ and thus $u - v \equiv 0$. □

6. The nonlinear problem

Lemma 31 *(Kato's inequality). For $f \in L^1(\Omega, \delta(x)dx)$ let $w \in L^1(\Omega, \delta(x)dx)$ weakly solve*

$$\begin{cases} (-\triangle|_\Omega)^s w = f & \text{in } \Omega \\ \dfrac{w}{h_1} = 0 & \text{on } \partial\Omega. \end{cases}$$

For any convex $\Phi : \mathbb{R} \to \mathbb{R}$, $\Phi \in C^2(\mathbb{R})$ *such that* $\Phi(0) = 0$ *and* $\Phi(w) \in L^1_{loc}(\Omega)$, *it holds*

$$(-\triangle|_\Omega)^s \Phi(w) \le \Phi'(w)(-\triangle|_\Omega)^s w.$$

Moreover, the same holds for $\Phi(t) = t^+ = t \wedge 0$.

Proof. Let us first assume that $f \in C^\alpha_{loc}(\Omega)$. In this case, by Lemma 21, $w \in C^{2s+\alpha}_{loc}(\Omega)$ and the equality $(-\triangle|_\Omega)^s w = f$ holds in a pointwise sense. Then

$$\begin{aligned}(-\triangle|_\Omega)^s \Phi \circ w(x) &= \int_\Omega [\Phi(w(x)) - \Phi(w(y))]\, J(x,y)\, dy + \kappa(x)\, \Phi(w(x)) \\ &= \Phi'(w(x)) \int_\Omega [w(x) - w(y)]\, J(x,y)\, dy + \kappa(x)\, \Phi(w(x)) \\ &\quad - \int_\Omega [w(x) - w(y)]^2\, J(x,y) \int_0^1 \Phi''(w(x) + t[w(y) - w(x)])(1-t)\, dt\, dy \\ &\le \Phi'(w(x))\, (-\triangle|_\Omega)^s w(x)\end{aligned}$$

where we have used that $\Phi'' \ge 0$ in $\mathbb{R}$ and that $\Phi'(t) \le t\Phi(t)$, which follows from $\Phi(0) = 0$.

We deal now with $f \in L^\infty(\Omega)$. Pick $\{f_j\}_{j\in\mathbb{N}} \subseteq C^\infty_c(\Omega)$ converging to f in $L^1(\Omega, \delta(x), dx)$ and bounded in $L^\infty(\Omega)$. The corresponding $\{w_j = \mathbb{G}^s_\Omega f_j\}_{j\in\mathbb{N}}$ converges to w in $L^1(\Omega, \delta(x)dx)$, is bounded in $L^\infty(\Omega)$ and without loss of generality we assume that $f_j \to f$ and $w_j \to w$ a.e. in Ω. We know that for any $\psi \in \mathcal{T}(\Omega)$, $\psi \ge 0$

$$\int_\Omega \Phi(w_j)\, (-\triangle|_\Omega)^s \psi \le \int_\Omega f_j\, \Phi'(w_j)\psi.$$

By the continuity of Φ and Φ' we have $\Phi(w_j) \to \Phi(w)$, $\Phi'(w_j) \to \Phi'(w)$ a.e. in Ω and that $\{\Phi(w_j)\}_{j\in\mathbb{N}}$, $\{\Phi'(w_j)\}_{j\in\mathbb{N}}$ are bounded in $L^\infty(\Omega)$. Since $\{f_j\}_{j\in\mathbb{N}}$ is converging to f in $L^1(\Omega, \delta(x)dx)$, then

$$\int_\Omega \Phi(w_j)\, (-\triangle|_\Omega)^s \psi \longrightarrow \int_\Omega \Phi(w)\, (-\triangle|_\Omega)^s \psi \qquad \text{and} \qquad \int_\Omega f_j\, \Phi'(w_j)\psi \longrightarrow \int_\Omega f\, \Phi'(w)\psi$$

by dominated convergence.

For a general $f \in L^1(\Omega, \delta(x)dx)$ define $f_{j,k} := (f \wedge j) \vee (-k)$, $j, k \in \mathbb{N}$. Also, split the expression of $\Phi = \Phi_1 - \Phi_2$ into the difference of two increasing function: this can be done in the following way. The function Φ' is continuous and increasing in $\mathbb{R}$, so that it can either have constant sign or there exists $t_0 \in \mathbb{R}$ such that $\Phi'(t_0) = 0$. If it has constant sign than Φ can be increasing or decreasing and we can choose respectively $\Phi_1 = \Phi$, $\Phi_2 = 0$ or $\Phi_1 = 0$, $\Phi_2 = -\Phi$. Otherwise we can take

$$\Phi_1(t) = \begin{cases} \Phi(t) & t > t_0 \\ \Phi(t_0) & t \le t_0 \end{cases} \qquad \text{and} \qquad \Phi_2(t) = \begin{cases} 0 & t > t_0 \\ \Phi(t_0) - \Phi(t) & t \le t_0 \end{cases}.$$

We already know that for any $\psi \in \mathcal{T}(\Omega)$, $\psi \ge 0$

$$\int_\Omega \Phi(w_{j,k})\, (-\triangle|_\Omega)^s \psi \le \int_\Omega f_{j,k}\, \Phi'(w_{j,k})\psi.$$

On the right-hand side we can use twice the monotone convergence, letting $j \uparrow \infty$ first and then $k \uparrow \infty$. On the left hand side, by writing $\Phi = \Phi_1 - \Phi_2$ again we can exploit several times the monotone convergence by splitting

$$\begin{aligned}\int_\Omega \Phi(w_{j,k})\, (-\triangle|_\Omega)^s \psi = \int_\Omega \Phi_1(w_{j,k})\, [(-\triangle|_\Omega)^s \psi]^+ - \int_\Omega \Phi_1(w_{j,k})\, [(-\triangle|_\Omega)^s \psi]^- \; + \\ - \int_\Omega \Phi_2(w_{j,k})\, [(-\triangle|_\Omega)^s \psi]^+ + \int_\Omega \Phi_2(w_{j,k})\, [(-\triangle|_\Omega)^s \psi]^-\end{aligned}$$

to deduce the thesis.

Finally, note that $\Phi(t) = t^+$ can be monotonically approximated by

$$\Phi_j(t) = \frac{1}{2}\sqrt{t^2 + \frac{1}{j^2}} + \frac{t}{2} - \frac{1}{2j}$$

which is convex, C^2 and $\Phi_j(0) = 0$. So

$$\int_\Omega \Phi_j(w)\,(-\triangle|_\Omega)^s\psi \le \int_\Omega f\,\Phi_j'(w)\psi.$$

Since $\Phi_j(t) \uparrow t^+$ and $2\Phi_j'(t) \uparrow 1 + \mathrm{sgn}(t) = 2\chi_{(0,+\infty)}(t)$, we prove the last statement of the Lemma. □

Theorem 32. *Let $f(x,t) : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be a Carathéodory function. Assume that there exists a subsolution and a supersolution $\underline{u}, \overline{u} \in L^1(\Omega, \delta(x)dx) \cap L^\infty_{loc}(\Omega)$ to*

$$\begin{cases} (-\triangle|_\Omega)^s u = f(x,u) & \text{in } \Omega \\ \dfrac{u}{h_1} = 0 & \text{on } \partial\Omega \end{cases} \tag{49}$$

Assume in addition that $f(\cdot, v) \in L^1(\Omega, \delta(x)dx)$ for every $v \in L^1(\Omega, \delta(x)dx)$ such that $\underline{u} \le v \le \overline{u}$ a.e. Then, there exist weak solutions $u_1, u_2 \in L^1(\Omega, \delta(x)dx)$ in $[\underline{u}, \overline{u}]$ such that any solution in the interval $[\underline{u}, \overline{u}]$ satisfies

$$\underline{u} \le u_1 \le u \le u_2 \le \overline{u} \qquad \text{a.e.}$$

Moreover, if the nonlinearity f is decreasing in the second variable, then the solution is unique.

Proof. According to Montenegro and Ponce [19], the mapping $v \mapsto F(\cdot, v)$, where

$$F(x,t) := f(x, [t \wedge \overline{u}(x)] \vee \underline{u}(x)), \qquad x \in \Omega, t \in \mathbb{R},$$

acts continuously from $L^1(\Omega, \delta(x)dx)$ into itself. In addition, the operator

$$\begin{aligned} \mathcal{K} :\ L^1(\Omega, \delta(x)dx) &\longrightarrow L^1(\Omega, \delta(x)dx) \\ v(x) &\longmapsto \mathcal{K}(v)(x) = \int_\Omega G^s_\Omega(x,y) F(y, v(y))dy \end{aligned}$$

is compact. Indeed, take a bounded sequence $\{v_n\}_{n\in\mathbb{N}}$ in $L^1(\Omega, \delta(x)dx)$. On a compact set $K \subset\subset \Omega$, $\underline{u}, \overline{u}$ are essentially bounded and so must be the sequence $\{F(\cdot, v_n)\}_{n\in\mathbb{N}}$. By Theorem 6 and Lemma 21, $\{\mathcal{K}(v_n)\}_{n\in\mathbb{N}}$ is bounded in $C^\alpha_{loc}(K) \cap L^p(\Omega, \delta(x)dx)$, $p \in [1, (N+1)/(N+1-2s))$. In particular, a subsequence $\{v_{n_k}\}_{k\in\mathbb{N}}$ converges locally uniformly to some v. By Hölder's inequality, we also have

$$\|v_{n_k} - v\|_{L^1(\Omega\setminus K, \delta(x)dx)} \le \|v_{n_k} - v\|_{L^p(\Omega\setminus K, \delta(x)dx)} \|\mathbb{1}_{\Omega\setminus K}\|_{L^{p'}(\Omega\setminus K, \delta(x)dx)}.$$

Hence,

$$\|v_{n_k} - v\|_{L^1(\Omega, \delta(x)dx)} \le \|v_{n_k} - v\|_{L^\infty(K, \delta(x)dx)} \|\delta\|_{L^1(\Omega)} + C\|\mathbb{1}_{\Omega\setminus K}\|_{L^{p'}(\Omega\setminus K, \delta(x)dx}.$$

Letting $k \to +\infty$ and then $K \to \Omega$, we deduce that $\mathcal{K}$ is compact and by the Schauder's Fixed Point Theorem, $\mathcal{K}$ has a fixed point $u \in L^1(\Omega, \delta(x)dx)$. We then may prove that $\underline{u} \le u \le \overline{u}$ by means of the Kato's Inequality (Lemma 31) as it is done in [19], which yields that u is a solution of (49).

The proof of the existence of the minimal and a maximal solution $u_1, u_2 \in L^1(\Omega, \delta(x)dx)$ can be performed in an analogous way as in [19], as the only needed tool is the Kato's Inequality.

As for the uniqueness, suppose f is decreasing in the second variable and consider two solutions $u, v \in L^1(\Omega, \delta(x)dx)$ to (49). By the Kato's Inequality Lemma 31, we have

$$(-\triangle|_\Omega)^s (u - v)^+ \le \chi_{\{u>v\}}[f(x,u) - f(x,v)] \le 0 \qquad \text{in } \Omega$$

which implies $(u-v)^+ \le 0$ by the Maximum Principle Lemma 30. Reversing the roles of u and v, we get also $(v-u)^+ \le 0$, thus $u \equiv v$ in Ω. □

6.1. *Proof of Theorem 8*

Problem (17) is equivalent to

$$\begin{cases} (-\triangle|_\Omega)^s v = g(x, \mathbb{P}^s_\Omega \zeta - v) & \text{in } \Omega \\ \dfrac{v}{h_1} = 0 & \text{on } \partial\Omega \end{cases} \tag{50}$$

that possesses $\overline{u} = \mathbb{P}^s_\Omega \zeta$ as a supersolution and $\underline{u} = 0$ as a subsolution. Indeed, by equation (38) we have

$$0 \le \mathbb{P}^s_\Omega \zeta \le \|\zeta\|_{L^\infty(\Omega)} h_1 \le C \|\zeta\|_{L^\infty(\Omega)} \delta^{-(2-2s)}$$

Thus any $v \in L^1(\Omega, \delta(x)dx)$ such that $0 \le v \le \mathbb{P}^s_\Omega \zeta$ satisfies

$$g(x, v) \le h(v) \le h(c\delta^{-(2-2s)}) \in L^1(\Omega, \delta(x)dx).$$

So, all hypotheses of Theorem 32 are satisfied and the result follows. □

7. Large solutions

Consider the sequence $\{u_j\}_{j\in\mathbb{N}}$ built by solving

$$\begin{cases} (-\triangle|_\Omega)^s u_j = -u_j^p & \text{in } \Omega \\ \dfrac{u_j}{h_1} = j & \text{on } \partial\Omega. \end{cases} \tag{51}$$

Theorem 8 guarantees the existence of such a sequence if $\delta^{-(2-2s)p} \in L^1(\Omega, \delta(x)dx)$, i.e. $p < 1/(1-s)$. We claim that the sequence is increasing in Ω: indeed the solution to problem (51) is a subsolution for the same problem with boundary datum $j+1$. In view of this, the sequence $\{u_j\}_{j\in\mathbb{N}}$ admits a pointwise limit, possibly infinite.

7.1. *Construction of a supersolution*

Lemma 33. *There exist $\delta_0, C > 0$ such that*

$$(-\triangle|_\Omega)^s \delta^{-\alpha} \ \ge\ -C\,\delta^{-\alpha p}, \qquad \text{for } \delta < \delta_0 \text{ and } \alpha = \frac{2s}{p-1}.$$

Proof. We use the expression in equation (3). Obviously,

$$(-\triangle|_\Omega)^s \delta^{-\alpha}(x) = \int_\Omega [\delta(x)^{-\alpha} - \delta(y)^{-\alpha}] J(x,y)\,dy + \delta(x)^{-\alpha}\kappa(x) \ge \int_\Omega [\delta(x)^{-\alpha} - \delta(y)^{-\alpha}] J(x,y)\,dy.$$

For any fixed $x \in \Omega$ close to $\partial\Omega$, split the domain Ω into three parts:

$$\Omega_1 = \left\{ y \in \Omega : \delta(y) \ge \frac{3}{2}\delta(x) \right\},$$

$$\Omega_2 = \left\{ y \in \Omega : \frac{1}{2}\delta(x) < \delta(y) < \frac{3}{2}\delta(x) \right\},$$

$$\Omega_3 = \left\{ y \in \Omega : \delta(y) \le \frac{1}{2}\delta(x) \right\}.$$

For $y \in \Omega_1$, since $\delta(y) > \delta(x)$, it holds $\delta(x)^{-\alpha} - \delta(y)^{-\alpha} > 0$ and we can drop the integral on Ω_1. Also, since it holds by equation (33)

$$J(x,y) \le \frac{C}{|x-y|^{N+2s}},$$

the integration on Ω_2 can be performed as in [2, *Second step* in Proposition 6] providing

$$PV\int_{\Omega_2}[\delta(x)^{-\alpha}-\delta(y)^{-\alpha}]\,J(x,y)\,dy\geq -C\,\delta(x)^{-\alpha-2s}=-C\,\delta(x)^{-\alpha p}.$$

To integrate on Ω_3 we exploit once again (33) under the form

$$J(x,y)\leq C\cdot\frac{\delta(x)\,\delta(y)}{|x-y|^{N+2+2s}}$$

to deduce

$$\int_{\Omega_3}[\delta(x)^{-\alpha}-\delta(y)^{-\alpha}]\,J(x,y)\,dy\geq -C\,\delta(x)\int_{\Omega_3}\frac{\delta(y)^{1-\alpha}}{|x-y|^{N+2+2s}}.$$

Again, a direct computation as in [2, *Third step* in Proposition 6] yields

$$\int_{\Omega_3}[\delta(x)^{-\alpha}-\delta(y)^{-\alpha}]\,J(x,y)\,dy\geq -C\,\delta(x)^{-\alpha-2s}=-C\,\delta(x)^{-\alpha p}.\qquad\square$$

Lemma 34. *If a function $v\in L^1(\Omega,\delta(x)dx)$ satisfies*

$$(-\triangle|_\Omega)^s v\in L^\infty_{loc}(\Omega),\qquad (-\triangle|_\Omega)^s v(x)\ \geq\ -C\,v(x)^p,\quad \text{when } \delta(x)<\delta_0, \tag{52}$$

for some $C,\delta_0>0$, then there exists $\overline{u}\in L^1(\Omega,\delta(x)dx)$ such that

$$(-\triangle|_\Omega)^s\overline{u}(x)\ \geq\ -\overline{u}(x)^p,\qquad \text{throughout } \Omega. \tag{53}$$

Proof. Let $\lambda:=C^{1/(p-1)}\vee 1$ and $\Omega_0=\{x\in\Omega:\delta(x)<\delta_0\}$, then

$$(-\triangle|_\Omega)^s\,(\lambda v)\ \geq\ -(\lambda v)^p\,,\qquad \text{in } \Omega_0.$$

Let also $\mu:=\lambda\|(-\triangle|_\Omega)^s v\|_{L^\infty(\Omega\setminus\Omega_0)}$ and define $\overline{u}=\mu\mathbb{G}^s_\Omega 1+\lambda v$. On $\overline{u}$ we have

$$(-\triangle|_\Omega)^s\overline{u}=\mu+\lambda(-\triangle|_\Omega)^s v\geq\lambda|(-\triangle|_\Omega)^s v|+\lambda(-\triangle|_\Omega)^s v\geq-\overline{u}^p\qquad\text{throughout }\Omega.\qquad\square$$

Corollary 35. *There exists a function $\overline{u}\in L^1(\Omega,\delta(x)dx)$ such that the inequality*

$$(-\triangle|_\Omega)^s\overline{u}\ \geq\ -\overline{u}^p,\qquad \text{in } \Omega,$$

holds in a pointwise sense. Moreover, $\overline{u}\asymp\delta^{-2s/(p-1)}$.

Proof. Apply Lemma 34 with $v=\delta^{-2s/(p-1)}$: the corresponding $\overline{u}$ will be of the form

$$\overline{u}=\mu\mathbb{G}^s_\Omega 1+\lambda\delta^{-2s/(p-1)}.\qquad\square$$

7.2. Existence

Lemma 36. *For any $j\in\mathbb{N}$, the solution u_j to problem (51) satisfies the upper bound*

$$u_j\ \leq\ \overline{u},\qquad \text{in } \Omega,$$

where $\overline{u}$ is provided by Corollary 35.

Proof. Write $u_j=jh_1-v_j$ where

$$\begin{cases}(-\triangle|_\Omega)^s v_j=\ \big(jh_1-v_j\big)^p & \text{in }\Omega\\ \dfrac{v_j}{h_1}=\ 0 & \text{on }\partial\Omega.\end{cases}$$

and $0\leq v_j\leq jh_1$. Since $\big(jh_1-v_j\big)^p\in L^\infty_{loc}(\Omega)$, we deduce that $v_j\in C^\alpha_{loc}(\Omega)$ for any $\alpha\in(0,2s)$. By bootstrapping $v_j\in C^\infty(\Omega)$ and, by Lemma 21, also $u_j\in C^\infty(\Omega)$. This says that u_j is a classical solution to problem (51). Now, we have that, by the boundary behavior of $\overline{u}$ stated in Corollary 35, $u_j \le \overline{u}$ close enough to $\partial\Omega$ (depending on the value of j) and

$$(-\triangle|_\Omega)^s\left(\overline{u}-u_j\right) \;\ge\; u_j^p-\overline{u}^p, \qquad \text{in } \Omega.$$

Since $u_j^p-\overline{u}^p \in C(\Omega)$ and $\lim_{x\to\partial\Omega} u_j^p-\overline{u}^p=-\infty$, there exists $x_0\in\Omega$ such that $u_j(x_0)^p-\overline{u}(x_0)^p=m=:\max_{x\in\Omega}\left(u_j(x)^p-\overline{u}(x)^p\right)$. If $m>0$ then $(-\triangle|_\Omega)^s(\overline{u}-u_j)(x_0)\ge m>0$: this is a contradiction, as Definition 3 implies. Thus $m\le 0$ and $u_j\le\overline{u}$ throughout Ω. □

Theorem 37. *For any* $p\in\left(1+s,\dfrac{1}{1-s}\right)$ *there exists a function* $u\in L^1(\Omega,\delta(x)dx)$ *solving*

$$\begin{cases} (-\triangle|_\Omega)^s u=-u^p & \text{in } \Omega\\ \delta^{2-2s}u=+\infty & \text{on } \partial\Omega\end{cases}$$

both in a distributional and pointwise sense.

Proof. Consider the sequence $\{u_j\}_{j\in\mathbb{N}}$ provided by Problem 51: it is increasing and locally bounded by Lemma 36, so it has a pointwise limit $u\le\overline{u}$, where $\overline{u}$ is the function provided by Corollary 35. Since $p>1+s$ and $\overline{u}\le C\delta^{-2s/(p-1)}$, then $u\in L^1(\Omega,\delta(x)dx)$. Pick now $\psi\in C_c^\infty(\Omega)$, and recall that $\delta^{-1}(-\triangle|_\Omega)^s\psi\in L^\infty(\Omega)$: we have, by dominated convergence,

$$\int_\Omega u_j(-\triangle|_\Omega)^s\psi \xrightarrow[j\uparrow\infty]{} \int_\Omega u(-\triangle|_\Omega)^s\psi, \qquad \int_\Omega u_j^p\psi \xrightarrow[j\uparrow\infty]{} \int_\Omega u^p\psi$$

so we deduce

$$\int_\Omega u(-\triangle|_\Omega)^s\psi \;=\; -\int_\Omega u^p\psi.$$

Note now that for any compact $K\subset\subset\Omega$, applying Lemma 21 we get for any $\alpha\in(0,2s)$

$$\|u_j\|_{C^\alpha(K)}\le C\left(\|u_j\|_{L^\infty(K)}^p+\|u_j\|_{L^1(\Omega,\delta(x)dx)}\right)\le C\left(\|\overline{u}\|_{L^\infty(K)}^p+\|\overline{u}\|_{L^1(\Omega,\delta(x)dx)}\right)$$

which means that $\{u_j\}_{j\in\mathbb{N}}$ is equibounded and equicontinuous in $C(K)$. By the Ascoli–Arzelà Theorem, its pointwise limit u will be in $C(K)$ too. Now, since

$$(-\triangle|_\Omega)^s u \;=\; -u^p \qquad \text{in } \mathcal{D}'(\Omega),$$

by bootstrapping the interior regularity in Lemma 21, we deduce $u\in C^\infty(\Omega)$. So, its spectral fractional Laplacian is pointwise well-defined and the equation is satisfied in a pointwise sense. Also,

$$\liminf_{x\to\partial\Omega}\frac{u(x)}{h_1(x)}\ge\liminf_{x\to\partial\Omega}\frac{u_j(x)}{h_1(x)}=j$$

and we obtain the desired boundary datum. □

Conflict of interest statement

No conflict.

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Appendix A

A.1. Another representation for the spectral fractional Laplacian

Lemma 38. *For any $u \in H(2s)$ and almost every $x \in \Omega$, there holds*

$$(-\triangle|_\Omega)^s u(x) = PV\int_\Omega [u(x)-u(y)]J(x,y)\,dy + \kappa(x)u(x),$$

where $J(x,y)$ and $\kappa(x)$ are given by (4).

Proof. Assume that $u = \varphi_j$ is an eigenfunction of the Dirichlet Laplacian associated to the eigenvalue λ_j. Then, $(-\triangle|_\Omega)^s u = \lambda_j^s u$, $e^{t\triangle|_\Omega}u = \int_\Omega p_\Omega(t,\cdot,y)\,u(y)\,dy = e^{-t\lambda_j}u$ and for all $x \in \Omega$

$$\begin{aligned}
&\frac{\Gamma(1-s)}{s}(-\triangle|_\Omega)^s u(x) = \\
&= \int_0^\infty \left(u(x) - e^{t\,\triangle|_\Omega}u(x)\right)\frac{dt}{t^{1+s}} \qquad \text{(A.1)}\\
&= \int_0^\infty \left(u(x) - \int_\Omega p_\Omega(t,x,y)u(y)\,dy\right)\frac{dt}{t^{1+s}}\\
&= \lim_{\epsilon\to 0}\int_0^\infty \int_{\Omega\setminus B(x,\epsilon)} p_\Omega(t,x,y)[u(x)-u(y)]\,dy\,\frac{dt}{t^{1+s}} + \int_0^\infty u(x)\left(1-\int_\Omega p_\Omega(t,x,y)\,dy\right)\frac{dt}{t^{1+s}}\\
&= PV\int_\Omega [u(x)-u(y)]J(x,y)\,dy + \kappa(x)u(x) \qquad \text{(A.2)}
\end{aligned}$$

By linearity, equality holds on the linear span of the eigenvectors. Now, if $u \in H(2s)$, a sequence $\{u_n\}_{n\in\mathbb{N}}$ of functions belonging to that span converges to u in $H(2s)$. In particular, $(-\triangle|_\Omega)^s u_n$ converges to $(-\triangle|_\Omega)^s u$ in $L^2(\Omega)$. Note also that for $v \in L^2(\Omega)$,

$$\frac{s}{\Gamma(1-s)}\left|\int_\Omega\int_0^\infty \left(u(x)-e^{t\,\triangle|_\Omega}u(x)\right)\frac{dt}{t^{1+s}}\cdot v(x)\,dx\right| = \left|\sum_{k=1}^{+\infty}\lambda_k^s\widehat{u}_k\widehat{v}_k\right| \le \|u\|_{H(2s)}\|v\|_{L^2(\Omega)}$$

so that we may also pass to the limit in $L^2(\Omega)$ when computing (A.1) along the sequence $\{u_n\}_{n\in\mathbb{N}}$. By the Fubini's theorem, for almost every $x \in \Omega$, all subsequent integrals are convergent and the identities remain valid. □

A.2. The reduction to the flat case

In this paragraph we are going to justify the computation of the asymptotic behavior of integrals of the type

$$\int_{A\cap\Omega} F(\delta(x),\delta(y),|x-y|)\,dy \qquad \text{as } \delta(x)\downarrow 0,$$

where A is a fixed neighborhood of x with $A\cap\partial\Omega \neq \emptyset$, by just looking at

$$\int_0^1 dt\int_B dy'\,F\left(\delta(x),t,\sqrt{|y'|^2+|t-\delta(x)|^2}\right).$$

The first thing to be proved is that

$$|x-y|^2 \asymp |x_0-y_0|^2+|\delta(x)-\delta(y)|^2,$$

where x_0, y_0 are respectively the projections of x, y on $\partial\Omega$.

Lemma 39. *There exists $\varepsilon=\varepsilon(\Omega)>0$ such that for any $x\in\Omega$, $x=x_0+\delta(x)\nabla\delta(x_0)$, $x_0\in\partial\Omega$, with $\delta(x)<\varepsilon$ and any $y\in\Omega$ with $\delta(y)<\varepsilon$ and $|y_0-x_0|<\varepsilon$*

$$\frac{1}{2}\left(|x_0-y_0|^2+|\delta(x)-\delta(y)|^2\right)\le|x-y|^2\le\frac{3}{2}\left(|x_0-y_0|^2+|\delta(x)-\delta(y)|^2\right).$$

Proof. Call $\Omega_\varepsilon=\{x\in\Omega:\delta(x)<\varepsilon\}$. Write $x=x_0+\delta(x)\nabla\delta(x_0)$, $y=y_0+\delta(y)\nabla\delta(y_0)$. Then

$$\begin{aligned}|x-y|^2=&\,|x_0-y_0|^2+|\delta(x)-\delta(y)|^2+\delta(y)^2|\nabla\delta(x_0)-\nabla\delta(y_0)|^2+2[\delta(x)-\delta(y)]\langle x_0-y_0,\nabla\delta(x_0)\rangle\ +\\&+\ 2\delta(y)\langle x_0-y_0,\nabla\delta(x_0)-\nabla\delta(y_0)\rangle+2\delta(y)[\delta(x)-\delta(y)]\langle\nabla\delta(x_0),\nabla\delta(x_0)-\nabla\delta(y_0)\rangle.\end{aligned}\tag{A.3}$$

Since, for $\varepsilon>0$ small, $\delta\in C^{1,1}(\Omega_\varepsilon)$ and

$$\begin{aligned}&|\nabla\delta(x)-\nabla\delta(y)|^2\le\|\delta\|^2_{C^{1,1}(\Omega_\varepsilon)}|x-y|^2\\&\langle x_0-y_0,\nabla\delta(x_0)\rangle=O(|x_0-y_0|^2)\\&|\langle x_0-y_0,\nabla\delta(x_0)-\nabla\delta(y_0)\rangle|\le\|\delta\|_{C^{1,1}(\Omega_\varepsilon)}|x_0-y_0|^2\\&|\delta(x)-\delta(y)|=\|\delta\|_{C^{1,1}(\Omega_\varepsilon)}|x-y|\\&|\langle\nabla\delta(x_0),\nabla\delta(x_0)-\nabla\delta(y_0)\rangle|\le\|\delta\|_{C^{1,1}(\Omega_\varepsilon)}|x_0-y_0|\end{aligned}$$

The error term we obtained in (A.3) can be reabsorbed in the other ones by choosing $\varepsilon>0$ small enough to have

$$\begin{aligned}&\delta(y)^2|\nabla\delta(x_0)-\nabla\delta(y_0)|^2+2|\delta(x)-\delta(y)|\cdot|\langle x_0-y_0,\nabla\delta(x_0)\rangle|+2\delta(y)|\langle x_0-y_0,\nabla\delta(x_0)-\nabla\delta(y_0)\rangle|\ +\\&\quad+\ 2\delta(y)|\delta(x)-\delta(y)|\cdot|\langle\nabla\delta(x_0),\nabla\delta(x_0)-\nabla\delta(y_0)\rangle|\le\frac{1}{2}\left(|x_0-y_0|^2+|\delta(x)-\delta(y)|^2\right).\qquad\square\end{aligned}$$

Lemma 40. *Let $F:(0,+\infty)^3\to(0,+\infty)$ be a continuous function, decreasing in the third variable. and $\Omega_\varepsilon=\{x\in\Omega:\delta(x)<\varepsilon\}$, with $\varepsilon=\varepsilon(\Omega)>0$ provided by the previous lemma. Consider $x=x_0+\delta(x)\nabla\delta(x_0)$, $x_0\in\partial\Omega$, and the neighborhood A of the point x, defined by $A=\{y\in\Omega_\varepsilon:y=y_0+\delta(y)\nabla\delta(y_0),|x_0-y_0|<\varepsilon\}$. Then there exist constants $0<c_1<c_2$, $c_1=c_1(\Omega)$, $c_2=c_2(\Omega)$ such that*

$$\begin{aligned}c_1\int_0^\varepsilon dt\int_{B'_\varepsilon}dy'\,F\left(\delta(x),t,c_2\sqrt{|y'|^2+|t-\delta(x)|^2}\right)&\le\int_A F(\delta(x),\delta(y),|x-y|)\,dy\ \le\\&\le\ c_2\int_0^\varepsilon dt\int_{B'_\varepsilon}dy'\,F\left(\delta(x),t,c_1\sqrt{|y'|^2+|t-\delta(x)|^2}\right).\end{aligned}$$

where the ’ superscript denotes objects that live in $\mathbb{R}^{N-1}$.

Proof. By writing $y=y_0+\delta(y)\nabla\delta(y_0)$, $y_0\in\partial\Omega$ and using the Fubini’s Theorem, we can split the integration into the variables y_0 and $t=\delta(y)$:

$$\int_A F(\delta(x),\delta(y),|x-y|)\,dy\le\int_{B_\varepsilon(x_0)\cap\partial\Omega}\left(\int_0^\varepsilon F(\delta(x),t,|x-y_0-t\nabla\delta(y_0)|)\,dt\right)d\sigma(y_0).$$

Using the monotony of F and the above lemma, we get

$$\int_A F(\delta(x),\delta(y),|x-y|)\,dy\le\int_{B_\varepsilon(x_0)\cap\partial\Omega}\left(\int_0^\varepsilon F\left(\delta(x),t,c\sqrt{|x_0-y_0|^2+|\delta(x)-t|^2}\right)\,dt\right)d\sigma(y_0)$$

where c is a universal constant. Representing $B_\varepsilon(x_0) \cap \partial\Omega$ via a diffeomorphism γ with a ball $B'_\varepsilon \subset \mathbb{R}^{N-1}$ centered at 0, we can transform the integration in the y_0 variable into the integration onto B'_ε. The volume element $|D\gamma|$ will be bounded above and below by

$$0 < c_1 \leq |D\gamma| \leq c_2,$$

in view of the smoothness assumptions on $\partial\Omega$. $\square$

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