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Everywhere differentiability of viscosity solutions to a class of Aronsson's equations

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Abstract

We show the everywhere differentiability of viscosity solutions to a class of Aronsson equations in $\mathbb{R}^n$ for $n \geq 2$, where the coefficient matrices A are assumed to be uniformly elliptic and $C^{1,1}$. Our result extends an earlier important theorem by Evans and Smart [18] who have studied the case $A = I_n$ which correspond to the ∞-Laplace equation. We also show that every point is a Lebesgue point for the gradient.

In the process of proving the results we improve some of the gradient estimates obtained for the infinity harmonic functions. The lack of suitable gradient estimates has been a major obstacle for solving the $C^{1,\alpha}$ problem in this setting, and we aim to take a step towards better understanding of this problem, too.

A key tool in our approach is to study the problem in a suitable intrinsic geometry induced by the coefficient matrix A. Heuristically, this corresponds to considering the question on a Riemannian manifold whose the metric is given by the matrix A.

Keywords: L^∞-variational problem; Absolute minimizer; Everywhere differentiability; Aronsson's equation

1. Introduction

For any open set $\Omega \subset \mathbb{R}^n$ with $n \geq 2$, we consider the Aronsson equation:

$$\mathcal{A}_H[u](x) := \langle D_x(H(x, Du(x))), D_p H(x, Du(x)) \rangle = 0 \quad \text{in} \quad \Omega, \tag{1.1}$$

where the Hamiltonian H is given by $H(x, p) = \langle A(x)p, p \rangle$. We denote the set of all uniformly elliptic matrices A of order n by $\mathscr{A}(\Omega)$. Our main result is the following theorem.

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Theorem 1.1. *Assume $A \in \mathscr{A}(\Omega) \cap C^{1,1}(\Omega)$. Then any viscosity solution $u \in C(\overline{\Omega})$ to the Aronsson equation (1.1) is everywhere differentiable in Ω.*

In order to show the robustness of the methodology, following Evans–Smart [18] we also show that every point is a Lebesgue point for the gradient.

Observe that when A is the identity matrix of order n, the Aronsson equation (1.1) becomes the infinity Laplace equation:

$$\Delta_\infty u := \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i x_j} = 0 \quad \text{in} \quad \Omega. \tag{1.2}$$

G. Aronsson [1–4] initiated the study of the infinity Laplace equation (1.2) by deriving it as the Euler–Lagrange equation, in the context of L^∞-variational problems, of absolute minimal Lipschitz extensions (AMLE) or equivalently absolute minimizers (AM) of

$$\inf\left\{ \operatorname*{esssup}_{x\in\Omega} |Du|^2 : \ u \in \mathrm{Lip}(\Omega)\right\}. \tag{1.3}$$

Employing the theory of viscosity solutions of elliptic equations, Jensen [20] has first proved the equivalence between AMLEs and viscosity solutions of (1.2), and the uniqueness of both AMLEs and infinity harmonic functions under the Dirichlet boundary condition. See [26] and [6] for alternative proofs. For further properties of infinity harmonic functions, we refer the readers to the paper by Crandall–Evans–Gariepy [13] and the survey articles by Aronsson–Crandall–Juutinen [7] and Crandall [12].

For L^∞-variational problems involving Hamiltonian functions $H = H(x, z, p) \in C^2(\Omega \times \mathbb{R} \times \mathbb{R}^n)$, Barron, Jensen and Wang [8] have proved that an absolute minimizer of

$$\mathscr{F}_\infty(u, \Omega) = \operatorname*{esssup}_{x\in\Omega} H(x, u(x), Du(x)) \tag{1.4}$$

is a viscosity solution of (1.1), provided the level sets of H are convex in p-variable. Recall that a Lipschitz function $u \in \mathrm{Lip}(\Omega)$ is an *absolute minimizer* for $\mathscr{F}_\infty$, if for every open subset $U \Subset \Omega$ and $v \in \mathrm{Lip}(U)$, with $v|_{\partial U} = u|_{\partial U}$, it holds

$$\mathscr{F}_\infty(u, U) \le \mathscr{F}_\infty(v, U).$$

See [15,5,21,22] for related works on both Aronsson's equations (1.1) and absolute minimizers of $\mathscr{F}_\infty$. Recently, Bjorland, Caffarelli and Figalli [9] (see also [11]) studied the infinity fractional Laplacian, that is, the L^∞-variational problems associated to non-local Hamiltonian functions.

The regularity for infinity harmonic functions (or viscosity solutions to (1.2)) has attracted great interest recently. When $n = 2$, Savin [27] has showed the interior C^1-regularity, and Evans–Savin [17] have established the interior $C^{1,\alpha}$-regularity. Wang and Yu [29] have established the C^1-boundary regularity and, moreover, they have also extended Savin's C^1-regularity to the Aronsson equation (1.1) for uniformly convex $H(p) \in C^2(\mathbb{R}^2)$ [28]. When $n \ge 3$, Evans and Smart [18,19] have established the interior everywhere differentiability of infinity harmonic functions, whereas Wang and Yu [29] have extended this to the boundary differentiability. For the inhomogeneous infinity Laplace equation, the everywhere differentiability has been shown by Lindgren in [23]. In this paper, we extend the Evans–Smart [18,19] differentiability result to cover also the case of the Aronsson equation (1.1) for $A \in \mathscr{A}(\Omega) \cap C^{1,1}(\Omega)$ and $n \ge 2$. The result is not merely a straightforward generalization of the known theory, since there are several new difficulties in running the arguments.

The Evans–Smart method heavily relies on a linear approximation property proved earlier by Crandall, Evans and Gariepy [13]. This result states that the difference quotient, corresponding to the differentiability condition, has a convergent subsequence. Then one needs to show the uniqueness of the limit to conclude the result. A linear approximation property also holds for the Aronsson equation (1.1); see Lemma 4.1 below. Lemma 4.1 was first proved by Yu [30] for some general Hamiltonian functions $H(x, p)$; later a proof was given in [22] when $H(x, p) = \langle A(x)p, p\rangle$ based on an intrinsic geometry induced by the coefficient A.

For showing the uniqueness of the aforementioned limit, we need to establish certain gradient estimates. The standard approach has been to study the ϵ-regularized equation. After introducing the coefficient matrix, the regularized equation does not necessarily have smooth enough solutions for the standard Bernstein type arguments which rely on differentiating the equation. In order to overcome this obstacle, we need to approximate the coefficient matrix, too.

We can prove the estimates required for the convergence results only if the coefficient matrix is close enough to the identity. As we want to consider the general case, we need to use a blow-up argument to reduce the problem to studying merely such matrices. This requires a careful analysis of the final reasoning for the differentiability result.

Finally, the introduction of the coefficient matrix into our gradient estimates causes several technical difficulties not present in the case of the ∞-Laplace equation. The equation includes terms with partial derivatives of the coefficient matrix. In order to control these new terms, we need to establish a series of new estimates.

In the process of proving the gradient estimates we are also able to improve the earlier estimates for the infinity harmonic functions by Evans and Smart, see Theorem 3.3 below. The lack of suitable gradient estimates has been a major obstacle in solving the $C^{1,\alpha}$ problem for the ∞-Laplace equation [17]. We aim to take a step towards better understanding of the problem also at this front.

As a final remark we would like to point out an interesting question related to the assumption $A \in C^{1,1}$. Already making sense of the equation (1.1) requires the coefficient matrix to be at least C^1. In the classical theory, on the other hand, this sort of higher regularity results are typically based on perturbation arguments which require the coefficient to be, say, Hölder continuous. In our setting this corresponds to assuming $A \in C^{1,\alpha}$. Now it is an interesting question whether the regularity assumption in Theorem 1.1 can be relaxed and, in particular, whether it holds for instance for merely $A \in \mathscr{A}(\Omega) \cap C^1(\Omega)$. As proved in Lemma 4.1, the linear approximation property only requires $A \in \mathscr{A}(\Omega) \cap C(\overline{\Omega})$.

2. Preliminaries

First of all, recall that the coefficient matrix $A = (a^{ij}(x))_{1\le i,j\le n}$ is called uniformly elliptic if there exists $L > 0$ such that

$$L^{-1}|p|^2 \le \langle A(x)p,\ p\rangle \le L|p|^2,\ x \in \Omega \text{ and } p \in \mathbb{R}^n. \tag{2.1}$$

Recall also the definition of the Hamiltonian

$$H(x,\ p) = \langle A(x)p, p\rangle = \sum_{i,\,j=1}^{n} a^{ij}(x)p_i p_j,\ x \in \Omega \text{ and }\ p \in \mathbb{R}^n. \tag{2.2}$$

In this section, we will describe a regularization scheme of the Aronsson equation (1.1). Let's recall the definition of viscosity solutions of the Aronsson equation (1.1).

Definition 2.1. A function $u \in C(\overline{\Omega})$ is a viscosity subsolution (supersolution) of the Aronsson equation (1.1) if, for every $x \in \Omega$ and every $\varphi \in C^2(\Omega)$ such that if $u - \varphi$ has a local maximum (minimum) at x then

$$\mathcal{A}_H[\varphi](x)\ \ge (\le)\ 0. \tag{2.3}$$

A function u is a viscosity solution of (1.1) if u is both viscosity subsolution and supersolution.

For $\epsilon > 0$ and a uniformly elliptic matrix $B \in \mathscr{A}(\Omega) \cap C^\infty(\Omega)$, set the Hamiltonian function H_B by

$$H_B(x, p) = \langle B(x)p, p\rangle,\ x \in \Omega \text{ and } p \in \mathbb{R}^n.$$

We consider an ϵ-regularized Aronsson equation (1.1) associated with B and H_B:

$$\begin{cases} -\mathcal{A}^\epsilon_{H_B}[u^\epsilon] := -\mathcal{A}_{H_B}[u^\epsilon] - \epsilon \operatorname{div}(B\nabla u^\epsilon) = 0 & \text{in} \quad \Omega, \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad u^\epsilon = u & \text{on} \quad \partial\Omega. \end{cases} \tag{2.4}$$

For (2.4), we have the following theorem.

Theorem 2.2. *For $\epsilon > 0$, $B \in \mathscr{A}(\Omega) \cap C^\infty(\Omega)$, and $u \in C^{0,1}(\Omega)$, there exists a unique solution $u^\epsilon \in C^\infty(\Omega) \cap C(\overline{\Omega})$ of the equation (2.4).*

Proof. Consider the minimization problem of the functional of exponential growth

$$c_\epsilon := \inf\Big\{\mathcal{I}_\epsilon[v] := \int_\Omega \exp\big(\frac{1}{\epsilon}H_B(x,\nabla v)\big)\,dx \;\Big|\; v \in \mathbf{K}_\epsilon\Big\},$$

where $\mathbf{K}_\epsilon$ is the set of admissible functions of the functional $\mathcal{I}_\epsilon$ defined by

$$\mathbf{K}_\epsilon = \Big\{w \in W^{1,1}(\Omega)\Big| \int_\Omega \exp\big(\frac{1}{\epsilon}H_B(x,\nabla w)\big)\,dx < +\infty,\; w = u \text{ on } \partial\Omega\Big\}.$$

Note that since $u \in \mathbf{K}_\epsilon$, $\mathbf{K}_\epsilon \neq \emptyset$. Let $\{u_m\} \subset \mathbf{K}_\epsilon$ be a minimizing sequence, i.e., $\lim_{m\to\infty} \mathcal{I}_\epsilon[u_m] = c_\epsilon$. Without loss of generality, we may assume that there exists $u^\epsilon \in \mathbf{K}_\epsilon$ such that $u_m \to u^\epsilon$ uniformly on Ω, and $Du_m \rightharpoonup Du^\epsilon$ in $L^q(\Omega)$ for any $1 \le q < +\infty$. Since $H_B(x,p) = \langle B(x)p, p\rangle$ is uniformly convex in p-variable, by the lower semicontinuity we have that

$$\begin{aligned}
\mathcal{I}_\epsilon[u^\epsilon] &= \int_\Omega \exp\big(\frac{1}{\epsilon}H_B(x,\nabla u^\epsilon)\big)\,dx = \sum_{k=0}^{\infty}\int_\Omega \frac{\big(\epsilon^{-1}H_B(x,\nabla u^\epsilon)\big)^k}{k!}\,dx \\
&\le \liminf_{m\to\infty}\sum_{k=0}^{\infty}\int_\Omega \frac{\big(\epsilon^{-1}H_B(x,\nabla u_m)\big)^k}{k!}\,dx \\
&= \liminf_{m\to\infty}\int_\Omega \exp\big(\frac{1}{\epsilon}H_B(x,\nabla u_m)\big)\,dx = \liminf_{m\to\infty}\mathcal{I}_\epsilon[u_m] = c_\epsilon.
\end{aligned}$$

Hence $c_\epsilon = \mathcal{I}_\epsilon[u^\epsilon]$ and u^ϵ is a minimizer of $\mathcal{I}_\epsilon$ over the set $\mathbf{K}_\epsilon$. Direct calculations imply that the Euler–Lagrange equation of u^ϵ is (2.4). The uniqueness of u^ϵ follows from the maximum principle that is applicable for the equation (2.4). The smoothness of u^ϵ follows from the theory of quasilinear uniformly elliptic equations, and the reader can find its proofs in the papers by Lieberman [24] pages 47–49 and [25] lemma 1.1 (see also the paper by Duc–Eells [16]). □

Note that any viscosity solution $u \in C(\overline{\Omega})$ of the Aronsson equation (1.1) is locally Lipschitz continuous, i.e. $u \in C^{0,1}_{\text{loc}}(\Omega)$ (see [10] and [22]). Since we consider the interior regularity of u, we may simply assume that $u \in C^{0,1}(\Omega)$.

Now we will indicate that under suitable conditions on A, any viscosity solution $u \in C^{0,1}(\Omega)$ of the Aronsson equation (1.1) can be approximated by smooth solutions u^ϵ of ϵ-regularized equations (2.4) associated with suitable H_B's. For this, we recall that for any $A \in \mathscr{A}(\Omega)\cap C^{1,1}(\Omega)$, it is a standard fact that there exists $\{A_\epsilon\} \subset \mathscr{A}(\Omega)\cap C^\infty(\Omega)$ such that

(i) $\|A_\epsilon\|_{C^{1,1}(\Omega)} \le 2\|A\|_{C^{1,1}(\Omega)}$ for all $\epsilon > 0$.
(ii) For any $\alpha \in (0,1)$, $A_\epsilon \to A$ in $C^{1,\alpha}(\Omega)$ as $\epsilon \to 0$.

Theorem 2.3. *For any $A \in \mathscr{A}(\Omega)\cap C^{1,1}(\Omega)$ with ellipticity constant $L < 2^{\frac{1}{5}}$ (see (2.1)), let $\{A_\epsilon\} \subset \mathscr{A}(\Omega)\cap C^\infty(\Omega)$ satisfy the properties (i) and (ii). Assume that $u \in C^{0,1}(\Omega)$ is a viscosity solution of the Aronsson equation (1.1), and $\{u^\epsilon\} \subset C^\infty(\Omega)\cap C(\overline{\Omega})$ are classical solutions of the ϵ-regularized equation (2.4) on Ω, with B and H_B replaced by A_ϵ and H_{A_ϵ} respectively. Then there exists a constant $\delta_0 = \delta_0(\Omega, \|A\|_{L^\infty(\Omega)}) > 0$ such that if $\|DA\|_{L^\infty(\Omega)} \le \delta_0$, then $u^\epsilon \to u$ in $C^0_{\text{loc}}(\Omega)$.*

Proof. From Theorem 3.1, we have that for any compact subset $K \Subset \Omega$,

$$\begin{aligned}
\|Du^\epsilon\|_{C(K)} &\le C\Big(\text{dist}(K,\partial\Omega), \|u\|_{C(\overline{\Omega})}, \|A_\epsilon\|_{C^{1,1}(\Omega)}\Big) \\
&\le C\Big(\text{dist}(K,\partial\Omega), \|u\|_{C(\overline{\Omega})}, \|A\|_{C^{1,1}(\Omega)}\Big), \;\forall\,\epsilon > 0.
\end{aligned}$$

This implies that there exists a $\hat{u} \in C^{0,1}_{\text{loc}}(\Omega)$ such that, after passing to a subsequence,

$$u^\epsilon \to \hat{u} \text{ in } C^0_{\mathrm{loc}}(\Omega). \tag{2.5}$$

Since $\{A_\epsilon\}$ satisfies (i) and (ii), there exists $\epsilon_0 > 0$ such that for any $0 < \epsilon \le \epsilon_0$, it holds that $\|A_\epsilon\|_{L^\infty(\Omega)} \le 2\|A\|_{L^\infty(\Omega)}$, and the ellipticity constant L_ϵ of A_ϵ satisfies $L_\epsilon \le 2^{\frac{1}{4}}$. Let $\delta_0 > 0$ be the constant given by Theorem 3.2 and assume $\|DA\|_{L^\infty(\Omega)} \le \frac{\delta_0}{2}$. Then there exists $0 < \epsilon_1 \le \epsilon_0$ such that $\|DA_\epsilon\|_{L^\infty(\Omega)} \le \delta_0$ for any $\epsilon < \epsilon_1$. Thus Theorem 3.2 below is applicable to u_ϵ for any $0 < \epsilon < \epsilon_1$ and we conclude that there exist $\gamma \in (0,1)$ and $C > 0$, independent of $0 < \epsilon < \epsilon_1$, such that

$$\left|u^\epsilon(x) - u(x_0)\right| \le C|x - x_0|^\gamma, \ \forall\, x \in \Omega, \ x_0 \in \partial\Omega. \tag{2.6}$$

From (2.5) and (2.6), we see that

$$|\hat{u}(x) - u(x_0)| \le C|x - x_0|^\gamma, \ \forall\, x \in \Omega, \ x_0 \in \partial\Omega.$$

This implies that $\hat{u} \in C(\overline{\Omega})$ and $\hat{u} \equiv u$ on $\partial\Omega$. By the compactness property of viscosity solutions of elliptic equations (see Crandall–Ishii–Lions [14]), we know that $\hat{u} \in C(\overline{\Omega})$ is a viscosity solution of the Aronsson equation (1.1) associated with A and H_A. Since $\hat{u} \equiv u$ on $\partial\Omega$, it follows from the uniqueness theorem of (1.1) (see [10] and [22]) that $\hat{u} = u$. This also implies that $u^\epsilon \to u$ in $C^0_{\mathrm{loc}}(\Omega)$ for $\epsilon \to 0$. □

3. A priori estimates

Motivated by [18,19], we will establish some necessary a priori estimates of smooth solutions $\{u^\epsilon\}$ of the equation (2.4) associated with $\{A_\epsilon\}$ satisfying (i) and (ii), which is the crucial ingredient to establish everywhere differentiability of viscosity solution of the Aronsson equation (1.1).

In this section, we will assume $A \in \mathscr{A}(\Omega) \cap C^\infty(\Omega)$, and $u^\epsilon \in C^\infty(\Omega) \cap C(\overline{\Omega})$ is a solution of the ϵ-regularized equation (2.4) with B and H_B replaced by A and H_A.

3.1. Lipschitz estimates

We begin with the following theorems.

Theorem 3.1. *For $u \in C^{0,1}(\Omega)$ and $A \in \mathscr{A}(\Omega) \cap C^\infty(\Omega)$, assume $u^\epsilon \in C^\infty(\Omega) \cap C(\overline{\Omega})$ is a solution of the ϵ-regularized equation (2.4), with B and H_B replaced by A and H_A. Then we have the estimates*

$$\max_{\overline{\Omega}} |u^\epsilon| \le \max_{\overline{\Omega}} |u|, \tag{3.1}$$

and for each open set $V \Subset \Omega$, there exists $C > 0$ depending on n, L, $\|u\|_{C(\overline{\Omega})}$, $\mathrm{dist}(V, \partial\Omega)$, and $\|A\|_{C^{1,1}(\Omega)}$ such that

$$\max_{\overline{V}} |Du^\epsilon| \le C. \tag{3.2}$$

Proof. The estimate (3.1) follows from the standard maximum principle of the equation (2.4). For (3.2), we proceed as follows. To simplify the presentation, we will use the Einstein summation convention. Denote $u^\epsilon_i = \frac{\partial}{\partial x_i} u^\epsilon$, $u^\epsilon_{ij} = \frac{\partial^2}{\partial x_i \partial x_j} u^\epsilon$, a^{ij} as the (i,j)th-entry of A, and $a^{ij}_k = \frac{\partial}{\partial x_k} a^{ij}$. Recall that

$$\mathcal{A}_H[u^\epsilon] = 2a^{ik} u^\epsilon_k u^\epsilon_{ij} a^{j\ell} u^\epsilon_\ell + a^{ij}_k u^\epsilon_i u^\epsilon_j a^{k\ell} u^\epsilon_\ell.$$

Taking $\frac{\partial}{\partial s}$ of the equation (2.4), we obtain

$$\begin{aligned} &2a^{ik} u^\epsilon_k u^\epsilon_{ijs} a^{j\ell} u^\epsilon_\ell + 4a^{ik}_s u^\epsilon_k u^\epsilon_{ij} a^{j\ell} u^\epsilon_\ell + 4a^{ik} u^\epsilon_{ks} u^\epsilon_{ij} a^{j\ell} u^\epsilon_\ell + a^{ij}_{ks} u^\epsilon_i u^\epsilon_j a^{k\ell} u^\epsilon_\ell + 2a^{ij}_k u^\epsilon_{is} u^\epsilon_j a^{k\ell} u^\epsilon_\ell \\ &\quad + a^{ij}_k u^\epsilon_i u^\epsilon_j a^{k\ell}_s u^\epsilon_\ell + a^{ij}_k u^\epsilon_i u^\epsilon_j a^{k\ell} u^\epsilon_{\ell s} + \epsilon \,\mathrm{div}(A D u^\epsilon_s) + \epsilon \,\mathrm{div}(A_s D u^\epsilon) = 0. \end{aligned} \tag{3.3}$$

Set

$$G^\epsilon_m := 4a^{im} u^\epsilon_{ij} a^{j\ell} u^\epsilon_\ell + 2a^{mj}_k u^\epsilon_j a^{k\ell} u^\epsilon_\ell + a^{ij}_k u^\epsilon_i u^\epsilon_j a^{km}, \tag{3.4}$$

and

$$F_s^\epsilon := 4a_s^{ik}u_k^\epsilon u_{ij}^\epsilon a^{j\ell}u_\ell^\epsilon + a_k^{ij}u_i^\epsilon u_j^\epsilon a_s^{k\ell}u_\ell^\epsilon + a_{ks}^{ij}u_i^\epsilon u_j^\epsilon a^{k\ell}u_\ell^\epsilon + \epsilon\,\mathrm{div}(A_s Du^\epsilon). \tag{3.5}$$

Define the operator L_ϵ by

$$L_\epsilon v := 2a^{ik}u_k^\epsilon v_{ij}a^{j\ell}u_\ell^\epsilon + \sum_{m=1}^n G_m^\epsilon v_m + \epsilon\,\mathrm{div}(ADv). \tag{3.6}$$

Then (3.3) can be written as

$$-L_\epsilon(u_s^\epsilon) = F_s^\epsilon. \tag{3.7}$$

Set $v^\epsilon := \frac{1}{2}|Du^\epsilon|^2$. Then

$$v_i^\epsilon = \sum_{s=1}^n u_s^\epsilon u_{si}^\epsilon \text{ and } v_{ij}^\epsilon = \sum_{s=1}^n \left[u_{si}^\epsilon u_{sj}^\epsilon + u_{sij}^\epsilon u_s^\epsilon\right],$$

so that by using the equation (3.7) we have

$$\begin{aligned} L_\epsilon v^\epsilon &= \sum_{s=1}^n \left[2a^{ik}u_k^\epsilon u_{si}^\epsilon u_{sj}^\epsilon a^{j\ell}u_\ell^\epsilon + u_s^\epsilon L_\epsilon u_s^\epsilon + \epsilon a^{ij}u_{si}^\epsilon u_{sj}^\epsilon\right] \\ &= 2|D^2u^\epsilon ADu^\epsilon|^2 + \sum_{s=1}^n \left[\epsilon a^{ij}u_{si}^\epsilon u_{sj}^\epsilon - u_s^\epsilon F_s^\epsilon\right]. \end{aligned} \tag{3.8}$$

Set $z^\epsilon := \frac{1}{2}(u^\epsilon)^2$. Then by the equation (2.4) we have

$$\begin{aligned} L_\epsilon z^\epsilon &= 2a^{ik}u_k^\epsilon u_{ij}^\epsilon u^\epsilon a^{j\ell}u_\ell^\epsilon + 2a^{ik}u_k^\epsilon u_i^\epsilon u_j^\epsilon a^{j\ell}u_\ell^\epsilon + \sum_{m=1}^n G_m^\epsilon u_m^\epsilon u^\epsilon + \epsilon u^\epsilon\,\mathrm{div}(ADu^\epsilon) + \epsilon a^{ij}u_i^\epsilon u_j^\epsilon \\ &= 2\langle Du^\epsilon, ADu^\epsilon\rangle^2 + \epsilon\langle ADu^\epsilon, Du^\epsilon\rangle + u^\epsilon \mathcal{A}_H^\epsilon[u^\epsilon] \\ &\quad + 4u^\epsilon a^{im}u_m^\epsilon u_{ij}^\epsilon a^{j\ell}u_\ell^\epsilon + 2u^\epsilon a_k^{mj}u_j^\epsilon a^{k\ell}u_\ell^\epsilon u_m^\epsilon \\ &= 2\langle Du^\epsilon, ADu^\epsilon\rangle^2 + \epsilon\langle ADu^\epsilon, Du^\epsilon\rangle \\ &\quad + 4u^\epsilon\langle ADu^\epsilon, D^2u^\epsilon ADu^\epsilon\rangle + 2u^\epsilon\langle\langle Du^\epsilon, DADu^\epsilon\rangle, ADu^\epsilon\rangle, \end{aligned}$$

where $\langle Du^\epsilon, DADu^\epsilon\rangle$ is interpreted as the vector $(\langle Du^\epsilon, A_k Du^\epsilon\rangle)_k$ with A_k being the element-wise derivative of A. Choose $\phi \in C_0^\infty(\Omega)$ such that

$$\phi = 1 \text{ in } V,\ 0 \le \phi \le 1,$$

and, for $\beta > 0$ to be determined later, define the auxiliary function w^ϵ by

$$w^\epsilon := \phi^2 v^\epsilon + \beta z^\epsilon.$$

If w^ϵ attains its maximum on $\partial\Omega$, then

$$\sup_{\overline{V}} v^\epsilon \le \sup_{\overline{V}} w^\epsilon(x) \le \max_{\overline{\Omega}} w^\epsilon = \max_{\partial\Omega} w^\epsilon = \frac{\beta}{2}\max_{\partial\Omega} u^2,$$

hence (3.2) holds. Thus we may assume w^ϵ attains its maximum at an interior point $x_0 \in \Omega$. This gives

$$Dw^\epsilon(x_0) = 0,\ D^2w^\epsilon(x_0) \le 0,$$

so that

$$-L_\epsilon w^\epsilon(x_0) = -(2a^{ik}u_k^\epsilon a^{j\ell}u_\ell^\epsilon + \epsilon a^{ij})w_{ij}^\epsilon\Big|_{x=x_0} \ge 0. \tag{3.9}$$

On the other hand, from (3.8) and (3.9) we have that, at $x = x_0$,

$$
\begin{aligned}
0 &\le -L_\epsilon w^\epsilon(x^0) = -L_\epsilon(\phi^2 v^\epsilon) - \beta L_\epsilon z^\epsilon \\
&= -\phi^2 L_\epsilon v^\epsilon - \beta L_\epsilon z^\epsilon - v^\epsilon L_\epsilon \phi^2 - 8\phi a^{ik} u_k^\epsilon a^{j\ell} u_\ell^\epsilon \phi_i \sum_{r=1}^n u_{rj}^\epsilon u_r^\epsilon - 4\epsilon\phi \sum_{m=1}^n \phi_i a^{ij} u_{mj}^\epsilon u_m^\epsilon \\
&= \left[-2\phi^2 |D^2 u^\epsilon A D u^\epsilon|^2 - \epsilon\phi^2 \sum_{s=1}^n a^{ij} u_{si}^\epsilon u_{sj}^\epsilon - 2\beta \langle Du^\epsilon, ADu^\epsilon\rangle^2 - \epsilon\beta\langle Du^\epsilon, ADu^\epsilon\rangle \right] \\
&\quad - \left[4\beta u^\epsilon \langle ADu^\epsilon, D^2u^\epsilon ADu^\epsilon\rangle + 2\beta u^\epsilon a_k^{mj} u_j^\epsilon u_m^\epsilon a^{k\ell} u_\ell^\epsilon \right] \\
&\quad - \left[8\phi a^{ik} u_k^\epsilon a^{j\ell} u_\ell^\epsilon \phi_i \sum_{r=1}^n u_{rj}^\epsilon u_r^\epsilon + 4\epsilon\phi \sum_{m=1}^n \phi_i a^{ij} u_{mj}^\epsilon u_m^\epsilon \right] + \phi^2 \sum_{s=1}^n u_s^\epsilon F_s^\epsilon - v^\epsilon L_\epsilon(\phi^2) \\
&= I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}
$$

We estimate $I_1, \cdots, I_5$ as follows. Since $\langle \xi, A\xi\rangle \ge \frac{1}{L}|\xi|^2$ for all $\xi \in \mathbb{R}^n$, we have

$$
\begin{aligned}
I_1 &= -2\phi^2 |D^2u^\epsilon ADu^\epsilon|^2 - \epsilon\phi^2 \sum_{s=1}^n a^{ij} u_{si}^\epsilon u_{sj}^\epsilon - 2\beta\langle Du^\epsilon, ADu^\epsilon\rangle^2 - \epsilon\beta\langle ADu^\epsilon, Du^\epsilon\rangle \\
&\le -2\phi^2 |D^2u^\epsilon ADu^\epsilon|^2 - \frac{\epsilon}{L}\phi^2 |D^2u^\epsilon|^2 - \frac{2\beta}{L^2}|Du^\epsilon|^4.
\end{aligned}
$$

Applying Young's inequality, we can estimate I_2 by

$$
\begin{aligned}
I_2 &= -4\beta u^\epsilon \langle ADu^\epsilon, D^2u^\epsilon ADu^\epsilon\rangle - 2\beta u^\epsilon a_k^{mj} u_j^\epsilon u_m^\epsilon a^{k\ell} u_\ell^\epsilon \\
&\le 4\beta |u^\epsilon||ADu^\epsilon||D^2u^\epsilon ADu^\epsilon| + C|Du^\epsilon|^3 \\
&\le \beta^{4/3} |D^2u^\epsilon ADu^\epsilon|^{4/3} + C|Du^\epsilon|^4 + C(\beta),
\end{aligned}
$$

where we have used (3.1). Henceforth $C > 0$ denotes constants depending only on n, L, $\|A\|_{C^{1,1}(\Omega)}$, $\|u\|_{C(\bar\Omega)}$, and $\operatorname{dist}(V, \partial\Omega)$.

Similarly, by Young's inequality we have

$$
\begin{aligned}
I_3 &= -8\phi a^{ik} u_k^\epsilon a^{j\ell} u_\ell^\epsilon \phi_i \sum_{r=1}^n u_{rj}^\epsilon u_r^\epsilon - 4\epsilon\phi \sum_{m=1}^n \phi_i a^{ij} u_{mj}^\epsilon u_m^\epsilon \\
&\le 8\phi \langle AD\phi, Du^\epsilon\rangle \cdot \langle Du^\epsilon, D^2u^\epsilon ADu^\epsilon\rangle + 4\epsilon\langle AD^2u^\epsilon Du^\epsilon, D\phi\rangle\phi \\
&\le C|D^2u^\epsilon ADu^\epsilon||Du^\epsilon|^2\phi + C\epsilon|D^2u^\epsilon Du^\epsilon|\phi \\
&\le \frac{1}{8}|D^2u^\epsilon ADu^\epsilon|^2\phi^2 + \frac{\epsilon}{16L}|D^2u^\epsilon|^2\phi^2 + C|Du^\epsilon|^4 + C.
\end{aligned}
$$

For I_4, by using $0 < \epsilon \le 1$, we have

$$
\begin{aligned}
I_4 &= \sum_{s=1}^n \Big[4\phi^2 u_s^\epsilon a_s^{ik} u_k^\epsilon u_{ij}^\epsilon a^{j\ell} u_\ell^\epsilon + \phi^2 u_s^\epsilon a_k^{ij} u_i^\epsilon u_j^\epsilon a_s^{k\ell} u_\ell^\epsilon \\
&\quad + \phi^2 u_s^\epsilon a_{sr}^{ij} u_i^\epsilon u_j^\epsilon a^{k\ell} u_\ell^\epsilon + \epsilon\phi^2 u_s^\epsilon \operatorname{div}(A_s Du^\epsilon) \Big] \\
&\le \frac{1}{8}|D^2u^\epsilon ADu^\epsilon|^2\phi^2 + C|Du^\epsilon|^4 + \frac{\epsilon}{16L}\phi^2|D^2u^\epsilon|^2 + C.
\end{aligned}
$$

Finally, for I_5, we have

$$\begin{aligned}I_5 &= 2v^\epsilon a^{ik}u_k^\epsilon(\phi^2)_{ij}a^{j\ell}u_\ell^\epsilon + 4v^\epsilon a^{ik}(\phi^2)_k u_{ij}^\epsilon a^{j\ell}u_\ell^\epsilon + 2v^\epsilon a_k^{ij}(\phi^2)_i u_j^\epsilon a^{k\ell}u_\ell^\epsilon \\ &\quad + v^\epsilon a_k^{ij}u_i^\epsilon u_j^\epsilon a^{k\ell}(\phi^2)_\ell + \epsilon v^\epsilon \operatorname{div}(AD\phi^2) \\ &\le C|Du^\epsilon|^4 + C|D^2u^\epsilon ADu^\epsilon||Du^\epsilon|^2\phi + C\epsilon|Du^\epsilon|^2 \\ &\le \frac{1}{8}|D^2u^\epsilon ADu^\epsilon|^2\phi^2 + C|Du^\epsilon|^4 + C.\end{aligned}$$

Combining all these estimates with (3.9) yields that, at $x = x_0$,

$$\begin{aligned}&2\phi^2|D^2u^\epsilon ADu^\epsilon|^2 + \frac{\epsilon}{L}\phi^2|D^2u^\epsilon|^2 + \frac{2}{L^2}\beta|Du^\epsilon|^4 \\ &\le |D^2u^\epsilon ADu^\epsilon|^2\phi^2 + C|Du^\epsilon|^4 + C\beta^{4/3}|D^2u^\epsilon ADu^\epsilon|^{4/3} + \frac{\epsilon}{8L}\phi^2|D^2u^\epsilon|^2 + C(\beta),\end{aligned}$$

so that

$$|D^2u^\epsilon ADu^\epsilon|^2\phi^2 + \frac{2}{L^2}\beta|Du^\epsilon|^4 \le C|Du^\epsilon|^4 + C\beta^{4/3}|D^2u^\epsilon ADu^\epsilon|^{4/3} + C(\beta).$$

We may choose $\beta > 1$ sufficiently large so that

$$|D^2u^\epsilon ADu^\epsilon|^2\phi^2 + \frac{\beta}{L^2}|Du^\epsilon|^4 \le C\beta^{4/3}|D^2u^\epsilon ADu^\epsilon|^{4/3} + C(\beta).$$

Multiplying both sides of this inequality by ϕ^4 and applying Young's inequality implies

$$\begin{aligned}|D^2u^\epsilon ADu^\epsilon|^2\phi^6 + \frac{\beta}{L^2}|Du^\epsilon|^4\phi^4 &\le C\beta^{4/3}|D^2u^\epsilon ADu^\epsilon|^{4/3}\phi^4 + C(\beta) \\ &\le \frac{1}{2}|D^2u^\epsilon ADu^\epsilon|^2\phi^6 + C(\beta).\end{aligned}$$

Hence we have

$$|Du^\epsilon|^4\phi^4\Big|_{x=x_0} \le C.$$

This finishes the proof, since $v^\epsilon = \frac{1}{2}|Du^\epsilon|^2$ attains its maximum at x^0. □

Next we will establish the boundary Hölder continuity estimate of u^ϵ.

Theorem 3.2. *With the same notations of Theorem 3.1, assume that in addition $L < 2^{1/4}$. Then there exist $\delta_0 > 0$, $\epsilon_0 > 0$, $\gamma \in (0, 1)$, and $C > 0$ depending only on Ω and $\|A\|_{L^\infty(\Omega)}$ such that if $\|DA\|_{L^\infty(\Omega)} \le \delta_0$ and $0 < \epsilon < \epsilon_0$, then*

$$|u^\epsilon(x) - u(y_0)| \le C|x - y_0|^\gamma,\ y_0 \in \partial\Omega,\ x \in \Omega. \tag{3.10}$$

Proof. To show (3.10), assume for simplicity that $y_0 = 0 \in \partial\Omega$. Define $w(x) = \lambda|x|^\gamma$, where $\lambda > 1$ is chosen such that

$$-w + u(0) \le u \le u(0) + w \text{ on } \partial\Omega.$$

This is always possible, since u is Lipschitz. Now we claim that w is a supersolution of the ϵ-regularized equation (2.4). In fact, direct calculations imply

$$\begin{aligned}-a^{ik}(x)w_k(x)w_{ij}(x)a^{j\ell}(x)w_\ell(x) &= -\frac{\lambda^2\gamma^2a^{ik}x_ka^{j\ell}x_\ell}{|x|^{4-2\gamma}}\cdot\lambda\gamma\left[(\gamma-2)\frac{x_ix_j}{|x|^{4-\gamma}} + \frac{\delta_{ij}}{|x|^{2-\gamma}}\right] \\ &= \lambda^3\gamma^3(2-\gamma)\frac{\langle x, Ax\rangle^2}{|x|^{8-3\gamma}} - \lambda^3\gamma^3\frac{\langle x, A^2x\rangle}{|x|^{6-3\gamma}} \\ &\ge \lambda^3\gamma^3\frac{2-\gamma}{L^2}|x|^{3\gamma-4} - \lambda^3\gamma^3L^2|x|^{3\gamma-4} \\ &= \lambda^3\gamma^3\left(\frac{2-\gamma}{L^2} - L^2\right)|x|^{3\gamma-4}.\end{aligned}$$

Note that we can choose $\gamma > 0$ so that $\widetilde{\gamma} := \frac{2-\gamma}{L^2} - L^2 > 0$, since $L < 2^{\frac{1}{4}}$. Next we estimate

$$-a_k^{ij}(x)w_i(x)w_j(x)a^{k\ell}(x)w_\ell(x) = -\lambda^3\gamma^3 a_k^{ij}(x)a^{k\ell}(x)\frac{x_i x_j x_\ell}{|x|^{6-3\gamma}}$$
$$\geq -\lambda^3\gamma^3\|A\|_{L^\infty(\bar{\Omega})}\|DA\|_{L^\infty(\bar{\Omega})}|x|^{3\gamma-3}$$

Finally, for the regularization term we can estimate

$$-\epsilon\,\mathrm{div}(ADw)(x) = -\epsilon\lambda a^{ij}\gamma\frac{\delta_{ij}}{|x|^{2-\gamma}} - \epsilon\lambda a^{ij}\gamma(\gamma-2)\frac{x_i x_j}{|x|^{4-\gamma}} - \epsilon\lambda\gamma a_j^{ij}\frac{x_i}{|x|^{2-\gamma}}$$
$$\geq -\epsilon\lambda L\gamma(n+\gamma-2)|x|^{\gamma-2} - 2\epsilon\lambda n\gamma\|DA\|_{L^\infty(\bar{\Omega})}|x|^{\gamma-1}.$$

Putting these estimates together, we have

$$-\mathcal{A}_H^\epsilon[w] \geq 2\lambda^3\gamma^3\widetilde{\gamma}|x|^{3\gamma-4} - \lambda^3\gamma^3\|A\|_{L^\infty(\Omega)}\|DA\|_{L^\infty(\Omega)}|x|^{3\gamma-3} - 2\epsilon\lambda L\gamma(n+\gamma-2)|x|^{\gamma-2}$$
$$- 2\epsilon\lambda n\gamma\|DA\|_{L^\infty(\Omega)}|x|^{\gamma-1}$$
$$\geq 2\lambda^3\gamma^3\widetilde{\gamma}|x|^{3\gamma-4} - \lambda^3\gamma^3\|A\|_{L^\infty(\Omega)}\|DA\|_{L^\infty(\Omega)}|x|^{3\gamma-3} - C\epsilon|x|^{3\gamma-4}.$$

Set

$$\delta_0 := \delta(\Omega, A) = \frac{\min_{x\in\bar{\Omega}}\frac{\widetilde{\gamma}}{2|x|}}{\|A\|_{L^\infty(\bar{\Omega})}}.$$

If $\|DA\|_{L^\infty(\Omega)} \leq \delta_0$ and $\epsilon_0 > 0$ is sufficiently small, then we have $\gamma \in (0, 1)$ that

$$-\mathcal{A}_H^\epsilon[w] \geq 0.$$

By the comparison principle, we conclude that $w + u(0) \geq u^\epsilon$ in Ω. Similarly, we have $-w + u(0) \geq u^\epsilon$ in Ω. Thus we obtain

$$|u^\epsilon(x) - u(0)| \leq \lambda|x|^\gamma, \quad x \in \Omega.$$

This completes the proof. □

3.2. *Flatness estimates*

In this section, we will prove refined a priori estimates of the ϵ-regularized equation (2.4) under a flatness assumption. Assume $u^\epsilon \in C^\infty(\Omega) \cap C(\overline{\Omega})$ is a smooth solution to the ϵ-regularized equation (2.4) associated with $A \in \mathscr{A}(\Omega) \cap C^\infty(\Omega)$.

Theorem 3.3. *Assume $B(0,3) \subset \Omega$. For any $0 < \lambda < 1$, if $A \in \mathscr{A}(\Omega) \cap C^\infty(\Omega)$ satisfies $A(0) = I_n$ and*

$$\|DA\|_{L^\infty(B(0,3))} + \|D^2A\|_{L^\infty(B(0,3))} \leq \lambda, \tag{3.11}$$

and if $u^\epsilon \in C^\infty(\Omega)$ is a smooth solution of (2.4) that satisfies

$$\max_{x\in B(0,2)} |u^\epsilon(x) - x_n| \leq \lambda, \tag{3.12}$$

then there exists a constant $C > 0$ independent of ϵ and λ such that

$$|Du^\epsilon(x)|^2 \leq u_n^\epsilon(x) + C\lambda^{1/2} \quad \text{for all } x \in B(0,1). \tag{3.13}$$

Proof. Set $\Phi(p) := (|p|^2 - p_n)_+^2 = \max\{|p|^2 - p_n, 0\}^2$. Let $\phi \in C_0^\infty(B(0,3))$ be such that

$$\phi = 1 \text{ in } B(0,1),\ \phi = 0 \text{ outside } B(0,2),\ 0 \leq \phi \leq 1, \text{ and } |D\phi| \leq 2.$$

Define

$$v^\epsilon = \phi^2\Phi(Du^\epsilon) + \beta(u^\epsilon - x_n)^2 + \lambda|Du^\epsilon|^2.$$

Applying Theorem 3.1, we have

$$|u^\epsilon| + |Du^\epsilon| \le C \text{ in } B(0, 2).$$

If $\max_{B(0,2)} v^\epsilon$ is attained on $\partial B(0, 2)$, then by (3.1), (3.11), and (3.12) we have

$$\max_{B(0,2)} v^\epsilon(x) = \max_{\partial B(0,2)} \big(\beta(u^\epsilon - x_n)^2 + \lambda|Du^\epsilon|^2\big) \le \beta\lambda^2 + C\lambda \le C\lambda,$$

and hence

$$\max_{B(0,1)} \big(|Du^\epsilon|^2 - u^\epsilon_{x_n}\big)_+{}^2 \le \max_{B(0,1)} \Phi(Du^\epsilon) \le C\lambda$$

so that (3.13) holds. Therefore we may assume that v^ϵ attains its maximum at an interior point $x_0 \in B(0, 2)$. If $\big(|Du^\epsilon|^2 - u^\epsilon_n\big)(x_0) \le 0$, then $\Phi(Du^\epsilon)(x_0) = 0$ and

$$\max_{B(0,1)} \Phi(Du^\epsilon) \le \max_{B(0,1)} v^\epsilon(x) = v^\epsilon(x_0) \le v^\epsilon(x^0) \le \beta\lambda^2 + C\lambda \le C\lambda$$

so that (3.13) also holds. So we can also assume

$$\big(|Du^\epsilon|^2 - u^\epsilon_n\big)(x_0) > 0.$$

To estimate $v^\epsilon(x_0)$, let L_ϵ and F^ϵ_s be given by (3.6) and (3.5). We need to compute $L_\epsilon v^\epsilon$ at x^0. Using

$$\mathcal{A}_H[u^\epsilon] + \epsilon\operatorname{div}(ADu^\epsilon) = 2a^{ik}u^\epsilon_k u^\epsilon_{ij} a^{j\ell} u^\epsilon_\ell + a^{ij}_k u^\epsilon_i u^\epsilon_j a^{k\ell} u^\epsilon_\ell + \epsilon\operatorname{div}(ADu^\epsilon) = 0,$$

we obtain

$$\begin{aligned}
-L_\epsilon((u^\epsilon - x_n)^2) &= -4a^{ik}u^\epsilon_k u^\epsilon_{ij} a^{j\ell} u^\epsilon_\ell (u^\epsilon - x_n) - 4a^{ik}u^\epsilon_k a^{j\ell}u^\epsilon_\ell (u^\epsilon_i - \delta_{in})(u^\epsilon_j - \delta_{jn}) \\
&\quad - 8a^{ik}(u^\epsilon_k - \delta_{kn})u^\epsilon_{ij} a^{j\ell} u^\epsilon_\ell (u^\epsilon - x_n) \\
&\quad - 4a^{ij}_k (u^\epsilon_i - \delta_{in}) u^\epsilon_j a^{k\ell} u^\epsilon_\ell (u^\epsilon - x_n) \\
&\quad - 2a^{ij}_k u^\epsilon_i u^\epsilon_j a^{k\ell}(u^\epsilon_\ell - \delta_{\ell n})(u^\epsilon - x_n) \\
&\quad - 2\epsilon(u^\epsilon - x_n)\operatorname{div}(ADu^\epsilon - ADx_n) - 2\epsilon\langle Du^\epsilon - e_n, A(Du^\epsilon - e_n)\rangle \\
&= -4\big(\langle Du^\epsilon, ADu^\epsilon\rangle - a^{nk}u^\epsilon_k\big)^2 - 2\epsilon\langle Du^\epsilon - e_n, A(Du^\epsilon - e_n)\rangle \\
&\quad - 8a^{ik}(u^\epsilon_k - \delta_{kn})u^\epsilon_{ij} a^{j\ell} u^\epsilon_\ell (u^\epsilon - x_n) \\
&\quad - 4a^{ij}_k (u^\epsilon_i - \delta_{in}) u^\epsilon_j a^{k\ell} u^\epsilon_\ell (u^\epsilon - x_n) \\
&\quad + 2a^{ij}_k u^\epsilon_i u^\epsilon_j a^{k\ell}\delta_{\ell n}(u^\epsilon - x_n) + 2\epsilon\sum_{i=1}^n a^{in}_i (u^\epsilon - x_n) \\
&= J_1 + J_2 + J_3 + J_4 + J_5 + J_6,
\end{aligned}$$

where we denote $e_n = (0, \dots, 0, 1)$.

Applying (3.12) and Theorem 3.1, we have by straightforward calculations that

$$|J_3| \le C\lambda|D^2u^\epsilon ADu^\epsilon|,$$

and

$$|J_4|, |J_5| \le C\lambda,$$

as well as

$$|J_6| \le C\epsilon\lambda.$$

Since $\|DA\|_{L^\infty} \le \lambda$ and $A(0) = I_n$, we have $|A - I_n| \le C\lambda$ on Ω and hence

$$\begin{aligned}\left|\langle Du^\epsilon, ADu^\epsilon\rangle - a^{nk}u_k^\epsilon\right| &\ge \left||Du^\epsilon|^2 - u_n^\epsilon\right| - \left|\langle Du^\epsilon, (A - I_n)Du^\epsilon\rangle\right| \\ &\quad - \left|a^{nn} - 1\right||u_n| - \sum_{k=1}^{n-1}\left|a^{nk}u_k^\epsilon\right| \\ &\ge \left||Du^\epsilon|^2 - u_n^\epsilon\right| - C\lambda.\end{aligned}$$

Hence we have that

$$J_1 = -4\big(\langle Du^\epsilon, ADu^\epsilon\rangle - a^{nk}u_k^\epsilon\big)2 \le -4\left||Du^\epsilon|^2 - u_n^\epsilon\right|^2 + C\lambda.$$

Since $\langle \xi, A\xi\rangle \ge \frac{1}{L}|\xi|^2$, we also have

$$J_2 \le -\frac{\epsilon}{L}\left|Du^\epsilon - e_n\right|^2.$$

Combining all these estimates on J_i's, we have

$$-L_\epsilon\big((u^\epsilon - x_n)^2\big) \le -4\big(|Du^\epsilon|^2 - u_n^\epsilon\big)2 - \frac{2\epsilon}{L}|Du^\epsilon - e_n|^2 + C\lambda(1 + |D^2u^\epsilon ADu^\epsilon|). \tag{3.14}$$

Moreover, similar to the proof of Theorem 3.1, we have

$$\begin{aligned}\frac{1}{2}L_\epsilon\big(|Du^\epsilon|^2\big) &= 2|D^2u^\epsilon ADu^\epsilon|^2 + \epsilon\sum_{s=1}^{n}\big(a^{ij}u_{si}^\epsilon u_{sj}^\epsilon - u_s^\epsilon F_s^\epsilon\big) \\ &\ge 2|D^2u^\epsilon ADu^\epsilon|^2 + \frac{\epsilon}{L}|D^2u^\epsilon|^2 - C|D^2u^\epsilon ADu^\epsilon||Du^\epsilon|^2 - C|Du^\epsilon|^4 \\ &\ge |D^2u^\epsilon ADu^\epsilon|^2 + \frac{\epsilon}{L}|D^2u^\epsilon|^2 - C.\end{aligned} \tag{3.15}$$

Next we need to estimate $L_\epsilon(\phi^2\Phi(Du^\epsilon))$. First recall

$$\begin{aligned}L_\epsilon(\Phi(Du^\epsilon)) &= 2a^{ik}u_k^\epsilon a^{j\ell}u_\ell^\epsilon(\Phi(Du^\epsilon))_{ij} + \epsilon\,\mathrm{div}(AD(\Phi(Du^\epsilon))) \\ &\quad + \big(4a^{is}u_{ij}^\epsilon a^{j\ell}u_\ell^\epsilon + 2a_k^{sj}u_j^\epsilon a^{k\ell}u_\ell^\epsilon + a_k^{ij}u_i^\epsilon u_j^\epsilon a^{ks}\big)(\Phi(Du^\epsilon))_s.\end{aligned}$$

As explained earlier, we may assume $|Du^\epsilon|^2 > u_n^\epsilon$ at $x^0 \in B(0, 2)$. With this assumption we have at $x = x^0$ that

$$(\Phi(Du^\epsilon))_s = 2\big(|Du^\epsilon|^2 - u_n^\epsilon\big)\Big(2\sum_{k=1}^{n}u_{ks}^\epsilon u_k^\epsilon - u_{ns}^\epsilon\Big),$$

and

$$\begin{aligned}(\Phi(Du^\epsilon))_{ij} &= 2\Big(2\sum_{s=1}^{n}u_{sj}^\epsilon u_s^\epsilon - u_{nj}^\epsilon\Big)\Big(2\sum_{s=1}^{n}u_{si}^\epsilon u_s^\epsilon - u_{ni}^\epsilon\Big) \\ &\quad + 2\big(|Du^\epsilon|^2 - u_n^\epsilon\big)\Big(2\sum_{s=1}^{n}(u_{si}^\epsilon u_{sj}^\epsilon + u_{sij}^\epsilon u_s^\epsilon) - u_{nij}^\epsilon\Big).\end{aligned}$$

Hence we obtain that, at $x = x_0$,

$$\begin{aligned}L_\epsilon(\Phi(Du^\epsilon)) &= 4a^{ik}u_k^\epsilon a^{j\ell}u_\ell^\epsilon\Big(2\sum_{s=1}^{n}u_{sj}^\epsilon u_s^\epsilon - u_{nj}^\epsilon\Big)\Big(2\sum_{s=1}^{n}u_{si}^\epsilon u_s^\epsilon - u_{ni}^\epsilon\Big) \\ &\quad + 4\big(|Du^\epsilon|^2 - u_n^\epsilon\big)a^{ik}u_k^\epsilon a^{j\ell}u_\ell^\epsilon\Big(2\sum_{s=1}^{n}(u_{si}^\epsilon u_{sj}^\epsilon + u_{sij}^\epsilon u_s^\epsilon) - u_{nij}^\epsilon\Big)\end{aligned}$$

$$
\begin{aligned}
&+2\epsilon a^{ij}\Big(2\sum_{s=1}^{n}u^\epsilon_{si}u^\epsilon_s-u^\epsilon_{ni}\Big)\Big(2\sum_{s=1}^{n}u^\epsilon_{sj}u^\epsilon_s-u^\epsilon_{nj}\Big)\\
&+2\epsilon\big(|Du^\epsilon|^2-u^\epsilon_n\big)a^{ij}\Big(2\sum_{s=1}^{n}(u^\epsilon_{si}u^\epsilon_{sj}+u^\epsilon_{sij}u^\epsilon_s)-u^\epsilon_{nij}\Big)\\
&+2\epsilon a^{ij}_j\big(|Du^\epsilon|^2-u^\epsilon_n\big)\Big(2\sum_{s=1}^{n}u^\epsilon_{sj}u^\epsilon_s-u^\epsilon_{nj}\Big)\\
&+2\big(|Du^\epsilon|^2-u^\epsilon_n\big)\sum_{m=1}^{n}G^\epsilon_m\Big(2\sum_{s=1}^{n}u^\epsilon_{sm}u^\epsilon_s-u^\epsilon_{nm}\Big)\\
=\,&4a^{ik}u^\epsilon_k a^{j\ell}u^\epsilon_\ell\Big(2\sum_{s=1}^{n}u^\epsilon_{sj}u^\epsilon_s-u^\epsilon_{nj}\Big)\Big(2\sum_{s=1}^{n}u^\epsilon_{si}u^\epsilon_s-u^\epsilon_{ni}\Big)\\
&+8\big(|Du^\epsilon|^2-u^\epsilon_n\big)a^{ik}u^\epsilon_k a^{j\ell}u^\epsilon_\ell\Big(\sum_{s=1}^{n}u^\epsilon_{si}u^\epsilon_{sj}\Big)\\
&+2\epsilon a^{ij}\Big(2\sum_{s=1}^{n}u^\epsilon_{si}u^\epsilon_s-u^\epsilon_{ni}\Big)\Big(2\sum_{s=1}^{n}u^\epsilon_{sj}u^\epsilon_s-u^\epsilon_{nj}\Big)\\
&+4\epsilon a^{ij}\Big(|Du^\epsilon|^2-u^\epsilon_n\Big)\Big(\sum_{s=1}^{n}u^\epsilon_{sj}u^\epsilon_{sj}\Big)\\
&+2\big(|Du^\epsilon|^2-u^\epsilon_n\big)\Big(2\sum_{s=1}^{n}u^\epsilon_s L_\epsilon(u^\epsilon_s)-L_\epsilon(u^\epsilon_n)\Big)\\
=\,&K_1+K_2+K_3+K_4+K_5.
\end{aligned}
\tag{3.16}
$$

Here G^ϵ_m is as defined in (3.4). Now we estimate $K_1,\dots,K_5$ separately as follows. For K_1, we have

$$K_1=4\Big[2\langle Du^\epsilon, D^2u^\epsilon ADu^\epsilon\rangle-\langle (D^2u^\epsilon)^n, ADu^\epsilon\rangle\Big]^2,$$

where $(D^2u^\epsilon)^n$ denotes the n^{th}-row of D^2u^ϵ. For K_2, we have

$$K_2=8(|Du^\epsilon|^2-u^\epsilon_n)|D^2u^\epsilon ADu^\epsilon|^2.$$

For K_3, we have

$$K_3\ge\frac{2\epsilon}{L}\sum_{i=1}^{n}\Big(2\sum_{s=1}^{n}u^\epsilon_{si}u^\epsilon_s-u^\epsilon_{ni}\Big)^2.$$

For K_4, we have

$$K_4\ge\frac{4\epsilon}{L}\big(|Du^\epsilon|^2-u^\epsilon_n\big)\big|D^2u^\epsilon\big|^2.$$

From (3.7), we have

$$K_5=2\big(|Du^\epsilon|^2-u^\epsilon_n\big)\Big(\sum_{s=1}^{n}2u^\epsilon_sF^\epsilon_s-F^\epsilon_n\Big),$$

so that we can apply Theorem 3.1 to estimate

$$\big|K_5\big|\le\big(|Du^\epsilon|^2-u^\epsilon_n\big)\big(C\lambda|D^2u^\epsilon ADu^\epsilon|+\frac{\epsilon}{4L}|D^2u^\epsilon|^2+C\lambda\big).$$

Putting these estimates into (3.16) gives

$$
\begin{aligned}
L_\epsilon(\Phi(Du^\epsilon)) \ge\ & 8\big(|Du^\epsilon|^2 - u_n^\epsilon\big)\Big(|D^2u^\epsilon ADu^\epsilon|^2 + \frac{\epsilon}{4L}|D^2u^\epsilon|^2\Big) \\
& + 4\Big[2\langle Du^\epsilon, D^2u^\epsilon ADu^\epsilon\rangle - \langle (D^2u^\epsilon)^n, ADu^\epsilon\rangle\Big]^2 \\
& + \frac{2\epsilon}{L}\sum_{i=1}^n\Big(2\sum_{s=1}^n u_{si}^\epsilon u_s^\epsilon - u_{ni}^\epsilon\Big)^2 \\
& - C\lambda(|Du^\epsilon|^2 - u_n^\epsilon)|D^2u^\epsilon ADu^\epsilon| - C\lambda. \qquad (3.17)
\end{aligned}
$$

It follows from (3.17) that

$$
\begin{aligned}
L_\epsilon\big(\phi^2\Phi(Du^\epsilon)\big) =\ & \phi^2 L_\epsilon\big(\Phi(Du^\epsilon)\big) + \Phi(Du^\epsilon)L_\epsilon\big(\phi^2\big) \\
& + 4a^{ik}u_k^\epsilon a^{jl}u_l^\epsilon \phi\phi_i(\Phi(Du^\epsilon))_j + 2\epsilon\phi a^{ij}\phi_i(\Phi(Du^\epsilon))_j \\
\ge\ & 8\phi^2\big(|Du^\epsilon|^2 - u_n^\epsilon\big)\big|D^2u^\epsilon ADu^\epsilon\big|^2 + \Phi\big(Du^\epsilon\big)L_\epsilon\big(\phi^2\big) \\
& + 4\phi^2\Big[2\langle Du^\epsilon, D^2u^\epsilon ADu^\epsilon\rangle - \langle (D^2u^\epsilon)^n, ADu^\epsilon\rangle\Big]^2 \\
& + 4a^{ik}u_k^\epsilon a^{j\ell}u_\ell^\epsilon \phi\phi_i(\Phi(Du^\epsilon))_j + \frac{2\epsilon}{L}\phi^2\sum_{i=1}^n\Big(2\sum_{s=1}^n u_{si}^\epsilon u_s^\epsilon - u_{ni}^\epsilon\Big)^2 \\
& + 2\epsilon\phi a^{ij}\phi_i(\Phi(Du^\epsilon))_j - C\lambda\phi^2\Big[1 + \big(|Du^\epsilon|^2 - u_n^\epsilon\big)\big|D^2u^\epsilon ADu^\epsilon\big|\Big].
\end{aligned}
$$

It is easy to see that

$$
\begin{aligned}
|L_\epsilon\big(\phi^2\big)| =\ & \Big|2a^{ik}u_k^\epsilon a^{j\ell}u_\ell^\epsilon(\phi^2)_{ij} + \epsilon\,\mathrm{div}(AD\phi^2) \\
& + \Big(4a^{is}u_{ij}^\epsilon a^{j\ell}u_\ell^\epsilon + 2a_k^{sj}u_j^\epsilon a^{k\ell}u_\ell^\epsilon + a_k^{ij}u_i^\epsilon u_j^\epsilon a^{ks}\Big)(\phi^2)_s\Big| \\
\le\ & C|Du^\epsilon|^2 + \phi\big|D^2u^\epsilon ADu^\epsilon\big| + C\epsilon \\
\le\ & \phi\big|D^2u^\epsilon ADu^\epsilon\big| + C,
\end{aligned}
$$

so that

$$
\Phi\big(Du^\epsilon\big)|L_\epsilon\big(\phi^2\big)| \le \big(|Du^\epsilon|^2 - u_n^\epsilon\big)2\big(\phi|D^2u^\epsilon ADu^\epsilon| + C\big).
$$

By Young's inequality, we have

$$
\begin{aligned}
& 4a^{ik}u_k^\epsilon a^{jl}u_l^\epsilon\phi\phi_i(\Phi(Du^\epsilon))_j \\
& = 8a^{ik}u_k^\epsilon a^{j\ell}u_\ell^\epsilon\phi\phi_i\big(|Du^\epsilon|^2 - u_n^\epsilon\big)\Big(2\sum_{s=1}^n u_{sj}^\epsilon u_s^\epsilon - u_{nj}^\epsilon\Big) \\
& = 8a^{ik}u_k^\epsilon\phi\phi_i(|Du^\epsilon|^2 - u_n^\epsilon)\cdot\Big(2\langle Du^\epsilon, D^2u^\epsilon ADu^\epsilon\rangle - \langle (D^2u^\epsilon)^n, ADu^\epsilon\rangle\Big) \\
& \le 4\phi^2\Big[2\langle Du^\epsilon, D^2u^\epsilon ADu^\epsilon\rangle - \langle (D^2u^\epsilon)^n, ADu^\epsilon\rangle\Big]^2 \\
& \quad + 16\Big[\langle D\phi, ADu^\epsilon\rangle(|Du^\epsilon|^2 - u_n^\epsilon)\Big]^2.
\end{aligned}
$$

Thus by Theorem 3.1, we obtain

$$
\begin{aligned}
& 4\phi^2\Big[2\langle Du^\epsilon, D^2u^\epsilon ADu^\epsilon\rangle - \langle (D^2u^\epsilon)^n, ADu^\epsilon\rangle\Big]^2 + 4a^{ik}u_k^\epsilon a^{j\ell}u_\ell^\epsilon\phi\phi_i(\Phi(Du^\epsilon))_j \\
& \ge -16\Big[\langle D\phi, ADu^\epsilon\rangle(|Du^\epsilon|^2 - u_n^\epsilon)\Big]^2 \\
& \ge -C\big(|Du^\epsilon|^2 - u_n^\epsilon\big)2.
\end{aligned}
$$

Similarly, by Young's inequality, we have that

$$2\epsilon\phi a^{ij}\phi_i\big(\Phi(Du^\epsilon)\big)_j = 4\epsilon\phi a^{ij}\phi_i\big(|Du^\epsilon|^2 - u_n^\epsilon\big)\Big(2\sum_{s=1}^n u_{sj}^\epsilon u_s^\epsilon - u_{nj}^\epsilon\Big)$$
$$\le C\epsilon|D\phi|^2\big(|Du^\epsilon|^2 - u_n^\epsilon\big)^2 + \frac{\epsilon}{L}\phi^2\sum_{i=1}^n\Big(2\sum_{s=1}^n u_{si}^\epsilon u_s^\epsilon - u_{ni}^\epsilon\Big)^2,$$

which gives

$$\begin{aligned}
&\frac{2\epsilon}{L}\sum_{i=1}^n\Big(2\sum_{s=1}^n u_{si}^\epsilon u_s^\epsilon - u_{ni}^\epsilon\Big)^2\phi^2 - 2\epsilon\phi a^{ij}\phi_i\big(\Phi(Du^\epsilon)\big)_j\\
&\quad\ge -C\epsilon|D\phi|^2\big(|Du^\epsilon|^2 - u_n^\epsilon\big)2\\
&\quad\ge -C\epsilon\big(|Du^\epsilon|^2 - u_n^\epsilon\big)2.
\end{aligned}$$

Putting all these estimates together and applying Young's inequality, we conclude that

$$\begin{aligned}
L_\epsilon\big(\phi^2\Phi(Du^\epsilon)\big) &\ge 8\phi^2\big(|Du^\epsilon|^2 - u_n^\epsilon\big)\big|D^2u^\epsilon ADu^\epsilon\big|^2 - C\big(|Du^\epsilon|^2 - u_n^\epsilon\big)2\\
&\quad - \big(|Du^\epsilon|^2 - u_n^\epsilon\big)2\big(\phi|D^2u^\epsilon ADu^\epsilon| + C\big)\\
&\quad - C\lambda\big(|Du^\epsilon|^2 - u_n^\epsilon\big)\big|D^2u^\epsilon ADu^\epsilon\big|\phi^2 - C\lambda\phi^2\\
&\ge -C(|Du^\epsilon|^2 - u_n^\epsilon)^3 - C(|Du^\epsilon|^2 - u_n^\epsilon)^2 - C\lambda(|Du^\epsilon|^2 - u_n^\epsilon) - C\lambda\phi^2\\
&\ge -C(|Du^\epsilon|^2 - u_n^\epsilon)^2 - C\lambda(|Du^\epsilon|^2 - u_n^\epsilon) - C\lambda.
\end{aligned} \tag{3.18}$$

Combining the estimates (3.14), (3.15), with (3.18) yields that, at $x = x_0$,

$$\begin{aligned}
0 &\le -L_\epsilon\big(v^\epsilon\big) = -L_\epsilon\big(\phi^2\Phi(Du^\epsilon)\big) - \beta L_\epsilon\big((u^\epsilon - x_n)^2\big) - \lambda L_\epsilon\big(|Du^\epsilon|^2\big)\\
&\le C\big(|Du^\epsilon|^2 - u_n^\epsilon\big)^2 + C\lambda\big(|Du^\epsilon|^2 - u_n^\epsilon\big) + C\lambda\\
&\quad - 4\beta\big(|Du^\epsilon|^2 - u_n^\epsilon\big)2 - \frac{2\epsilon\beta}{L}\big|Du^\epsilon - e_n\big|^2 + C\beta\lambda + C\beta\lambda\big|D^2u^\epsilon ADu^\epsilon\big|\\
&\quad + 2\lambda\Big(-|D^2u^\epsilon ADu^\epsilon|^2 - \frac{\epsilon}{L^2}|D^2u^\epsilon|^2 + C\Big).
\end{aligned}$$

Thus we have that, at $x = x_0$,

$$\begin{aligned}
&(4\beta - C)\big(|Du^\epsilon|^2 - u_n^\epsilon\big)2 + 2\lambda\big|D^2u^\epsilon ADu^\epsilon\big|^2 + \frac{2\lambda\epsilon}{L^2}\big|D^2u^\epsilon\big|^2\\
&\quad\le C\lambda\big(|Du^\epsilon|^2 - u_n^\epsilon\big) + C(1+\beta)\lambda + C\beta\lambda\big|D^2u^\epsilon ADu^\epsilon\big|.
\end{aligned}$$

Choosing $\beta > C$ and applying Young's inequality, we obtain

$$\beta\big(|Du^\epsilon|^2 - u_n^\epsilon\big)2 \le C\lambda + 2\beta^2\lambda.$$

Thus we conclude that, at $x = x_0$,

$$\big(|Du^\epsilon|^2 - u_n^\epsilon\big)2 \le C\lambda.$$

This completes the proof. □

4. Differentiability

This section is devoted to the proof of Theorem 1.1. In order to do it, we need some lemmas. The first lemma is the linear approximation property, which was proved by Yu for some general Hamiltonian functions $H(x, p)$ (see [30] Theorem 2.9 and Remark 2.11); later a proof based on the intrinsic distance was given in [22] Theorem 6 for the special case $H(x, p) = \langle A(x)p, p\rangle$. Below for reader's convenience, we sketch the proof of Lemma 4.1 based on the intrinsic distance and [22].

Lemma 4.1. *Let $A \in \mathscr{A}(\Omega) \cap C(\overline{\Omega})$ and $u \in C^{0,1}(\Omega)$ be an absolute minimizer of $\mathscr{F}_\infty$ with respect to A in Ω. Then for each $x \in \Omega$ and every sequence $\{r_j\}_{j\in\mathbb{N}}$ converging to 0, there exists a subsequence $\mathbf{r} = \{r_{j_k}\}_{k\in\mathbb{N}}$ and a vector $\mathbf{e}_{x,\mathbf{r}} \in \mathbb{R}^n$ such that*

$$\lim_{k\to\infty} \sup_{y\in B(0,\,1)} \left| \frac{u(x + r_{j_k} y) - u(x)}{r_{j_k}} - \langle \mathbf{e}_{x,\,\mathbf{r}},\ y\rangle \right| = 0, \tag{4.1}$$

and $H(x,\ \mathbf{e}_{x,\,\mathbf{r}}) = \mathrm{Lip}_{d_A} u(x)$. Here

$$\mathrm{Lip}_{d_A} u := \limsup_{y\to x} \frac{|u(y) - u(x)|}{d_A(x,\,y)},$$

and

$$d_A(x,\,y) := \sup\Big\{ w(x) - w(y) :\ w \in C^{0,1}(\Omega) \text{ satisfies } H(z,\,Dw(z)) \le 1 \text{ a.e. } z \in \Omega \Big\}.$$

Sketch of the proof of Lemma 4.1. Without loss of generality, assume $x = 0 \in \Omega$ and $u(0) = 0$. We also assume $\mathrm{Lip}_{d_A} u(0) > 0$, since the case $\mathrm{Lip}_{d_A} u(0) = 0$ is trivial.

For any fixed $r_0 \in \big(0,\, d_A(0,\,\partial\Omega)\big)$, assume that $r_{j+1} < r_j < r_0$ for all j. For each $j \in \mathbb{N}$, define

$$u_j(y) = \frac{1}{r_j} u(r_j y),\ \ A_j(y) = A(r_j y),\ \ y \in B\big(0, r_j^{-1} r_0\big),$$
$$A_\infty(y) = A(0),\ \ y \in \mathbb{R}^n,$$

and

$$H_j(x,\xi) = \langle A_j(x)\xi,\ \xi\rangle,\ \ x \in B\big(0, r_j^{-1} r_0\big),\ \ \xi \in \mathbb{R}^n.$$

Also let d_j denote the intrinsic distance d_{A_j} corresponding to A_j.

Recall that by [22] Lemma 15 there exists $u_\infty \in W^{1,\infty}(\mathbb{R}^n)$ and a subsequence $\{r_{j_k}\}_{k\in\mathbb{N}}$ of $\{r_j\}_{j\in\mathbb{N}}$ such that u_{j_k} converges to u_∞ locally uniformly in $\mathbb{R}^n$, and weak* in $W^{1,\,\infty}(\mathbb{R}^n)$. Moreover, by [22] Lemma 19 that there exists a vector $\mathbf{e} \in \mathbb{R}^n$ such that

$$u_\infty(x) = \langle e, x\rangle,\ \ x \in \mathbb{R}^n,\ \text{ and } H_\infty(\mathbf{e})\big(\equiv H_\infty(0,\mathbf{e})\big) = \mathrm{Lip}_{d_\infty} u_\infty(0).$$

From this, we conclude that

$$\sup_{y\in B(0,1)} \Big| \frac{1}{r_{j_k}} u(r_{j_k} y) - \langle \mathbf{e},\ y\rangle \Big| = \sup_{y\in B(0,1)} \big| u_{j_k}(y) - \langle \mathbf{e},\ y\rangle \big| = \sup_{y\in B(0,1)} \big| u_{j_k}(y) - u_\infty(y) \big| \to 0$$

as $k \to \infty$, and $H_\infty(\mathbf{e}) = \mathrm{Lip}_{d_A} u(0)$. This completes the proof. $\square$

Given a pair of functions $A \in \mathscr{A}(\Omega) \cap C(\overline{\Omega})$ and $u \in C^{0,1}(\Omega)$, and a pair of $0 \ne r \in \mathbb{R}$ and $x_0 \in \Omega$, we define

$$A_{x_0,r}(y) = A(x_0 + ry),\ \ u_{x_0,r}(y) = \frac{u(x_0 + ry) - u(x_0)}{r},\ \ \ y \in \Omega_{x_0,r} := r^{-1}\big(\Omega \setminus \{x_0\}\big).$$

Similarly, for any $x_0 \in \Omega$ and any non-singular matrix $M \in \mathbb{R}^{n\times n}$, we define

$$A_{x_0,M}(y) = A(x_0 + My),\ \ u_{x_0,M}(y) = M^{-1}\big(u(x_0 + My) - u(x_0)\big),$$

for $y \in \Omega_{x_0,M} := M^{-1}\big(\Omega \setminus \{x_0\}\big)$.

The following scaling invariant property of absolute miminizers of $\mathscr{F}_\infty$ is a simple consequence of change of variables, whose proof is left for the readers.

Lemma 4.2. *For any $x_0 \in \Omega$, $r \ne 0$, and a non-singular matrix $M \in \mathbb{R}^{n\times n}$, if $u \in C^{0,1}(\Omega)$ is an absolute minimizer of $\mathscr{F}_\infty$, with respect to A, in Ω, then $u_{x_0,r}$ is an absolute minimizer of $\mathscr{F}_\infty$, with respect to $A_{x_0,r}$, in $\Omega_{x_0,r}$, and $u_{x_0,M}$ is an absolute minimizer of $\mathscr{F}_\infty$, with respect to $A_{x_0,M}$, in $\Omega_{x_0,M}$.*

We also need the following lemma, which was proved in [19].

Lemma 4.3. *For* $\mathbf{b}\in\mathbb{S}^{n-1}$ *and* $\eta>0$, *if* $v\in C^2(B(0,1))$ *satisfies*

$$\sup_{x\in B(0,1)}\left|v(x)-\langle\mathbf{b},x\rangle\right|\le\eta,$$

then there exists a point $x_0\in B(0,1)$ *such that*

$$\left|Dv(x_0)-\mathbf{b}\right|\le 4\eta.$$

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. For every point $x_0\in\Omega$, we will show that there exists a vector $Du(x_0)\in\mathbb{R}^n$ such that

$$|u(x_0+h)-u(x_0)-\langle Du(x_0),\ h\rangle|=o(|h|),\ \forall\, h\in\mathbb{R}^n. \tag{4.2}$$

From Lemma 4.2, we may assume that $x_0=0$, $u(x_0)=0$, and $A(x_0)=I_n$. By Theorem 4.1, in order to prove (4.2), it suffices to show that for every pair of sequences $\mathbf{r}=\{r_j\}$ and $\mathbf{s}=\{s_k\}$ that converge to 0, if

$$\lim_{j\to\infty}\sup_{y\in B(0,\,3r_j)}\frac{1}{r_j}|u(y)-\langle\mathbf{a},\ y\rangle|=0 \tag{4.3}$$

and

$$\lim_{k\to\infty}\sup_{y\in B(0,\,3s_k)}\frac{1}{s_k}|u(y)-\langle\mathbf{b},\ y\rangle|=0 \tag{4.4}$$

for some $\mathbf{a},\ \mathbf{b}\in\mathbb{R}^n$, then $\mathbf{a}=\mathbf{b}$.

Since $H(0,\mathbf{a})=\langle\mathbf{a},\ \mathbf{a}\rangle=\langle\mathbf{b},\ \mathbf{b}\rangle=H(0,\mathbf{b})=\mathrm{Lip}_{d_A}u(0)$, we have $|\mathbf{a}|=|\mathbf{b}|$. We prove the above claim by contradiction. Suppose that $0\ne\mathbf{a}\ne\mathbf{b}$. Then, without loss of generality, we may assume that $\mathbf{a}=e_n$. For, otherwise, let M be a nonsingular matrix such that $M\mathbf{a}=\mathbf{e}_n$. Set $v(y)=\frac{u(|\mathbf{a}|M^Ty)}{|\mathbf{a}|}$ and $\widetilde{A}(y)=A(|\mathbf{a}|M^Ty)M$. Then by Lemma 4.2 v is an absolute minimizer of $\mathscr{F}_\infty$, with respect to $\widetilde{A}$. It is clear that (4.3) holds with u and $\mathbf{a}$ replaced by v and e_n respectively.

Since $|\mathbf{b}|=|e_n|=1$ and $\mathbf{b}\ne e_n$, we have

$$\theta:=1-b_n>0.$$

Let $C>0$ be the constant in (3.13) and choose $\lambda>0$ such that

$$C\lambda^{\frac12}=\frac{\theta}{4}.$$

Choose $r\in\{r_j\}$ such that

$$\sup_{y\in B(0,\,3r)}\frac{1}{r}|u(y)-y_n|\le\frac{\lambda}{4}, \tag{4.5}$$

and

$$\begin{cases}\frac{2}{1+2^{1/5}}|\xi|^2\le\left\langle A(x)\xi,\ \xi\right\rangle\le\frac{1+2^{1/5}}{2}|\xi|^2,\ x\in B(0,3r),\ \xi\in\mathbb{R}^n,\\ r\left\|DA\right\|_{L^\infty(B(0,3))}+r^2\left\|D^2A\right\|_{L^\infty(B(0,3))}\le\frac12\min\left\{\delta(B(0,\ 3)),\lambda\right\},\end{cases} \tag{4.6}$$

where $\delta(B(0,3))$ is the constant given by Theorem 3.2.

For $x\in B(0,3)$, let $\widetilde{A}(x)=A(rx)$ and $\widetilde{u}(x)=\frac1r u(rx)$. Since $D\widetilde{A}(x)=r(DA)(rx)$ and $D^2\widetilde{A}(x)=r^2(D^2A)(rx)$ for $x\in B(0,3)$, it follows from (4.6) that

$$\begin{cases}\frac{2}{1+2^{1/5}}|\xi|^2\le\left\langle\widetilde{A}(x)\xi,\xi\right\rangle\le\frac{1+2^{1/5}}{2}|\xi|^2,\ x\in B(0,3),\ \xi\in\mathbb{R}^n,\\ \left\|D\widetilde{A}\right\|_{L^\infty(B(0,3))}+\left\|D^2\widetilde{A}\right\|_{L^\infty(B(0,3))}\le\frac12\min\left\{\delta(B(0,\ 3)),\lambda\right\}.\end{cases}$$

Let $\widetilde{A}_\epsilon\in\mathscr{A}(\Omega)\cap C^\infty(\Omega)$ such that

(i) $\|\widetilde{A}_\epsilon\|_{C^{1,1}(B(0,3))} \le 2\|\widetilde{A}\|_{C^{1,1}(B(0,3))}$ for all $\epsilon > 0$,
(ii) for any $0 < \alpha < 1$, $\widetilde{A}_\epsilon \to \widetilde{A}$ in $C^{1,\alpha}(B(0,3))$ as $\epsilon \to 0$.

Then there exists an $\epsilon_0 > 0$ such that for $\epsilon < \epsilon_0$

$$\begin{cases} \frac{2}{1+2^{1/4}}|\xi|^2 \le \langle \widetilde{A}_\epsilon(x)\xi, \xi\rangle \le \frac{1+2^{1/4}}{2}|\xi|^2, \ x \in B(0,3),\ \xi \in \mathbb{R}^n, \\ \|D\widetilde{A}_\epsilon\|_{L^\infty(B(0,3))} + \|D^2\widetilde{A}_\epsilon\|_{L^\infty(B(0,3))} \le \min\{\delta(B(0,\,3)), \lambda\}. \end{cases} \tag{4.7}$$

Let $\widetilde{u}^\epsilon \in C^{0,1}(B(0,3))$ be the unique solution of (2.4) associated with $\widetilde{A}_\epsilon$ and $H_{\widetilde{A}_\epsilon}$, with u and Ω replaced by $\widetilde{u}$ and $B(0,3)$ respectively. Then, by Theorem 3.2, we have that $\widetilde{u}^\epsilon \to \widetilde{u}$ uniformly in $B(0,3)$. By Lemma 4.2, $\widetilde{u}$ is an absolute minimizer of $\mathscr{F}_\infty$ with respect to $\widetilde{A}$. From (4.5), we also have

$$\sup_{y\in B(0,3)} |\widetilde{u}(y) - y_n| \le \frac{\lambda}{4}.$$

Hence there exists $\epsilon_1 \in (0, \epsilon_0)$ such that for all $\epsilon < \epsilon_1$,

$$\sup_{y\in B(0,3)} \left|\widetilde{u}^\epsilon(y) - y_n\right| \le \frac{\lambda}{2}. \tag{4.8}$$

Setting $\widetilde{s}_k = s_k/r$. Then we have

$$\lim_{k\to\infty} \sup_{y\in B(0,3\widetilde{s}_k)} \frac{1}{\widetilde{s}_k} |\widetilde{u}(y) - \langle \mathbf{b},\ y\rangle| = 0.$$

Choose $\eta = \frac{\theta}{48}$ and pick $s \in \{\widetilde{s}_k\}$, with $0 < s < 1$, so that

$$\sup_{y\in B(0,\ s)} \frac{1}{s} |\widetilde{u}(y) - \langle \mathbf{b},\ y\rangle| \le \frac{\eta}{2}.$$

By Theorem 3.2, there exists $\epsilon_2 > 0$ such that for all $\epsilon < \epsilon_2$,

$$\sup_{y\in B(0,\ s)} \frac{1}{s} \left|\widetilde{u}^\epsilon(y) - \langle \mathbf{b},\ y\rangle\right| \le \eta.$$

Applying Lemma 4.3 to $\frac{1}{s}\widetilde{u}^\epsilon(s\cdot)$, we can find a point $x_0 \in B(0,s)$ such that

$$\left|D\widetilde{u}^\epsilon(x_0) - \mathbf{b}\right| \le 4\eta,$$

which, combined with $|\mathbf{b}| = 1$, yields

$$\begin{cases} \widetilde{u}^\epsilon_n(x_0) \le b_n + 4\eta \le 1 - \theta + 4\eta, \\ |D\widetilde{u}^\epsilon(x_0)| \ge 1 - 4\eta. \end{cases} \tag{4.9}$$

From (4.8), we can apply Theorem 3.3 to conclude

$$\left|D\widetilde{u}^\epsilon(x_0)\right|^2 \le \widetilde{u}^\epsilon_n(x_0) + C\lambda^{1/2} \le \widetilde{u}^\epsilon_n(x_0) + \frac{\theta}{4}.$$

This, combined with (4.9), implies that

$$(1 - 4\eta)^2 \le 1 - \theta + 4\eta + \frac{\theta}{4},$$

so that

$$\theta \le 12\eta + \frac{\theta}{4} \le \frac{\theta}{2},$$

this is impossible. Thus $\mathbf{a} = \mathbf{b}$, and there is a unique tangent plane at 0 and u is differentiable at 0. The proof is complete. □

5. Lebesgue points of the gradient

In this last section, we show that every point is a Lebesgue point for the gradient, which extends the property on infinity harmonic functions by [18].

Theorem 5.1. *Let $A \in \mathscr{A}(\Omega) \cap C^{1,1}(\Omega)$ and u be a viscosity solution of the Aronsson equation (1.1). Then every point in Ω is a Lebesgue point of Du.*

For the intrinsic distance d_A associated with A, define the intrinsic ball

$$B_{d_A}(x,r) := \left\{ y \;\middle|\; d_A(x,y) < r \right\}$$

for $x \in \Omega$ and $0 < r < d_A(x, \partial\Omega)$. For $E \subset \mathbb{R}^n$, define $\fint_E f = \frac{1}{|E|}\int_E f$.

Lemma 5.2. *For $0 < \lambda < 1$, let $A \in \mathscr{A}(\Omega) \cap C^{1,1}(\Omega)$ such that $A(0) = I_n$ and $\|DA\|_{L^\infty(\Omega)} \le \lambda^2$. Assume $u \in C^{0,1}(\Omega)$ is an absolute minimizer of $\mathscr{F}_\infty$ with respect to A, and satisfies, for $B_{d_A}(0,3) \subset \Omega$,*

$$\max_{x \in B_{d_A}(0,3)} \left| u(x) - u(0) - \langle \mathbf{a}, x\rangle \right| \le \lambda.$$

Then there exists a constant $C > 0$ depending on $|\mathbf{a}|$ such that

$$\fint_{B_{d_A}(0,1)} \left| Du(x) - \mathbf{a} \right|^2 dx \le C\lambda. \tag{5.1}$$

Proof. Since

$$(1 + C\lambda^2)^{-1}|\xi|^2 \le \langle A(x)\xi, \xi\rangle \le (1 + C\lambda^2)|\xi|^2, \ \forall x \in \Omega, \ \xi \in \mathbb{R}^n,$$

we have

$$(1 + C\lambda)^{-1}|x - y| \le d_A(x, \ y) \le (1 + C\lambda)|x - y|, \ \forall x, y \in \Omega.$$

It suffices to show that

$$\fint_{B(0,1+C\lambda)} \left| Du(x) - \mathbf{a} \right|^2 dx \le C\lambda. \tag{5.2}$$

By the same argument as in the proof of [18] Theorem 4.1, (5.2) follows if

$$\sup_{x \in B(0,1+C\lambda)} |Du(x)| \le |\mathbf{a}| + C\lambda. \tag{5.3}$$

To prove (5.3), let

$$S_r^+ u(x) := \max_{d_A(z,x)=r} \frac{u(z) - u(x)}{r}.$$

A simple modification of the proof of [22] Theorem 2 shows that $S_r^+ u(x)$ is monotone increasing with respect to r, and

$$\sqrt{\langle A(x)Du(x), Du(x)\rangle} = \mathrm{Lip}_{d_A} u(x) = \lim_{r \to 0} S_r^+ u(x).$$

This implies

$$|Du(x)| \le (1 + C\lambda) S_1^+ u(x), \ x \in B(0, 1 + C\lambda).$$

For $x \in B(0, 1 + C\lambda)$, if $B_{d_A}(x,1) \subset B_{d_A}(0,3)$ and $d_A(z,x) = 1$, then we have

$$|u(x) - u(z)| \le |u(x) - u(0) - \langle \mathbf{a}, x\rangle| + |u(z) - u(0) - \langle \mathbf{a}, z\rangle| + |\langle \mathbf{a}, x - z\rangle|$$
$$\le 2\lambda + |\mathbf{a}||x - z| \le |\mathbf{a}| + C\lambda,$$

which implies that

$$S_1^+ u(x) \le |\mathbf{a}| + C\lambda, \ \forall x \in B(0, 1 + C\lambda).$$

Hence we have that

$$|Du(x)| \le |\mathbf{a}| + C\lambda, \ \forall \, x \in B(0, 1 + C\lambda).$$

The proof is completed by applying the argument in Theorem 4.1 of [18]. □

Proof of Theorem 5.1. We want to show that for every $x_0 \in \Omega$ and for every $\epsilon > 0$, there exists $r_0 > 0$ such that

$$⨍_{B_{d_A}(x_0,r)} \left|Du(x) - Du(x_0)\right|^2 dx \le \epsilon,$$

for every $r \le r_0$. As before, by Lemma 4.2, we may assume that $x_0 = 0$, $u(0) = 0$ and $A(0) = I_n$. For an arbitrary $0 < \lambda < 1$, since u is differentiable at 0, there exists $r_0 < \lambda^2$ such that

$$\max_{z \in B_{d_A}(0,\, 3r)} \left|u(x) - \langle Du(0), x\rangle\right| \le \lambda r, \ \ 0 < r \le r_0. \tag{5.4}$$

Set $A_r(x) = A(rx)$ and $u_r(x) = \dfrac{u(rx)}{r}$. Then u_r is an absolute minimizer of $\mathscr{F}_\infty$ associated to A_r. Observe that $\|DA_r\| \le r\|DA\|$ and by [22] Lemma 5.4, $d_{A_r}(rx, ry) = r d_A(x, y)$. Hence $B_{d_A}(0, r) = r B_{d_{A_r}}(0, 1)$. Therefore, (5.4) implies

$$\max_{x \in B_{d_{A_r}}(0,\, 3)} \left|u_r(x) - \langle Du(0), x\rangle\right| \le \lambda, \ \ 0 < r \le r_0. \tag{5.5}$$

Now we can apply Lemma 5.2 to conclude that

$$⨍_{B_{d_A}(0,r)} \left|Du(x) - Du(0)\right|^2 dx = ⨍_{B_{d_{A_r}}(0,1)} \left|Du_r(x) - Du(0)\right|^2 dx$$
$$\le C\lambda,$$

for every $r \le r_0$ and λ small enough. This completes the proof. □

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References

[1] G. Aronsson, Minimization problems for the functional $\sup_x F(x, f(x), f'(x))$, Ark. Mat. 6 (1965) 33–53.
[2] G. Aronsson, Minimization problems for the functional $\sup_x F(x, f(x), f'(x))$. II, Ark. Mat. 6 (1966) 409–431.
[3] G. Aronsson, Extension of functions satisfying Lipschitz conditions, Ark. Mat. 6 (1967) 551–561.
[4] G. Aronsson, Minimization problems for the functional $\sup_x F(x, f(x), f'(x))$. III, Ark. Mat. 7 (1969) 509–512.
[5] S.N. Armstrong, M.G. Crandall, V. Julin, C.K. Smart, Convexity criteria and uniqueness of absolutely minimizing functions, Arch. Ration. Mech. Anal. 200 (2) (2011) 405–443.
[6] S. Armstrong, C. Smart, An easy proof of Jensen's theorem on the uniqueness of infinity harmonic functions, Calc. Var. Partial Differ. Equ. 37 (3–4) (2010) 381–384.
[7] G. Aronsson, M.G. Crandall, P. Juutinen, A tour of the theory of absolutely minimizing functions, Bull., New Ser., Am. Math. Soc. 41 (2004) 439–505.
[8] E.N. Barron, R.R. Jensen, C.Y. Wang, The Euler equation and absolute minimizers of L^∞ functionals, Arch. Ration. Mech. Anal. 157 (2001) 255–283.

[9] C. Bjorland, L. Caffarelli, A. Figalli, Non-local tug-of-war and the infinity fractional Laplacian, Commun. Pure Appl. Math. 65 (3) (2012) 337–380.

[10] T. Bieske, F. Federica, J. Manfredi, The Carnot–Carathéodory distance and the infinite Laplacian, J. Geom. Anal. 19 (2009) 737–754.

[11] A. Chambolle, E. Lindgren, R. Monneau, The Hölder infinite Laplacian and Hölder extensions, ESAIM Control Optim. Calc. Var. 18 (3) (2012) 799–835.

[12] M.G. Crandall, A visit with the ∞-Laplace equation, in: Calculus of Variations and Nonlinear Partial Differential Equations, in: Lecture Notes in Math., vol. 1927, Springer, Berlin, 2008, pp. 75–122.

[13] M.G. Crandall, L.C. Evans, R.F. Gariepy, Optimal Lipschitz extensions and the infinity Laplacian, Calc. Var. Partial Differ. Equ. 13 (2001) 123–139.

[14] M. Crandall, H. Ishii, P. Lions, User's guide to viscosity solutions of second order partial differential equations, Bull., New Ser., Am. Math. Soc. 27 (1) (1992) 1–67.

[15] M.G. Crandall, C.Y. Wang, Y.F. Yu, Derivation of the Aronsson equation for C^1-Hamiltonians, Trans. Am. Math. Soc. 361 (1) (2009) 103–124.

[16] D.M. Duc, J. Eells, Regularity of exponentially harmonic functions, Int. J. Math. 2 (4) (1991) 395–408.

[17] L.C. Evans, O. Savin, $C^{1,\alpha}$ regularity for infinity harmonic functions in two dimensions, Calc. Var. Partial Differ. Equ. 32 (2008) 325–347.

[18] L.C. Evans, C.K. Smart, Everywhere differentiability of infinity harmonic functions, Calc. Var. Partial Differ. Equ. 42 (2011) 289–299.

[19] L.C. Evans, C.K. Smart, Adjoint methods for the infinity Laplacian partial differential equation, Arch. Ration. Mech. Anal. 201 (2011) 87–113.

[20] R. Jensen, Uniqueness of Lipschitz extensions: minimizing the sup norm of the gradient, Arch. Ration. Mech. Anal. 123 (1993) 51–74.

[21] R. Jensen, C.Y. Wang, Y.F. Yu, Uniqueness and nonuniqueness of viscosity solutions to Aronsson's equation, Arch. Ration. Mech. Anal. 190 (2) (2008) 347–370.

[22] P. Koskela, N. Shanmugalingam, Y. Zhou, Intrinsic geometry and analysis of diffusion process and L^∞-variational problem, Arch. Ration. Mech. Anal. 214 (1) (2014) 99–142.

[23] E. Lindgren, On the regularity of solutions of the inhomogeneous infinity Laplace equation, Proc. Am. Math. Soc. 142 (1) (2014) 277–288.

[24] G. Lieberman, On the regularity of the minimizer of a functional with exponential growth, Comment. Math. Univ. Carol. 33 (1) (1992) 45–49.

[25] G. Lieberman, Gradient estimates for a class of elliptic systems, Ann. Mat. Pura Appl. (4) CLXIV (1993) 103–120.

[26] Y. Peres, O. Schramm, S. Sheffield, D.B. Wilson, Tug-of-war and the infinity Laplacian, J. Am. Math. Soc. 22 (2009) 167–210.

[27] O. Savin, C^1 regularity for infinity harmonic functions in two dimensions, Arch. Ration. Mech. Anal. 176 (2005) 351–361.

[28] C.Y. Wang, Y.F. Yu, C^1-regularity of the Aronsson equation in $\mathbb{R}^2$, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 25 (2008) 659–678.

[29] C.Y. Wang, Y.F. Yu, C^1-boundary regularity of planar infinity harmonic functions, Math. Res. Lett. 19 (4) (2012) 823–835.

[30] Y.F. Yu, L^∞ variational problems and weak KAM theory, Commun. Pure Appl. Math. 60 (8) (2007) 1111–1147.